The Quantum Spectral Method:

From Atomic Orbitals to Classical Self-Force

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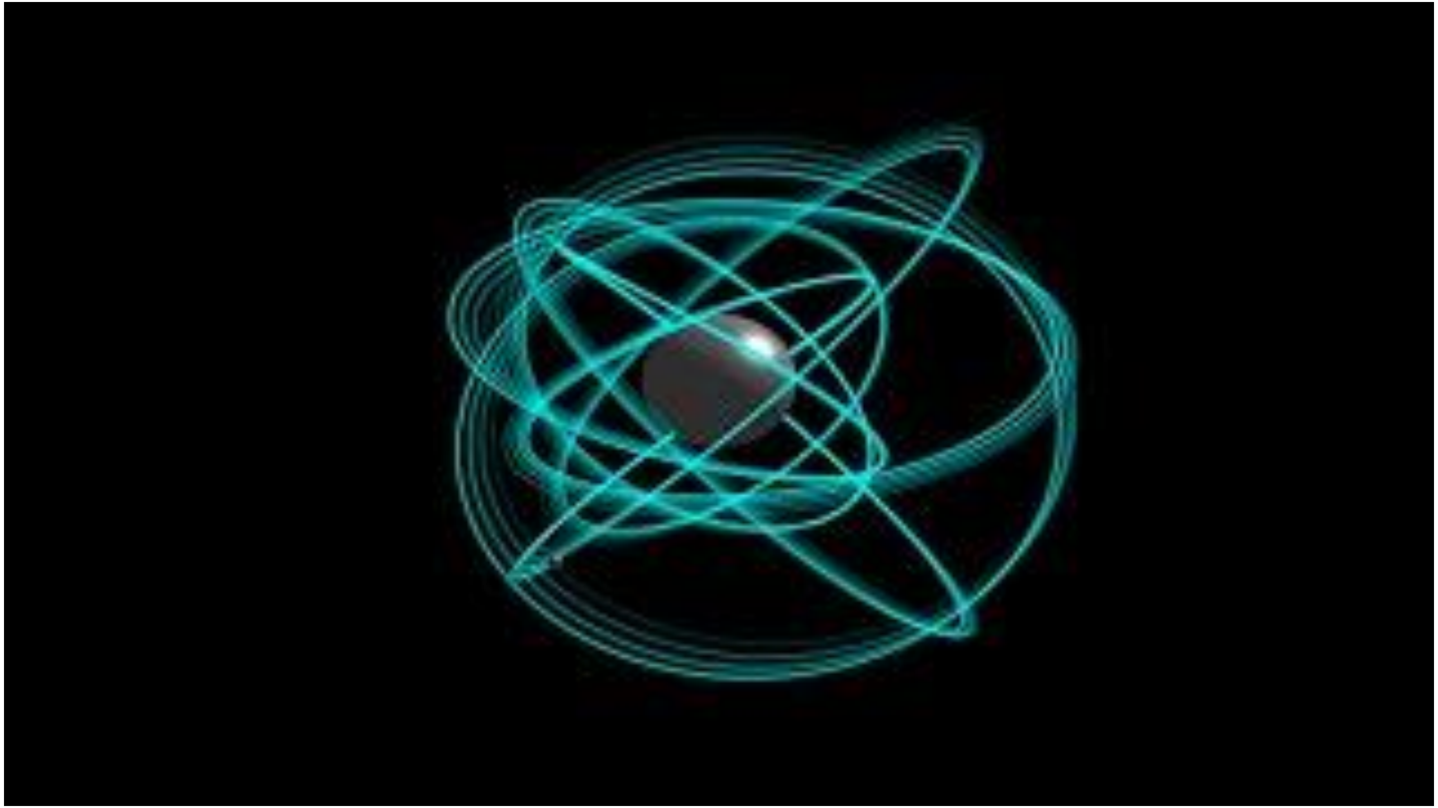
Extreme-Mass-Ratio Black-Hole Inspirals



Overarching Goal of Research Program

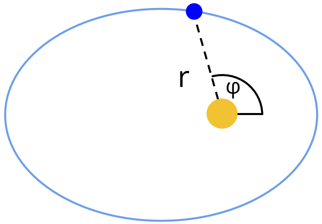
Compute the gravitational radiation-reaction force in Black Hole Extreme Mass-Ratio Inspirals analytically, to $O(G^n)$ in the coupling and $O((m/M)^2)$ in the mass ratio & use it to generate predictions for the LISA future space-based gravitational-wave detector

Using existing methods, this task seems very hard to impossible. Equivalent to a set of (n-1)-loop computations in terms of Feynman diagrams.

See e.g. Plefka et al. PhysRevLett.132.241402 and Porto et al. PhysRevLett.132.221401 for tour-de-force calculations at $O(G^{4\text{-}5})$

With the Quantum Spectral Method outlined in this talk (and a lot of hard work) this may be possible

Radiation-Reaction Leads to Inspirals: EM Example

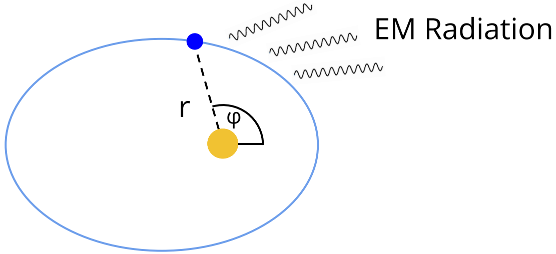


Keplerian Motion

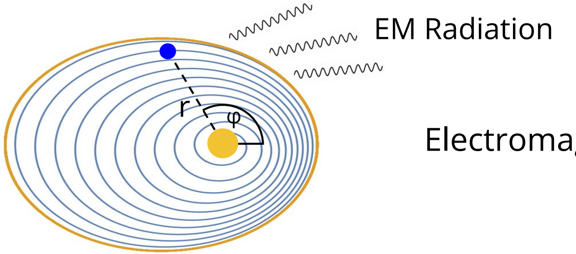
$$r(t) = \frac{p}{1 + e\cos(\varphi(t))}$$

Keplerian parameterization

Radiation-Reaction Leads to Inspirals: EM Example



Radiation-Reaction Leads to Inspirals: EM Example



Electromagnetic Inspiral

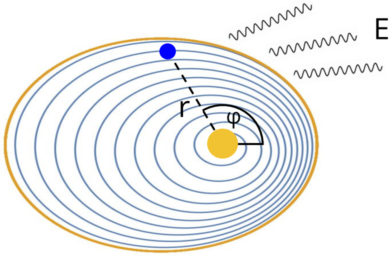
$$r(t) = \frac{p(t)}{1 + e(t)\cos(\varphi(t))}$$

quasi-Keplerian parameterization

p(t), e(t) change slowly with time due to radiation-reaction

Their change is calculated from $\mu \ddot{\vec{r}} = \vec{f}_{\rm rad}(t)$ radiation-reaction force

Radiation-Reaction Leads to Inspirals: EM Example



Electromagnetic Inspiral

When $\epsilon \equiv (T_{\rm period}/T_{\rm rad}) \ll 1$, can use Post-Adiabatic (PA) perturbation theory, i.e:

Calculate the orbit-averaged $\vec{f}_{\rm rad}(t)$ on the momentarily tangent Keplerian orbit with (p(t), e(t))

$$\vec{f}_{\rm rad}(t) = \vec{f}_{\rm rad}\left[p(t), e(t)\right] \quad \text{to order } O(\varepsilon)$$

For the next order, include

$$\vec{f}_{\rm rad}(t) = \left\{ \vec{f}_{\rm rad}(p,e) + \frac{\partial \vec{f}_{\rm rad}(p,e)}{\partial p}\,\dot{p} + \frac{\partial \vec{f}_{\rm rad}(p,e)}{\partial e}\,\dot{e} \right\}_{p=p(t),e=e(t)}$$

Moral of the Story

To set up a Post-Adiabatic inspiral calculation, enough to know:

- O(ε): the orbit-averaged radiation reaction force on a Keplerian orbit $\vec{f}_{\rm rad}[p,e]$
- O(ε²): the orbit-averaged partials $\frac{\partial \vec{f}_{\rm rad}}{\partial p}, \frac{\partial \vec{f}_{\rm rad}}{\partial e}$ on a Keplerian orbit

In relevant examples, $\vec{f}_{\rm rad}(p,e)$ currently known either numerically, or perturbatively in G at great effort

Can we compute it analytically, to all-order in the coupling? Can we generalize to GR?

The Quantum Spectral Method (QSM)

Khalaf, OT '23

A framework to analytically compute all time-dependent observables for any conservative motion in a spherically symmetric potential

- Conservative trajectories
- Emitted radiation from conservative trajectories
- Radiation reaction

In this talk: framework, non-relativistic electromagnetic (EM) inspirals

Out soon: EM unbound trajectories, special relativistic generalization

In the works: scalar and gravitational radiation-reaction in Schwarzschild, beyond spherically symmetric potentials and Kerr inspirals

Talk Plan

- Refresher: classical motion in a spherically symmetric potential
- Quantum-to-Classical relations, a first glance
- The Master Equation of the QSM and its WKB proof
- Applications
    - Time dependent Keplerian motion
    - Emitted EM field from a Keplerian electron
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- Summary and future work

Refresher: Classical Motion in a Spherically Symmetric Potential

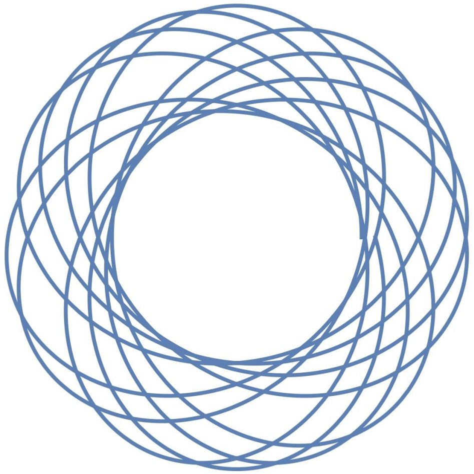
$$H(\vec{r}, \vec{p}) = \frac{\vec{p}^{\,2}}{2\mu} + V(r)$$

Motion is planar → can set θ=π/2 and forget about θ, set $\vec{L} = L\hat{z}$ for angular momentum

Azimuthal motion:

$$\dot{\varphi} = \frac{L}{\mu r^2}$$

from angular momentum conservation



bound, precessing orbit

Refresher: Classical Motion in a Spherically Symmetric Potential

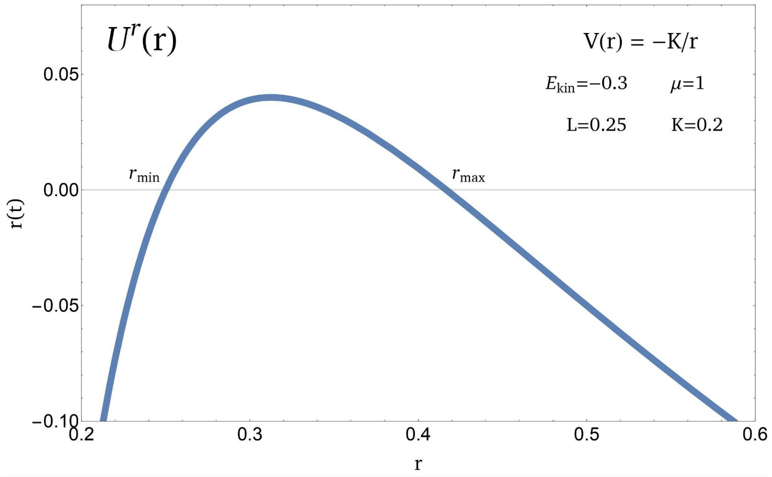
Radial motion: $\dot{r} = \pm\frac{1}{\mu}\sqrt{U^r(r)}$, $U^r(r) = 2\mu\left(E - V(r) - \frac{L^2}{2\mu r^2}\right)$

Radial action:

$$S^r(r) = \pm\int_{r_{\rm min}}^{r} dr'\,\sqrt{U^r(r')}$$

For E<0, bound motion

$0 \leq r_{\rm min} \leq r_{\rm max}$ real roots of $U^r(r)$



Doubly Periodic Motion

$$T^r = 2\int_{r_{\min}}^{r_{\max}} \frac{\mu}{\sqrt{U^r(r)}}\, dr$$

radial period

$$T^\varphi = T^r \left[\frac{L}{\pi} \int_{r_{\min}}^{r_{\max}} \frac{1}{r^2\sqrt{U^r(r)}} dr \right]^{-1}$$

azimuthal period

$$\Upsilon^i = \frac{2\pi}{T^i}$$

Fundamental frequencies

Action-angle variables:

$$\alpha^r = \Upsilon^r t \quad \xleftrightarrow{\text{conjugate}} \quad J_r = \frac{1}{\pi} S^r(r_{\max})$$

$$\alpha^\varphi = \Upsilon^\varphi t \quad \xleftrightarrow{\text{conjugate}} \quad J_\varphi = L$$

Implication: all observables are doubly-periodic in the angles α^r, α^φ

Time Dependent Observables

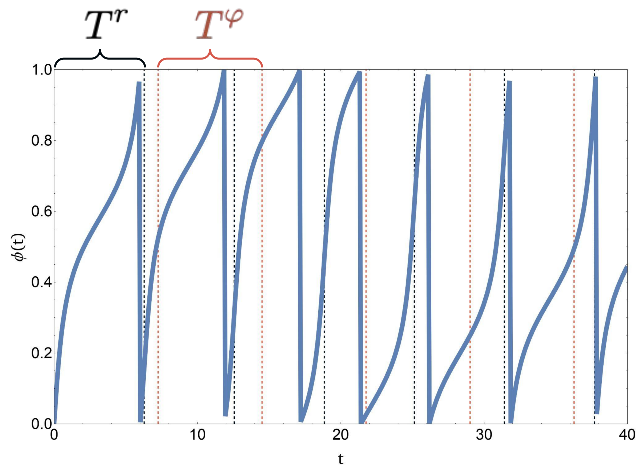
Every time-dependent observable O(t) is a double Fourier series in α^r, α^φ

$$\mathcal{O}\left[\alpha^r, \alpha^\varphi\right] = \sum_{\Delta j_r,\, \Delta l} \mathcal{O}_{\Delta j_r, \Delta l} \exp\left\{-i\Delta j_r \alpha^r - i\Delta l \alpha^\varphi\right\}$$

The coefficients given by inverse-Fourier

$$\mathcal{O}_{\Delta j_r, \Delta l} = \int \frac{d\alpha^r}{2\pi} \int \frac{d\alpha^\varphi}{2\pi} \mathcal{O}\left[\alpha^r, \alpha^\varphi\right] \exp\left\{i\Delta j_r \alpha^r + i\Delta l \alpha^\varphi\right\}.$$

In most cases known only numerically



for example φ(t) with precession

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Quantum-to-Classical Relations, a First Glance

Consider $H(\vec{r},\vec{p}) = \frac{\vec{p}^2}{2\mu} + V(r)$ as a *quantum* Hamiltonian, with the eigenbasis $|j_r, l, m\rangle$

$$H\,|j_r, l, m\rangle = E_{j_r,l}\,|j_r, l, m\rangle$$

$$L^2\,|j_r, l, m\rangle = \hbar^2 l(l+1)\,|j_r, l, m\rangle$$

$$L_z\,|j_r, l, m\rangle = \hbar m\,|j_r, l, m\rangle$$

In the classical limit, $j_r \to \hbar^{-1} J_r \;\;,\;\; l \to \hbar^{-1} L \;\;,\;\; m \to \hbar^{-1} M \equiv \hbar^{-1} L_z$

and without loss of generality $L=L_z$.

Quantum-to-Classical Relations, a First Glance

Khalaf, OT '23

First non-trivial result:

$$\Upsilon^r_{J_r,L} = \lim_{\hbar\to 0} \frac{E_{j_r,l} - E_{j_r-\Delta j_r,l}}{\hbar \Delta j_r}$$

$$\Upsilon^\varphi_{J_r,L} = \lim_{\hbar\to 0} \frac{E_{j_r,l} - E_{j_r,l-\Delta l}}{\hbar \Delta l}$$

with

$$j_r = \hbar^{-1} J_r \quad , \quad l = \hbar^{-1} L$$

Classical fundamental frequencies

Differences of quantum energies

This is true for Coulomb, Schwarzschild, etc… any spherically symmetric motion

Proof of Quantum-to-Classical Relations

Radial Schrodinger: $$-\frac{\hbar^2}{2m}\left[r^{-2}\partial_r(r^2\partial_r R) - \frac{l(l+1)}{r^2}R\right] - V(r)R = E_{j_r,l}R$$

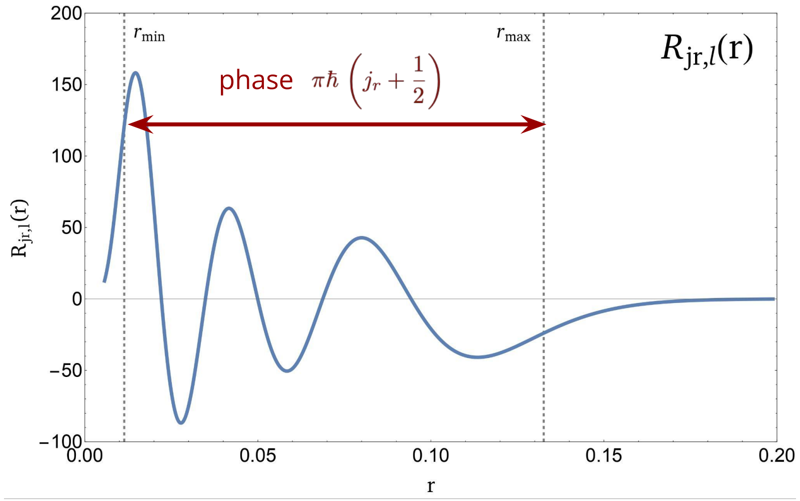
WKB solution: $$R^{\mathrm{WKB}}_{j_r,l}(r) \propto \sin\left[\hbar^{-1}S^r_{j_r,l}(r) + \frac{\pi}{4}\right]$$

$S^r_{j_r,l}(r)$ Classical radial action

WKB quantization condition:

$$J_r(E=E_{j_r,l}) \equiv \frac{1}{\pi}S^r_{j_r,l}(r_{\max}) = \hbar\left(j_r + \frac{1}{2}\right)$$

→ $$\lim_{\hbar\to 0}\frac{J_r(E=E_{j_r,l}) - J_r(E=E_{j_r-\Delta j_r,l})}{\hbar\Delta j_r} = 1$$



Proof of Quantum-to-Classical Relations

$$\lim_{\hbar\to 0}\frac{J_r(E=E_{j_r,l})-J_r(E=E_{j_r-\Delta j_r,l})}{\hbar\Delta j_r}=1$$

$$\underbrace{\frac{\partial J_r(E_{J_r,L})}{\partial E_{J_r,L}}}_{1/\Upsilon^r_{J_r,L}}\lim_{\hbar\to 0}\frac{E_{j_r,l}-E_{j_r-\Delta j_r,l}}{\hbar\Delta j_r}=1$$

Radial frequency

$$\Upsilon^r_{J_r,L}=\lim_{\hbar\to 0}\frac{E_{j_r,l}-E_{j_r-\Delta j_r,l}}{\hbar\Delta j_r}$$

Q.E.D. Similarly for $\Upsilon^{\varphi}_{J_r,L}$

Coulomb/Kepler example:

$$E_{j_r,l}=-\frac{\mu K^2}{2\hbar^2 n^2}\quad,\quad n=j_r+l+1$$

hydrogen atom energies

$$\lim_{\hbar\to 0}\frac{E_{j_r,l}-E_{j_r-\Delta j_r,l}}{\hbar\Delta j_r}=\Upsilon^r_{J_r,L}=\sqrt{(-2E)^3\,\frac{\mu}{K^2}}$$

Keplerian fundamental frequency

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The Quantum Spectral Method (QSM)

Khalaf, OT '23

Recall that any observable O(t) for motion in a spherically symmetric potential is a double Fourier series:

$$\mathcal{O}\left[\alpha^r, \alpha^\varphi\right] = \sum_{\Delta j_r,\, \Delta l} \mathcal{O}_{\Delta j_r, \Delta l} \exp\left\{-i \Delta j_r \alpha^r - i \Delta l \alpha^\varphi\right\}$$

Where the Fourier coefficients encode the classical dynamics and are usually only computable numerically

The QSM provides a way to compute them analytically to all orders in the coupling

The Master Equation of the QSM

$$\mathcal{O}^{J_r,L}_{\Delta j_r,\Delta l} = \lim_{\hbar\to 0} \sum_{\Delta m} \langle j_r - \Delta j_r, l - \Delta l, l - \Delta m | \, \mathcal{O} \, | j_r, l, l \rangle$$

with

$$j_r = \hbar^{-1} J_r \quad , \quad l = \hbar^{-1} L \quad , \quad m = \hbar^{-1} L_z = \hbar^{-1} L$$

- The classical Fourier coefficient is the $\hbar \to 0$ limit of the quantum matrix element for the operator O
- Correspondence principle: quantum numbers go to infinity, their products with $\hbar$ are the finite, conserved action variables of the classical system J_r and L
- Δj_r and Δl do not go to infinity and remain the integer indices of the double Fourier series for O(t)

WKB Proof for a Scalar Operator

(see paper for general proof)

$$\lim_{\hbar\to 0}\sum_{\Delta m}\langle j_r-\Delta j_r,l,l|\ \mathcal{O}(r)\ |j_r,l,l\rangle=\lim_{\hbar\to 0}\int_{r_{\min}}^{r_{\max}}dr\,r^2\,\mathcal{O}(r)\,R^*_{j_r-\Delta j_r,l-\Delta l}R_{j_r,l}(r)$$

With the WKB radial wavefunction $\lim_{\hbar\to 0}R_{j_r,l}(r)=\sqrt{\frac{\mu}{T^r_{j_r,l}}}\frac{2}{r\,[U^r_{j_r,l}(r)]^{1/4}}\sin\left(\frac{1}{\hbar}S^r_{j_r,l}(r)+\frac{\pi}{4}\right)$

substituting it, we have

$$\lim_{\hbar\to 0}\langle\mathcal{O}\rangle=\lim_{\hbar\to 0}\frac{2\mu}{T^r_{j_r,l}}\int_{r_{\min}}^{r_{\max}}dr\,\frac{\mathcal{O}(r)}{\sqrt{U^r_{j_r,l}(r)}}\cos\left(\frac{S^r_{j_r,l}(r)-S^r_{j_r-\Delta j_r,l}(r)}{\hbar}\right)$$

where we dropped the phase with the sum of S's which tends to 1

WKB Proof for a Scalar Operator

$$\lim_{\hbar\to 0}\langle\mathcal{O}\rangle = \lim_{\hbar\to 0}\frac{2\mu}{T^r_{j_r,l}}\int_{r_{\min}}^{r_{\max}} dr\,\frac{\mathcal{O}(r)}{\sqrt{U^r_{j_r,l}(r)}}\cos\left(\frac{S^r_{j_r,l}(r)-S^r_{j_r-\Delta j_r,l}(r)}{\hbar}\right)$$

Trick - change variables $r \to \alpha_r$ using $\frac{dr}{d\alpha^r} = \frac{T^r_{J_r,L}}{2\pi}\frac{\sqrt{U^r(r)}}{\mu}$:

$$\lim_{\hbar\to 0}\langle\mathcal{O}\rangle = \lim_{\hbar\to 0}\int_0^{2\pi}\frac{d\alpha^r}{2\pi}\,\mathcal{O}(\alpha^r)\,\cos\left(\frac{S^r_{j_r,l}(\alpha^r)-S^r_{j_r-\Delta j_r,l}(\alpha^r)}{\hbar}\right)$$

This is starting to look like the definition of the inverse Fourier transform defining $\mathcal{O}_{\Delta j_r}$

Exactly what we need to prove the Master Equation

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WKB Proof for a Scalar Operator

$$\lim_{\hbar \to 0} \langle \mathcal{O} \rangle = \lim_{\hbar \to 0} \int_0^{2\pi} \frac{d\alpha^r}{2\pi} \mathcal{O}(\alpha^r) \cos\left(\frac{S^r_{j_r,l}(\alpha^r) - S^r_{j_r - \Delta j_r, l}(\alpha^r)}{\hbar} \right)$$

Let's drive this point home. The difference of radial actions is

$$\lim_{\hbar \to 0} \left(\frac{S^r_{j_r,l}(\alpha^r) - S^r_{j_r - \Delta j_r, l}(\alpha^r)}{\hbar} \right) = \lim_{\hbar \to 0} \Delta j_r \frac{\partial S^r_{J_r,L}(\alpha^r)}{\partial E_{J_r,L}} \frac{E_{j_r,l} - E_{j_r - \Delta j_r, l}}{\hbar \Delta j_r} = \Delta j_r \alpha^r \qquad \alpha^r = \Upsilon^r t$$

t by classical EOM

Υ^r by Quantum-Classical relation

$$\lim_{\hbar \to 0} \langle \mathcal{O} \rangle = \lim_{\hbar \to 0} \int_0^{2\pi} \frac{d\alpha^r}{2\pi} \mathcal{O}(\alpha^r) \cos\left(\Delta j_r \alpha^r \right) \equiv \mathcal{O}_{\Delta j_r}$$

Q.E.D. Master Equation

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Using the QSM for Novel Analytical Results

Master equation: $$\mathcal{O}^{J_r,L}_{\Delta j_r,\Delta l} = \lim_{\hbar\to 0} \sum_{\Delta m} \langle j_r - \Delta j_r, l - \Delta l, l - \Delta m|\ \mathcal{O}\ |j_r, l, l\rangle$$

For any O in any spherically symmetric potential

Important: to prove the QSM, we used WKB. But to apply it, we need to use the full quantum matrix elements.

If we use WKB too early, we go back to the inverse Fourier we don't know how to compute

Algorithm:

- Find full quantum eigenstates $|j_r, l, m\rangle$
- Compute $\langle j_r - \Delta j_r, l - \Delta l, l - \Delta m|\ \mathcal{O}\ |j_r, l, l\rangle$ for the desired O
- Set $j_r = \hbar^{-1} J_r \ ,\ \ l = \hbar^{-1} L \ ,\ \ m = \hbar^{-1} L_z = \hbar^{-1} L$
- Take ħ→0 limit and get $\mathcal{O}^{J_r,L}_{\Delta j_r,\Delta l}$

Why Can We Compute Analytically What Seemed Impossible?

Matrix element

$$\sum_{\Delta m} \langle j_r - \Delta j_r, l - \Delta l, l - \Delta m| \, \mathcal{O} \, |j_r, l, l\rangle$$

Computable as an overlap integral over special functions - can use their special properties

$$j_r = \hbar^{-1} J_r \quad , \quad l = \hbar^{-1} L \quad , \quad m = \hbar^{-1} L_z = \hbar^{-1} L$$

ħ→0

$$\mathcal{O}_{\Delta j_r, \Delta l} = \int \frac{d\alpha^r}{2\pi} \int \frac{d\alpha^\varphi}{2\pi} \, \mathcal{O}\left[\alpha^r, \alpha^\varphi\right] \exp\left\{i\Delta j_r \alpha^r + i\Delta l \alpha^\varphi\right\}.$$

"Impossible" classical integral

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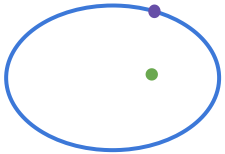
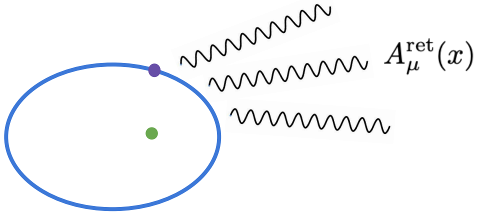
Applications of the QSM so Far

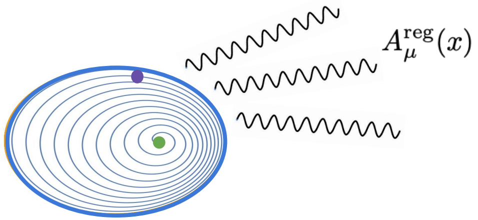
1. Proof-of-principle: time-dependent Keplerian motion



2. First all-multipole analytical result for A^μ from a Keplerian electron



3. First all-multipole analytical result for EM self-force on an inspiralling classical electron + inspiralling adiabatic trajectory and EM waveform

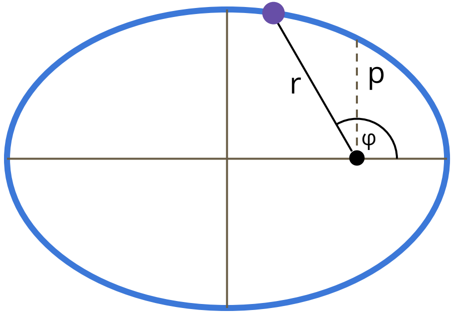


First Application: Time-Dependent Keplerian Motion

$$H(\vec{r},\vec{p}) = \frac{\vec{p}^2}{2\mu} - \frac{K}{r} \longrightarrow r(\varphi) = \frac{p}{1 + e\cos\varphi}$$

$$K \equiv kQq$$

elliptic orbit



eccentricity $e = \sqrt{1 + \frac{2EL^2}{\mu K^2}}$

$p = \frac{L^2}{K\mu}$ semi-latus rectum

$$\Upsilon^r = \Upsilon^\varphi = \sqrt{(-2E)^3 \frac{\mu}{K^2}}$$

Degeneracy of Coulomb potential (no precession)

For any O(t): $\mathcal{O}(t) = \sum_{\Delta n, \Delta l} \mathcal{O}_{\Delta n, \Delta l} \exp\{-i\Delta n \alpha\}$ $\quad \alpha^r = \alpha^\varphi \equiv \alpha \quad \Delta n \equiv \Delta j_r + \Delta l$

Let's use the QSM to find r(t), the time-dependent radius for Keplerian motion

First Application: Time-Dependent Keplerian Motion

Quantum Eigenfunctions:
(hydrogen atom)

$$R_{n,l}(r) = \frac{1}{(2l+1)!}\sqrt{\frac{(n+l)!}{(n-l-1)!2n}}\left(\frac{2K\mu}{\hbar^2 n}\right)^{l+3/2}\exp\left(-\frac{K\mu}{\hbar^2 n}r\right) r^l\ {}_1F_1\left(-n+l+1; 2l+2; \frac{2K\mu}{\hbar^2 n}r\right)$$

Master equation:

$$r_{\Delta n} = \lim_{\hbar\to 0}\langle n', l, l|\, r\, |n, l, l\rangle \qquad \Delta n = n - n'$$

Quantum matrix element:

(that's where the special function magic comes in)

$$\langle n', l|\, r\, |n, l\rangle = \frac{\hbar^2}{\mu K}\frac{(-1)^{n'-l}\, 2^{2l+2}\, (nn')^{l+2}\, (n-l-1)}{(2l+1)!\, (n+n')^{2l+4}}\left(\frac{n-n'}{n+n'}\right)^{n-n'-2}\sqrt{\frac{(n+l)!\,(n'+l)!}{(n-l-1)!\,(n'-l-1)!}}\times$$

$$\left[{}_2F_1\left(l-n'+1; n+l; 2l+2; \frac{4nn'}{(n+n')^2}\right) - \frac{n+l+1}{n-l-1}\left(\frac{n-n'}{n+n'}\right)^2 {}_2F_1\left(l-n'+1; n+l+2; 2l+2; \frac{4nn'}{(n+n')^2}\right)\right]$$

First Application: Time-Dependent Keplerian Motion

Classical limit:

$$\lim_{\hbar \to 0} \langle n', l | \, r \, | n, l \rangle = -\frac{p}{1-e^2} \frac{e}{\Delta n^2} \frac{d}{de} J_{\Delta n}(e \Delta n) \qquad \Delta n \neq 0$$

$$\lim_{\hbar \to 0} \langle n, l | \, r \, | n, l \rangle = \frac{p}{1-e^2} \left(1 + \frac{e^2}{2} \right)$$

$$r(\alpha) = \frac{p}{1-e^2} \left[1 + \frac{e^2}{2} - 2e \sum_{\Delta n = 1}^{\infty} \frac{1}{(\Delta n)^2} \frac{d J_{\Delta n}(\Delta n \, e)}{de} \cos(\Delta n \alpha) \right]$$

Time - dependent Keplerian motion

Calculated with the QSM

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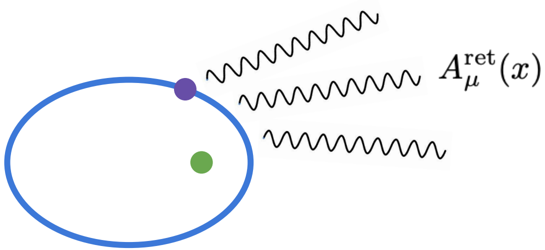
All-Multipole EM Emission

Switching gears: up until now we computed the Keplerian trajectory

We can use QSM to compute a more interesting quantity: the radiated EM field

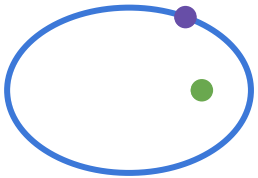


What is the generated electric field from a Keplerian electron at all orders in the multipole expansion?

All-Multipole EM Emission

Current density from a Keplerian electron:

$$J^{\mu}\left(x'\right) = \frac{q}{\mu} p^{\mu}_{\mathrm{Kep}}\left(t'\right) \delta^{(3)}\left[\vec{x}' - \vec{r}_{\mathrm{Kep}}\left(t'\right)\right]$$

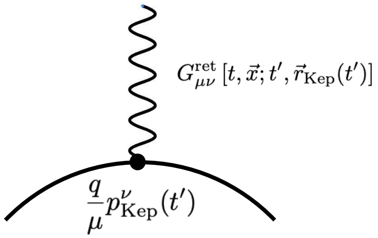


Radiated field:

$$A^{\mathrm{ret}}_{\mu}(x) = \int d^4x \, G^{\mathrm{ret}}_{\mu\nu}(x, x') J^{\nu}(x')$$

Retarded EM Green's function

$$A^{\mu}_{\mathrm{ret}} =$$



All-Multipole EM Emission

EM retarded Green's function:

(Jackson E&M)

$$G^{\mu\nu}_{\rm ret}(t,\vec{x};t',\vec{x}') = \eta^{\mu\nu}\frac{\Theta(t-t')}{4\pi R}\,\delta(t-t'-R)$$

Multipole expansion:

$$G^{\mu\nu}_{\rm ret}(t,\vec{x};t',\vec{x}') = \eta^{\mu\nu}\frac{\Theta(t-t')}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega(t-t')}\left\{ i\omega \sum_{l_\gamma=0}^{\infty} j_{l_\gamma}(\omega r_<)\, h^{(1)}_{l_\gamma}(\omega r_>) \sum_{m_\gamma=-l_\gamma}^{l_\gamma} Y^{m_\gamma}_{l_\gamma}(\theta',\varphi')Y^{m_\gamma *}_{l_\gamma}(\theta,\varphi)\right\}$$

We substitute the multipole-expanded Green's function and Keplerian current into

$$A^{\rm ret}_\mu(x) = \int d^4x\, G^{\rm ret}_{\mu\nu}(x,x')J^\nu(x')$$

and integrate over some delta functions

All-Multipole EM Emission

Fourier series in Keplerian angle variable α=Yt

Shape of multipole (Jackson E&M)

$$A^\mu_{\text{ret}} = \frac{iq}{\mu} \sum_{l_\gamma=0}^{\infty} \sum_{m_\gamma=-l_\gamma}^{l_\gamma} \sum_{\Delta n} e^{-i\Delta n\, \alpha}\, \omega_{\Delta n}\, h^{(1)}_{l_\gamma}(\omega_{\Delta n} r)\, Y^{m_\gamma *}_{l_\gamma}(\theta, \varphi)\, \mathcal{M}^{\mu, N, L}_{l_\gamma, m_\gamma, \Delta n}$$

$$\mathcal{M}^{\mu, N, L}_{l_\gamma, m_\gamma, \Delta n} = \lim_{\hbar \to 0} \sum_{\Delta l,\, \Delta m} \langle n', l', m'|\, j_l(\omega r)\, Y^{m_\gamma}_{l_\gamma}(\hat{r})\, p^\mu\, |n, l, l\rangle$$

$$N = J_r + L$$

Fourier coefficients of the Keplerian electron (with the QSM)

we skip the details of the QSM calculation but give a just taste...

All-Multipole EM Emission: a Taste of the QSM Calculation

$$\lim_{\hbar \to 0} \langle l', m' | \, Y_{l_\gamma}^{m_\gamma}(\hat{r}) \, | l, l \rangle = \delta_{l', m'} \delta_{-\Delta l, m_\gamma} \, Y_{l_\gamma}^{-m_\gamma}(\pi/2, 0)$$

$$\langle n', l' | \, j_{l_\gamma}(\omega_{\Delta n} \, r) \, | n, l \rangle = \quad 2^{l_\gamma} \sum_{j=0}^{\infty} \frac{(-1)^j \, (j + l_\gamma)!}{j! \, (2j + 2l_\gamma + 1)!} \, \omega_{\Delta n}^{2j + l_\gamma} \, \langle n', l' | \, r^{2j + l_\gamma} \, | n, l \rangle$$

$$\lim_{\hbar \to 0} \langle n', l' | \, r^j \, | n, l \rangle = \left(\frac{p}{1 - e^2} \right)^j (-\eta)^{-\Delta n - \Delta l} \left(\frac{\eta e}{2} \right)^{j+1} \sum_{m=0}^{\infty} L_{m + \Delta l + \Delta n}^{j+1-m-\Delta n} \left(\frac{\eta e \Delta n}{2} \right) L_m^{j+1-m-\Delta l} \left(-\frac{\eta e \Delta n}{2} \right) \eta^{-2m}$$

(special function magic)

Laguerre polynomials

$$\eta = \frac{1 - \sqrt{1 - e^2}}{e}$$

These are non-perturbative results - equivalent to a sum over all perturbative diagrams / amplitudes in the probe limit

All-Multipole EM Emission: Results

A_t radiated over one period by an electron undergoing Keplerian

The observation point is on the x-axis, far away from the electron's orbit

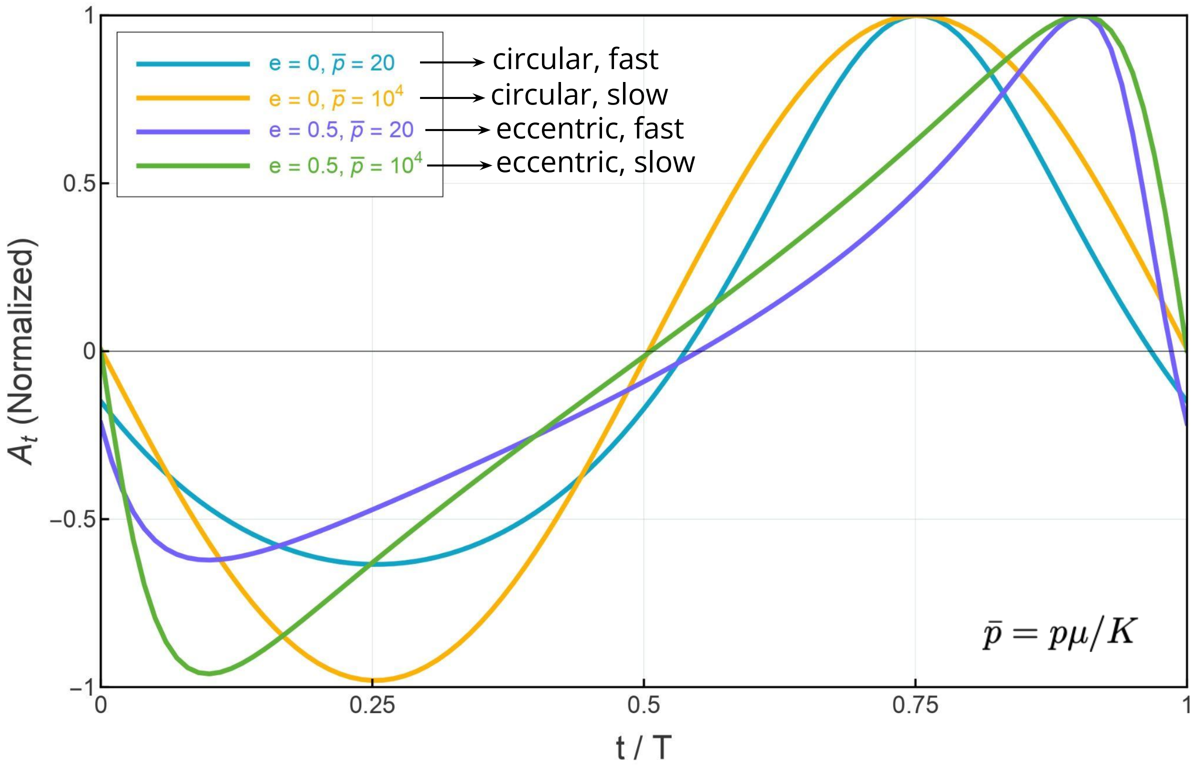
Slow circular Δn=Δl=1

Fast circular Δn=Δl=anything

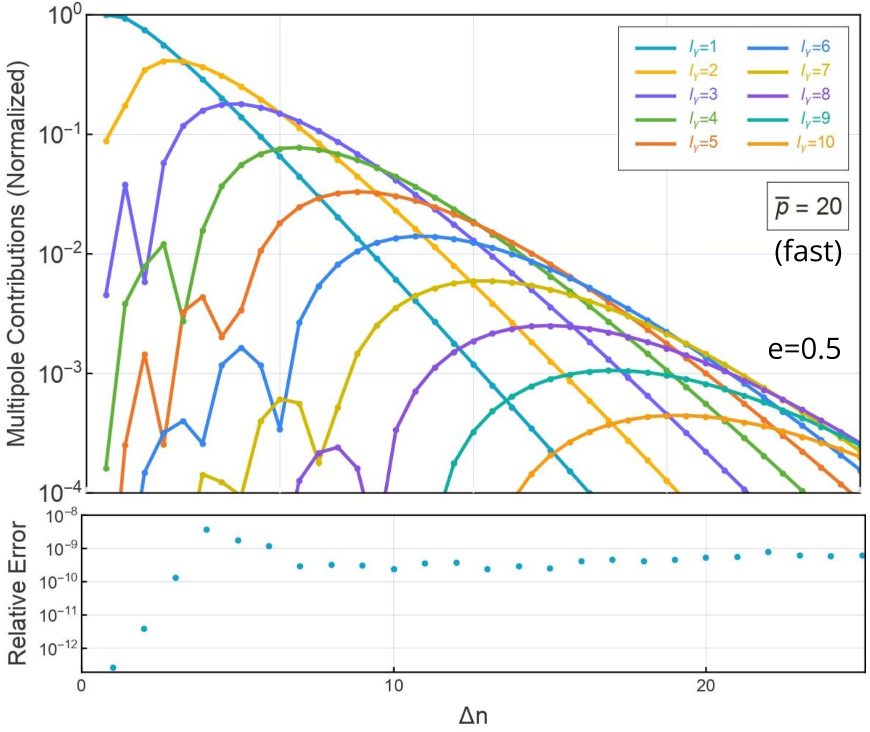
Slow Eccentric Δl=1, Δn=anything

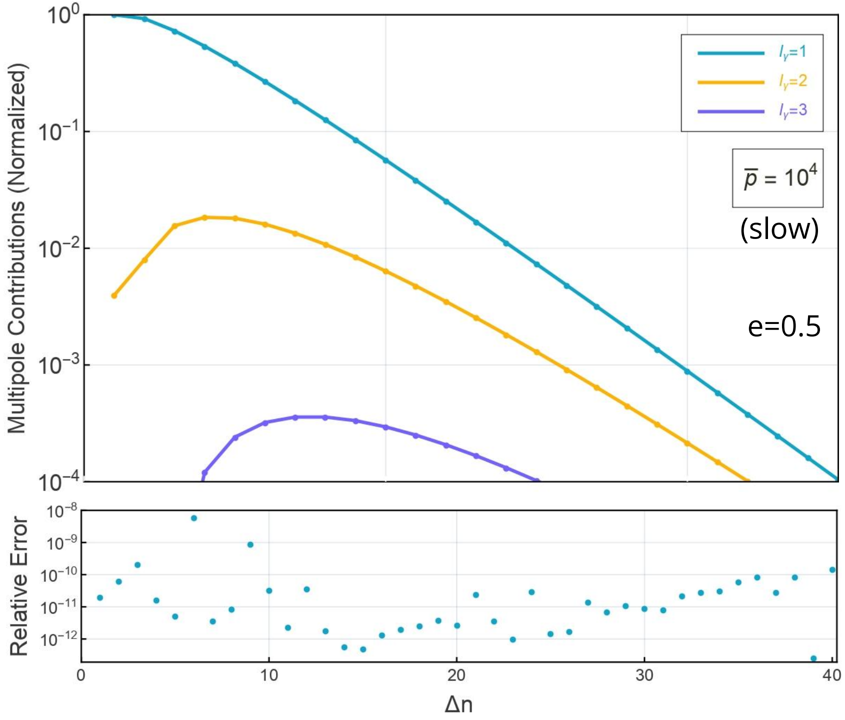
Fast Eccentric Δl=anything, Δn=anything

Asymmetry in **Fast Orbits** due to doppler



All-Multipole EM Emission: Results





Top: Multipole contributions (without spherical hankel factor)

Bottom: Relative error with respect to the (numerical) classical integrals

Talk Plan

- ✓ Refresher: classical motion in a spherically symmetric potential
- ✓ Quantum-to-Classical relations, a first glance
- ✓ The Master Equation of the QSM and its WKB proof
- Applications
  - ✓ Time dependent Keplerian motion
  - ✓ Emitted EM field from a Keplerian electron
  - ➡ Adiabatic EM inspirals
- Summary and future work

Adiabatic EM Inspirals

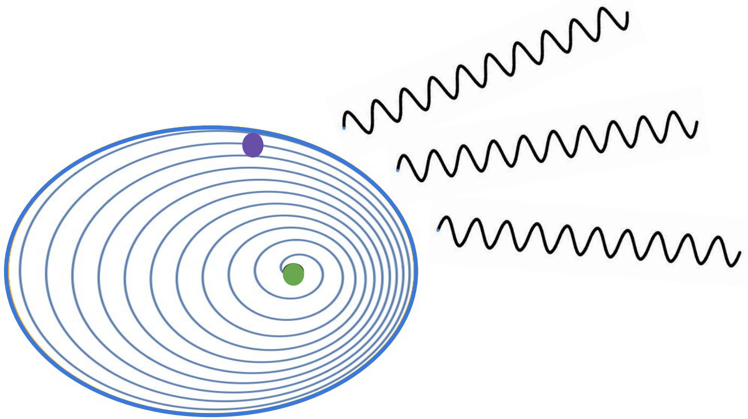
In the beginning of the talk we learned how to calculate an O(ε) (adiabatic) EM inspiral

$$r(t) = \frac{p(t)}{1 + e(t)\cos(\varphi(t))} \qquad p(t) = \frac{[L(t)]^2}{K\mu} \qquad e(t) = \sqrt{1 + \frac{2E(t)[L(t)]^2}{\mu K^2}}$$

E(t), L(t) calculated from

$$\mu \ddot{\vec{r}} = \underbrace{\vec{f}_{\mathrm{rad}}[A^{\mu}_{\mathrm{rad}}(t)]}$$

The Lorentz force from the emitted EM field that we already calculated



Let's write E(t), L(t) explicitly from the QSM!

Third Application: EM Self-Force

With the A_μ calculated with the QSM, we get the energy and angular momentum loss:

$$\frac{dE}{dt} = -\lim_{\hbar\to 0} \sum_{\Delta n>0,\Delta l,\Delta m} (E_n - E_{n'})\, \Gamma_{s.e.}$$

$$\frac{dL}{dt} = -\lim_{\hbar\to 0} \sum_{\Delta n>0,\Delta l,\Delta m} \hbar\, (l - l')\, \Gamma_{s.e.}$$

where $$\Gamma_{s.e.} = -\frac{2q^2\omega_{\Delta n}}{\hbar\mu^2} \sum_{l_\gamma=0}^{\infty} \sum_{m_\gamma=-l_\gamma}^{l_\gamma} \mathcal{M}_\mu^* \mathcal{M}^\mu + \mathcal{O}(\hbar^0)$$

is the rate for quantum spontaneous emission

In this way we recover the radiation-reaction as the classical limit of spontaneous emission

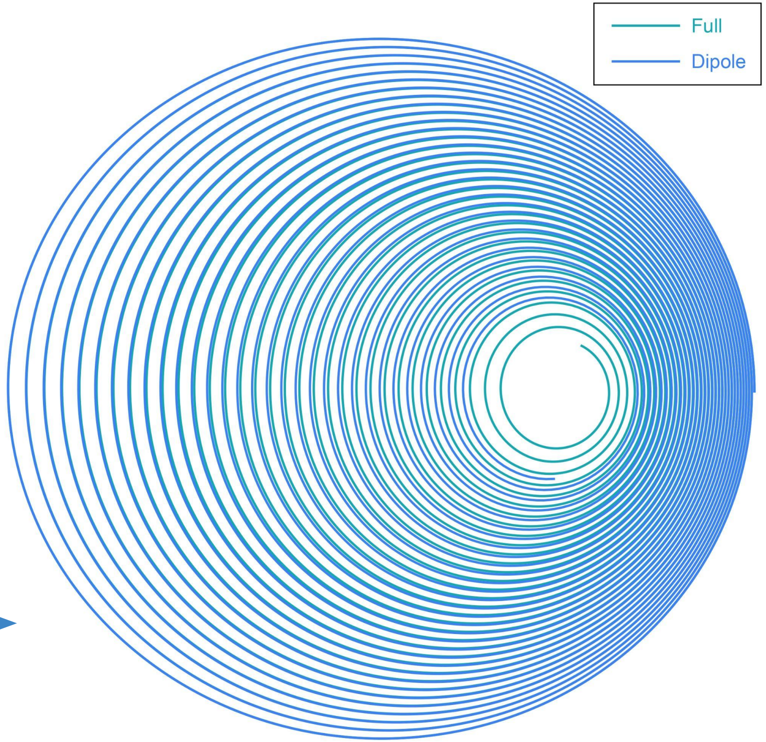
Adiabatic EM Inspiral

Using our energy and angular momentum loss, we calculate an adiabatic EM inspiral

$$r(t) = r\left[\alpha(t)\right]|_{E=E(t),L=L(t)}$$

$$\varphi(t) = \varphi\left[\alpha(t)\right]|_{E=E(t),L=L(t)}$$

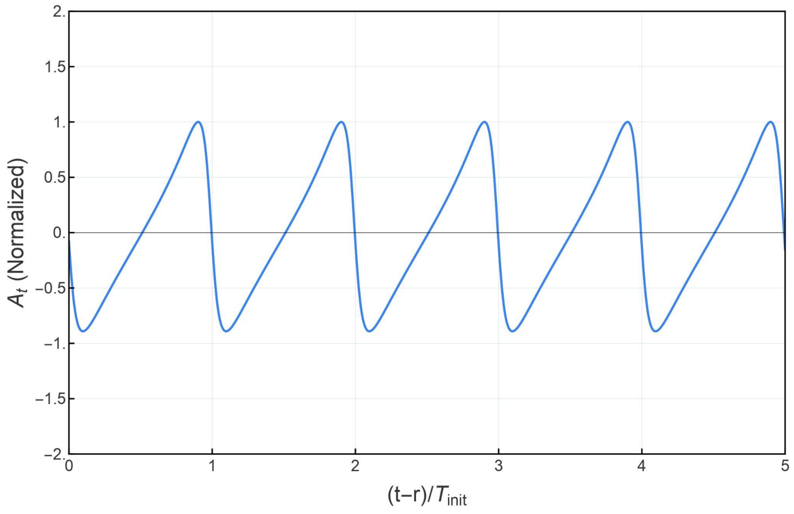
Here we took a very fast inspiral just for the visual effect →

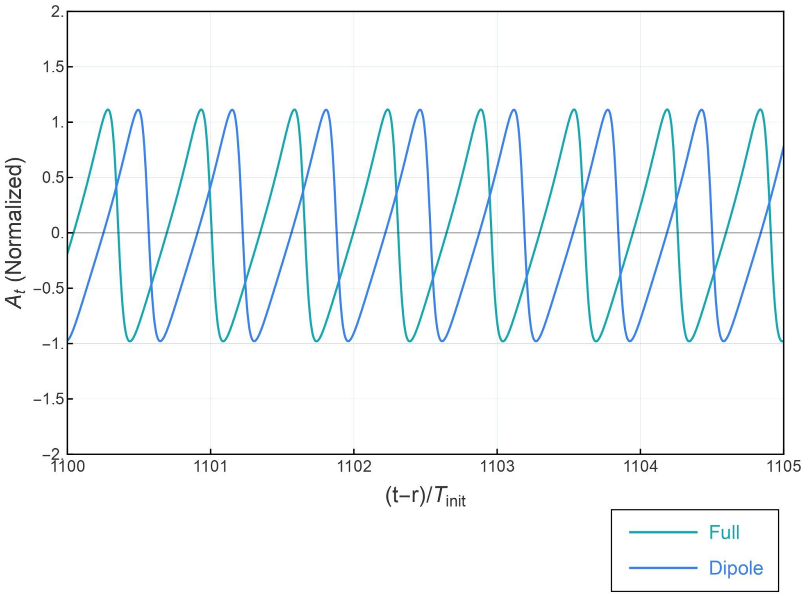


Adiabatic EM Inspiral

In the far field approximation, we also calculate the EM waveform as:

$$A^{\mu}_{\text{ret}} = \left\{ \frac{q}{\mu r} \sum_{l_\gamma=0}^{\infty} \sum_{m_\gamma=-l_\gamma}^{l_\gamma} \sum_{\Delta n} e^{-i\Delta n\,\alpha} (-i)^{l_\gamma} Y_{l_\gamma}^{m_\gamma *}(\theta,\varphi) \sum_{\Delta l, \Delta m} \mathcal{M}^{\mu}_{\Delta, l_\gamma, m_\gamma}(\omega_{\Delta n}, N, L) \right\}_{\text{ret}} \qquad N = J_r + L$$

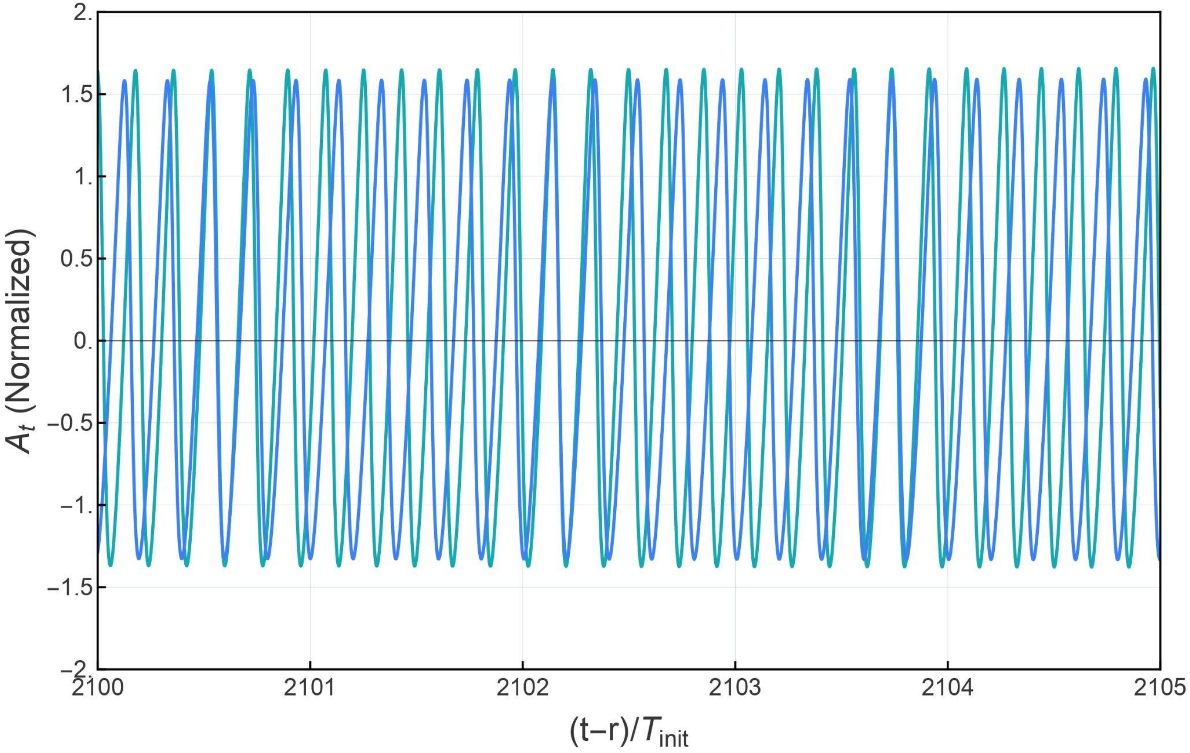




$$Z = Q/q = 4\pi$$

Adiabatic EM Inspiral

In the far field approximation, we also calculate the EM waveform:



After many cycles, the electron in the full calculation is close to the origin than the one in the dipole

An amplitude difference in addition to phase

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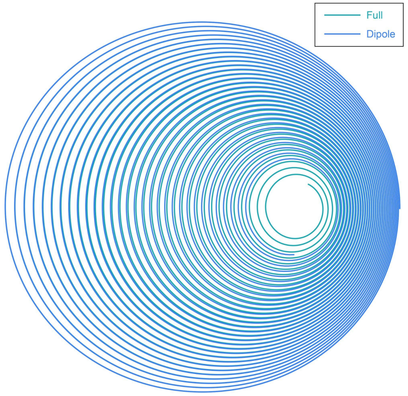
Summary

The QSM is a method to obtain the Fourier coefficients of classical observables:

"The Δj_r, Δl Fourier coefficient of the *classical observable* O is the classical limit of the Δj_r, Δl transition mediated by the *quantum operator* O"

We applied it for the analytical calculation of:

- Time-dependent Keplerian motion
- All-multipole EM radiation from a Keplerian orbit
- EM self-force and an adiabatic EM inspiral



Upcoming: Bound-Unbound Universality with Relativistic QSM

In this paper we applied the QSM for non-relativistic bound motion

In an upcoming paper, we generalize to relativistic and/or unbound motion

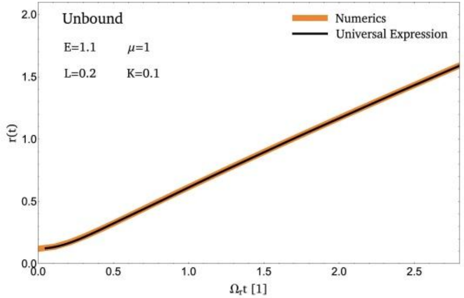
Khalaf, OT, Shen, 2411.xxxx

Relativistic: Schrodinger → Klein-Gordon

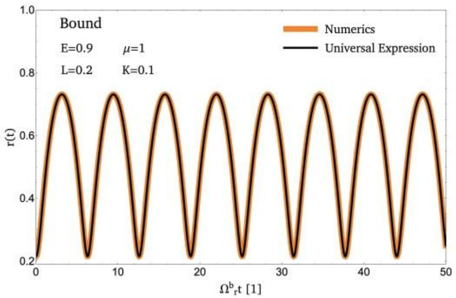
Unbound: Fourier series → Laplace transform

$$r(t;E,L) = i\frac{p}{e^2-1}\frac{e}{\gamma_\infty^2}\mathcal{P}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{1}{(\Delta j_r)^2\sin(\pi\Delta j_r)}\,\mathrm{Re}\left[\frac{d}{de}J_{-\Delta j_r}\left(e\gamma_\infty^2\Delta j_r\right)\right]\,e^{\Delta j_r\Omega_r t}\,d\Delta j_r\,.$$

The unbound expressions can be analytically continued to bound, picking up poles



(a) Unbound

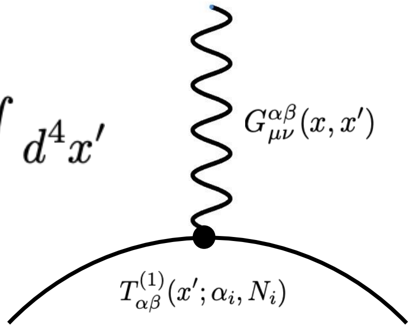


(b) Bound

Future: Towards Gravitational Self-Force with the QSM

First order gravitational self-force

$$h^{(1)}_{\mu\nu}(x;\alpha_i,N_i) = \int d^4x'$$



The circular line means integrating the Green's function along the worldline of the osculating BH geodesic

Currently this is done numerically, but we are working towards an analytical result with the QSM

We already have the eigenstates $|n,l,m\rangle$ in Schwarzschild/Kerr and can reproduce geodesics

"Wait, Didn't People Use Amplitudes for Inspirals?"

Scattering Amplitudes / NREFT

Perturbative in G or α

All orders in m/M or q/Q

Good for LIGO

Expand in m/M or q/Q
Resum in G or α

Expand in G or α
Resum in m/M or q/Q

Post Adiabatic / Self-Force

All-orders in G or α

Perturbative in m/M or q/Q

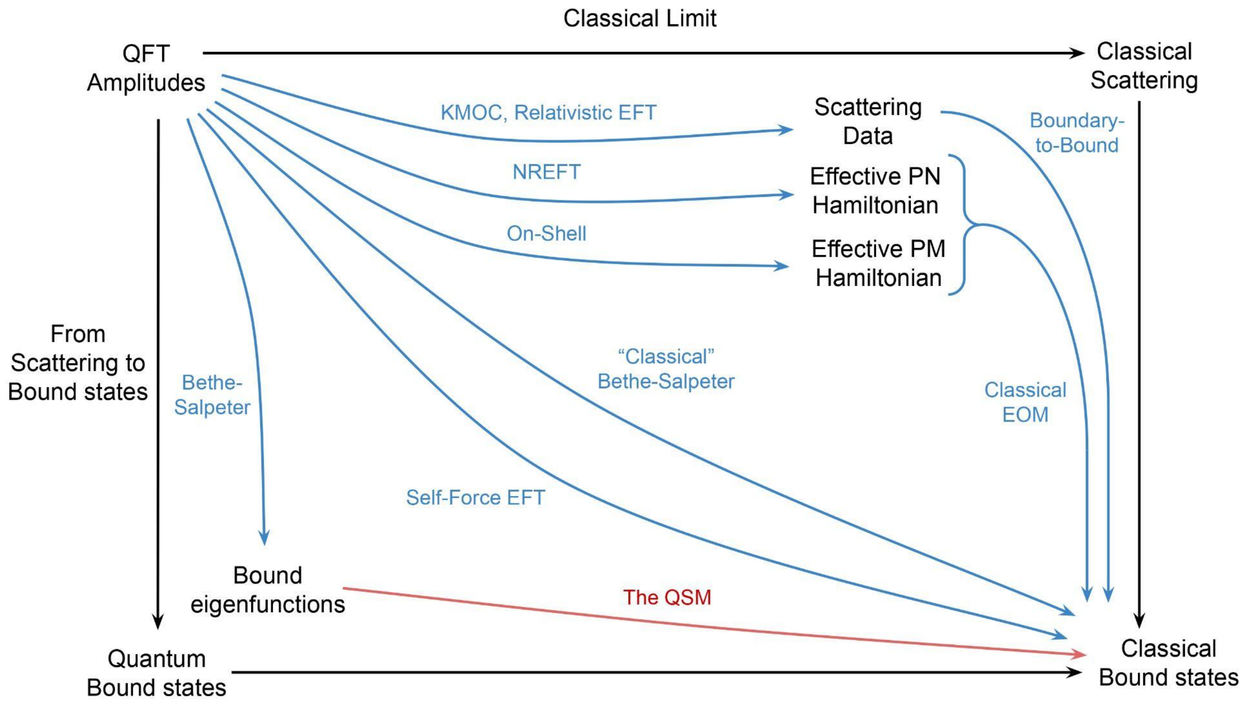
Good for LISA

Goldberger, Rothstein, Porto, Bern, Cheung, Kosower, O'Connell, Huang, Shen……

Poisson, Pound, Barack, Wardell, Warburton, Miller, van de Meent……

We are here - analytically!

The QSM in the Landscape of Quantum-to-Classical Methods



Thank You!