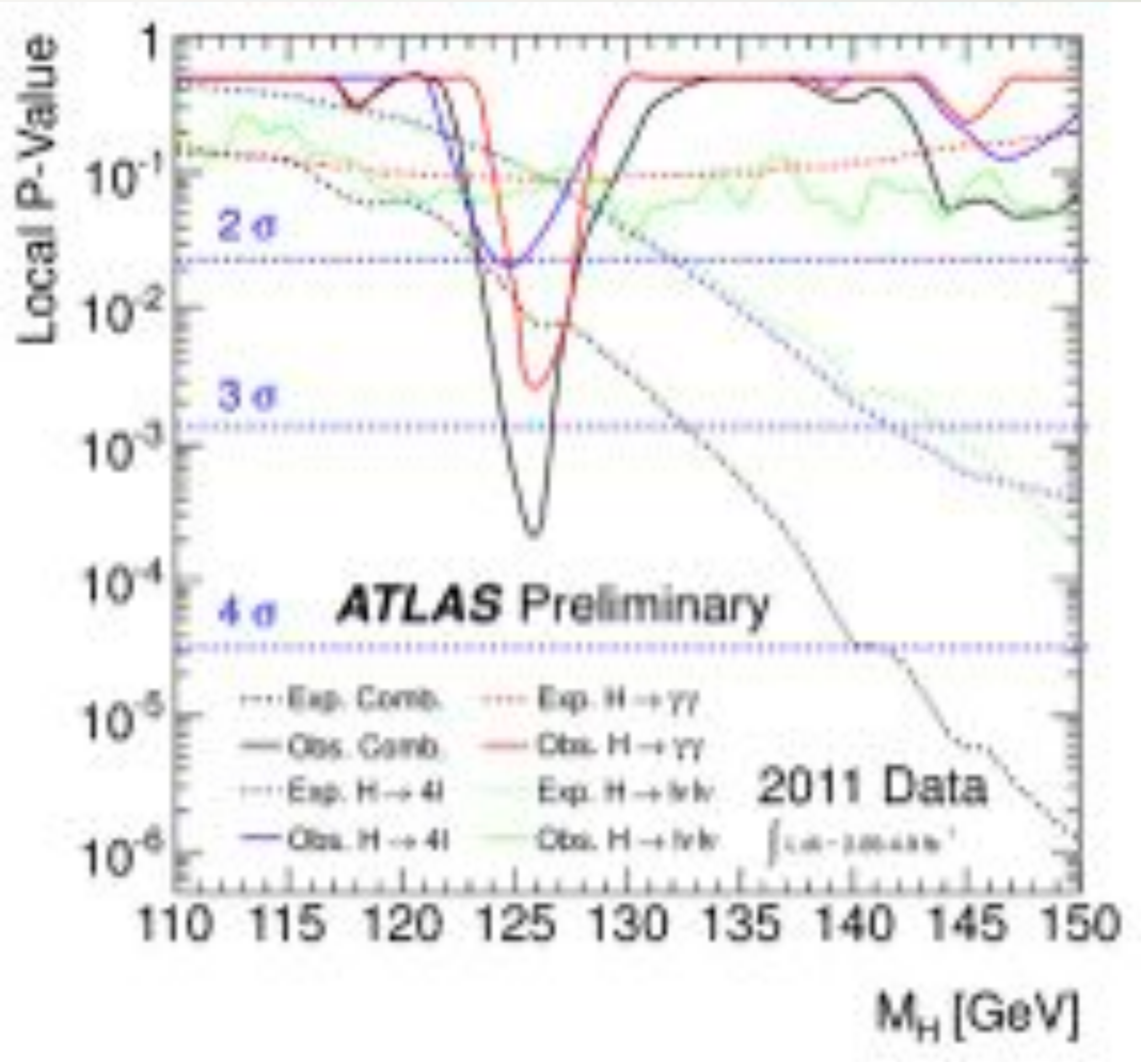
the Holographic S-Matrix

Jared Kaplan
SLAC

1107.1499 - Fitzpatrick, JK, Penedones, Raju, Van Rees
1111.6972 - Fitzpatrick, JK (Analyticity)
1112.4845 - Fitzpatrick, JK (Unitarity)



We seem to be on the cusp of a Higgs discovery, awaiting information on its production/decay, as well as the results of other searches for new physics!

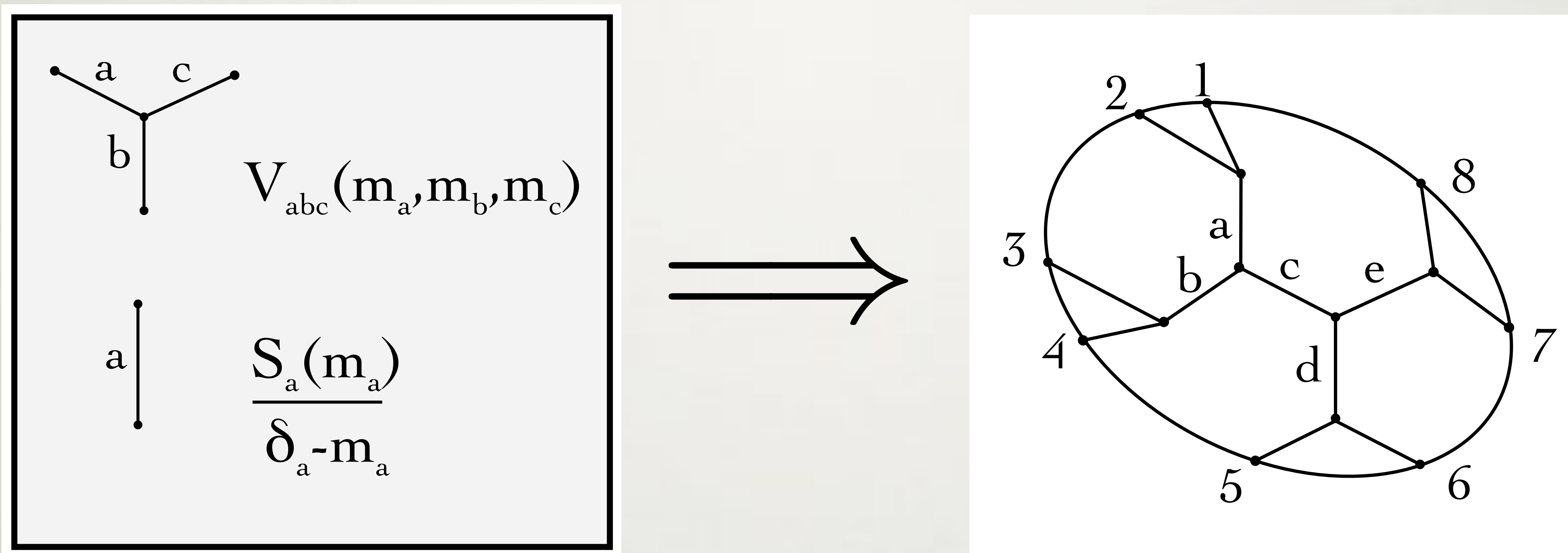
Important work to be done improving searches, understanding implications, focusing in on leftover (new?) model space.

But amidst the LHC, also other exciting topics for **Field Theorists**...

Motivations: Improved Understanding of CFTs

- Want to write CFT correlation functions in a form that clarifies the physics.
- Hope to greatly simplify calculation of correlators, OPE coefficients, etc.
- Why do CFTs have local AdS duals?

Motivations: Improved Understanding of CFTs

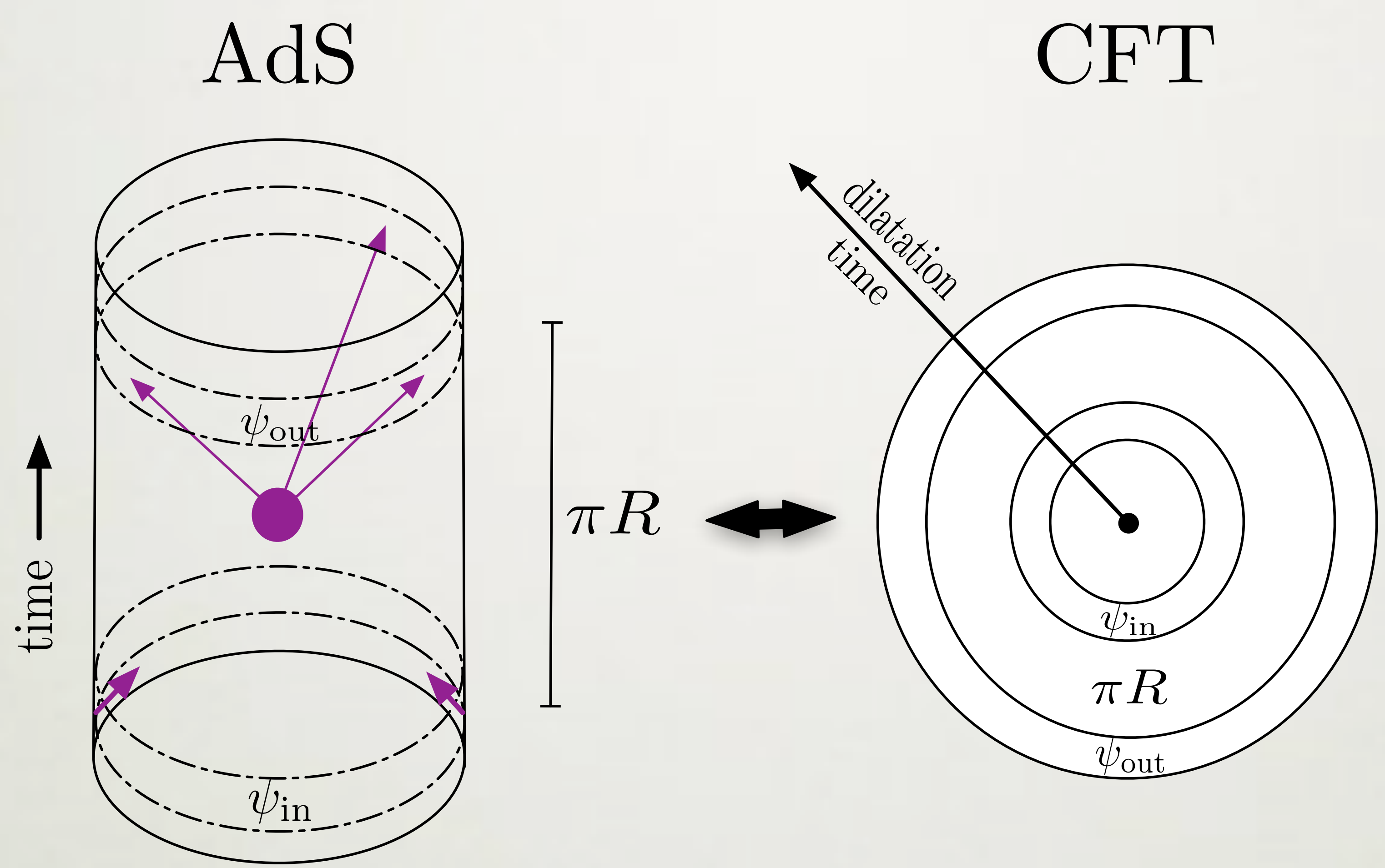


Key Point: argue that `Mellin space' is the analog of momentum space for Conformal Field Theories.

Motivations: Holographic Theory of Flat Spacetime

- Scattering Amplitudes are very simple.
- In gravity, local observables not gauge invariant, need boundary observables.
- Want a holographic **theory** of the flat space **S-Matrix** where bulk **locality** and **unitarity** emerge simply and robustly.

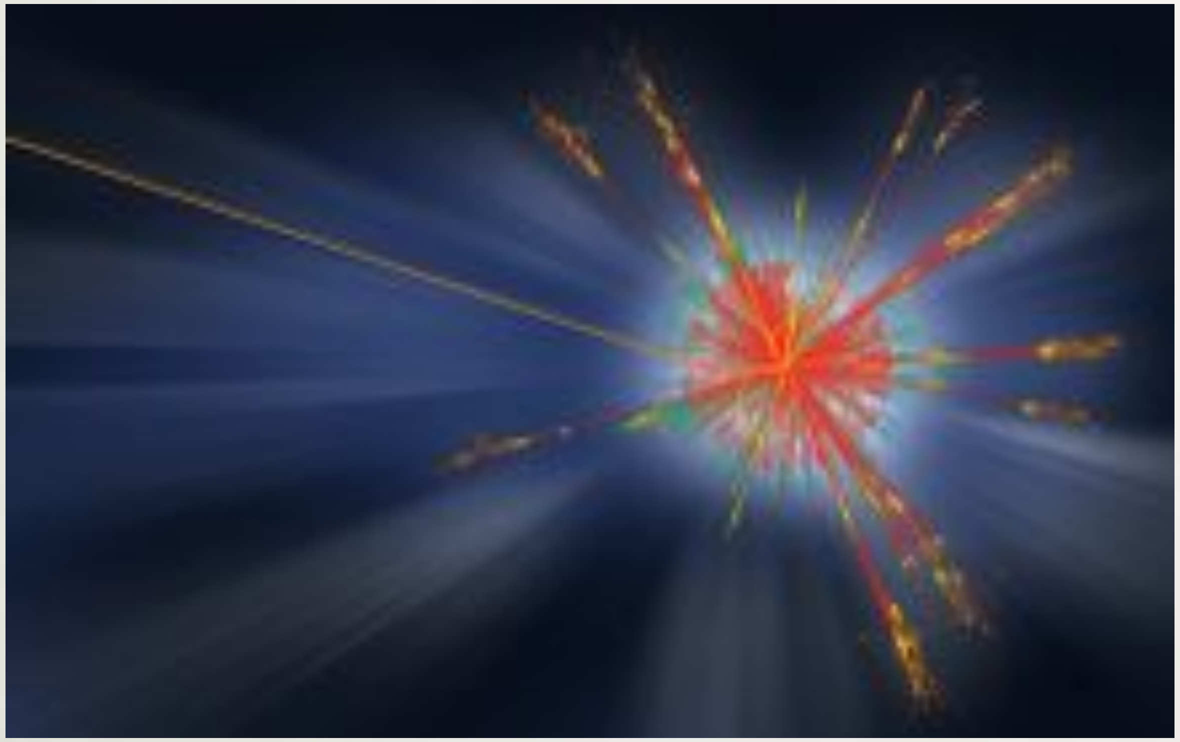
Motivations: Holographic Theory of Flat Spacetime



Key Point: $M(\delta_{ij}) \xrightarrow{R\to\infty} S(s,t,u)$

Mellin Amplitude becomes the flat space S-Matrix!

Motivations: Hawking Evaporation



(an ATLAS picture of BH production and decay.)

- Expect generic BHs decay via Hawking Radiation
- BH formation/decay = a Scattering process
- Thermodynamics from statistics of S-Matrix
- Compute S-Matrix from flat limit of AdS/CFT!

Simple, Sharp Questions about Black Holes?

Expect Transplanckian S-Matrix has

$$\langle n_{out} \rangle \approx \frac{E}{T_{BH}} \approx \left(\frac{E}{M_{pl}} \right)^{\frac{D-2}{D-3}}$$

Only gravity has scattering amplitudes like this.
Reproducing it with AdS/CFT is a sharp question
that should have a **generic** understanding.

Planck scale should emerge as a dimension in the CFT.

A first step: we will derive a new bound on CFT correlators due to BH intermediate states.

Outline

I. Mellin Space as Momentum Space for CFTs, or CFT correlators as scattering amplitudes

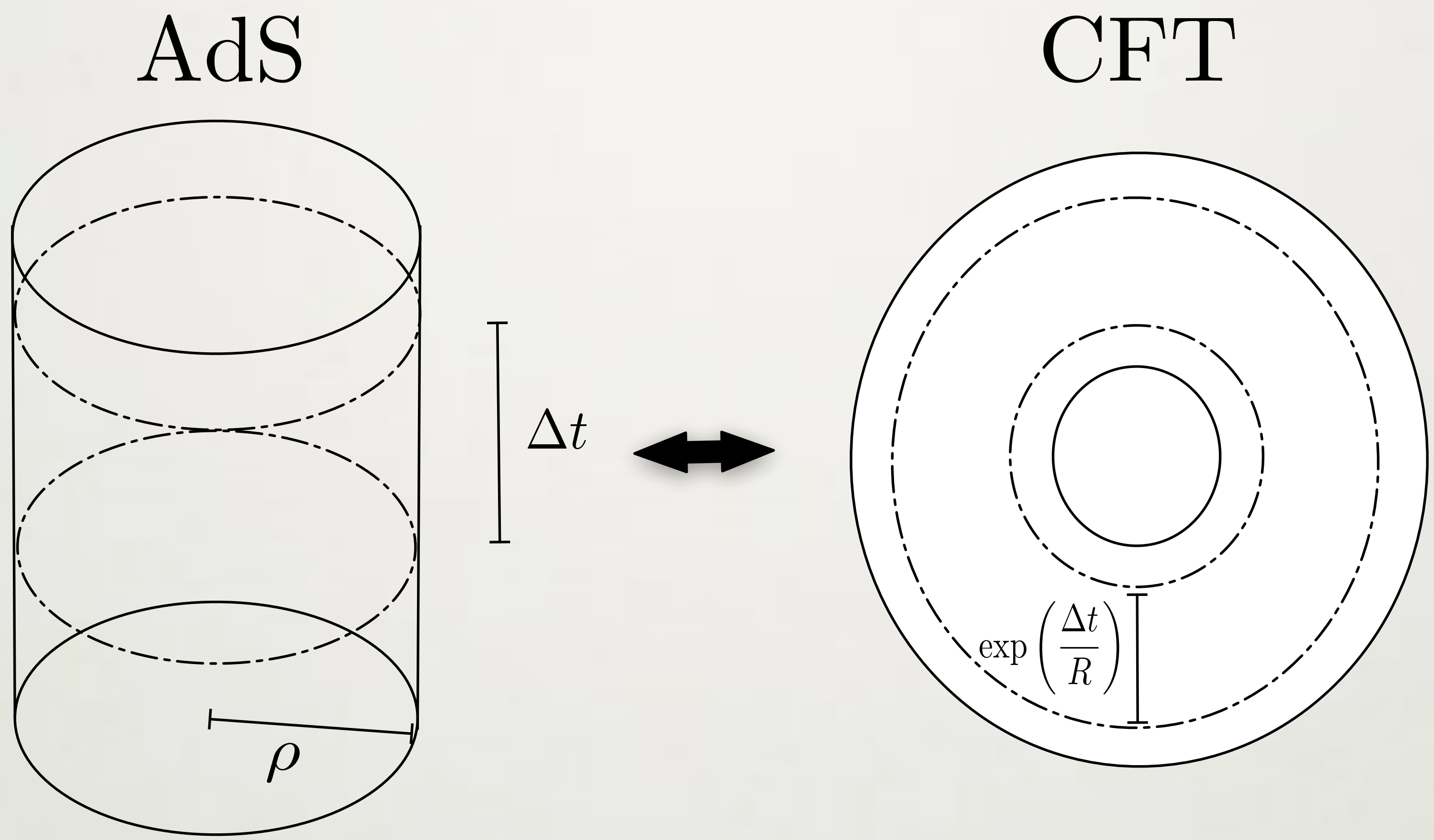
II. Mellin Amplitude as Holographic S-Matrix

III. Analyticity (locality!?) from Mellin-space Meromorphy, some loop level examples

IV. S-Matrix Unitarity from CFT Unitarity

V. S-Matrix program as the Bootstrap program and a peak at Black Holes

AdS/CFT Preliminary



With AdS in Global Coordinates

$$ds^2 = \frac{1}{\cos^2 \rho}(dt^2 + d\rho^2 + \sin^2 \rho d\Omega^2)$$

the **Dilatation Operator generates time translations.**

Let's Try to think of CFT Correlators as Scattering Amplitudes.

What is the CFT Analog of "Free Particles"?

Scattering amplitudes involve states composed of particles that are asymptotically free.

The CFT analog is the large N expansion, because given operators $\mathcal{O}_1$ and $\mathcal{O}_2$, there must exist

$$\text{“}\mathcal{O}_1\mathcal{O}_2\text{”}$$

with dimension $\approx \Delta_1 + \Delta_2$

How should we compute Correlators?

Previous computations in AdS used position space. **Analogous** to computing Feynman diagrams as...

$$\int d^d x D_F(x_1 - x) D_F(x_2 - x) D_F(x_3 - x) D_F(x_4 - x)$$

Even the 4-pt function is a box integral!!

In AdS, computations have been even worse, with very few results beyond 4-pt.

(We will see how to compute at n-pt, easily.)

Review: The Advantages of Momentum Space

In flat space we go to momentum space, which has several familiar advantages.

Eq. of Motion become **algebraic**

$$\nabla^2 = -p^2$$

because the Laplacian acts very simply on the momentum space representation.

We find a similar simplification in **Mellin space**, because the **Conformal Casimir** acts nicely.

Factorization and Momentum Space

Also, flat space scattering amplitudes **Factorize**

$$M(p_i) \to M_L(p_{i_L}, P_L)\frac{1}{P_L^2}M_R(-p_L, p_{i_R})$$

Involves **analyticity** and **unitarity**,
since factorization poles follow from the exchange of single-particle states.

Also, there are **purely algebraic Feynman Rules**.

So position space obscures a lot of physics!

So what is the Mellin Amplitude?

A CFT Correlator written in Mellin Space (Mack):

$$A_n(x_i) = \int [d\delta] M_n(\delta_{ij}) \prod_{i<j}^{n} (x_i - x_j)^{-2\delta_{ij}} \Gamma(\delta_{ij})$$

$$\sum_{j \neq i} \delta_{ij} = \Delta_i$$

Roughly speaking, the δ_{ij} variables are a space of relative scaling dimensions between operators.

The Mellin Amplitude for scalar operators is **Conformally Invariant**.

Mellin Space ~ Space of Mandelstam Invariants

δ_{ij} are symmetric, and with $\delta_{ii} = 0$

You can **always pretend** that $\delta_{ij} =$ "$p_i \cdot p_j$" with

$$\sum_{i=1}^{n} p_i = 0 \quad \text{and} \quad p_i^2 = \Delta_i$$

(**fictitious!**) momentum conservation and on-shell-ness

We will often see combinations in propagators such as

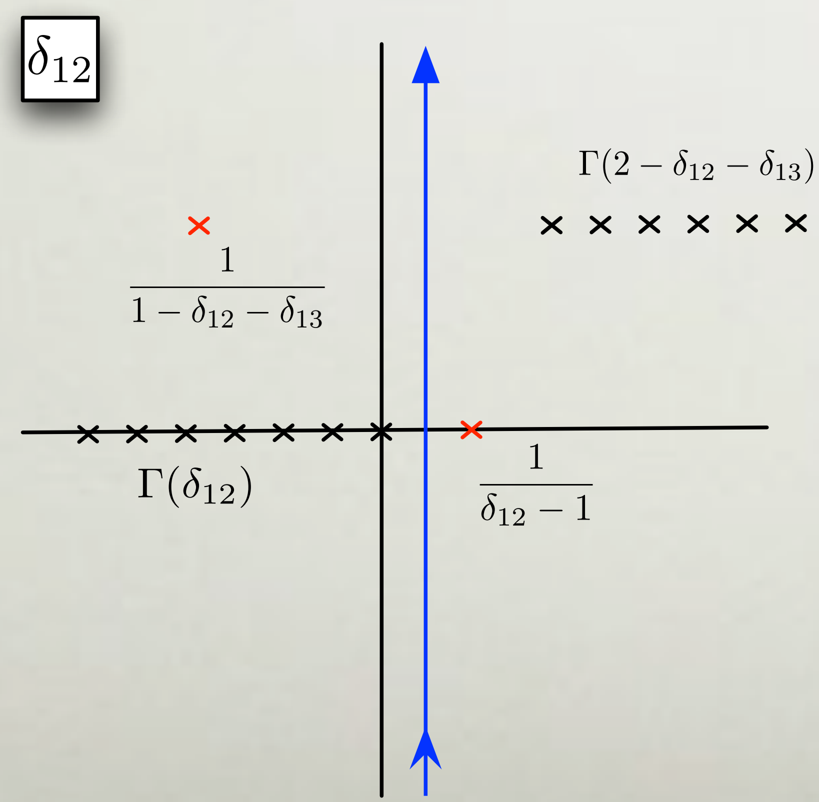
$$\sum_{i,j=1}^{K} \delta_{ij} = (p_1 + ... + p_K)^2$$

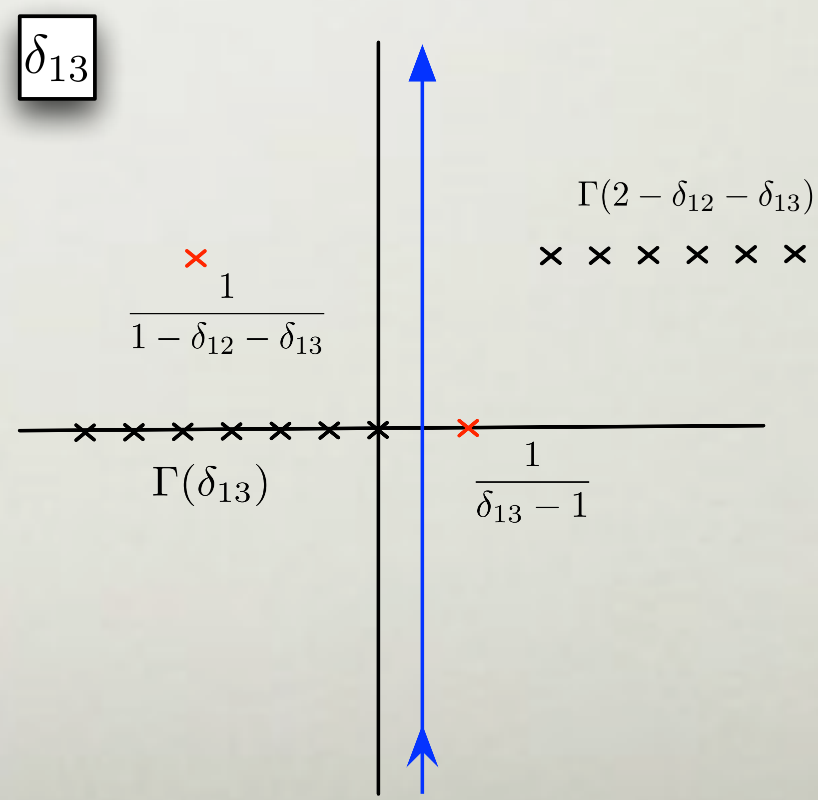
A Simple Example

$\mu\phi^3$ theory in AdS at tree level in 3-D

$$M(\delta_{ij}) = \frac{R^3\mu^2}{2(4\pi)^3}\left(\frac{1}{\delta_{12}-1}+\frac{1}{\delta_{13}-1}+\frac{1}{\delta_{14}-1}\right)$$

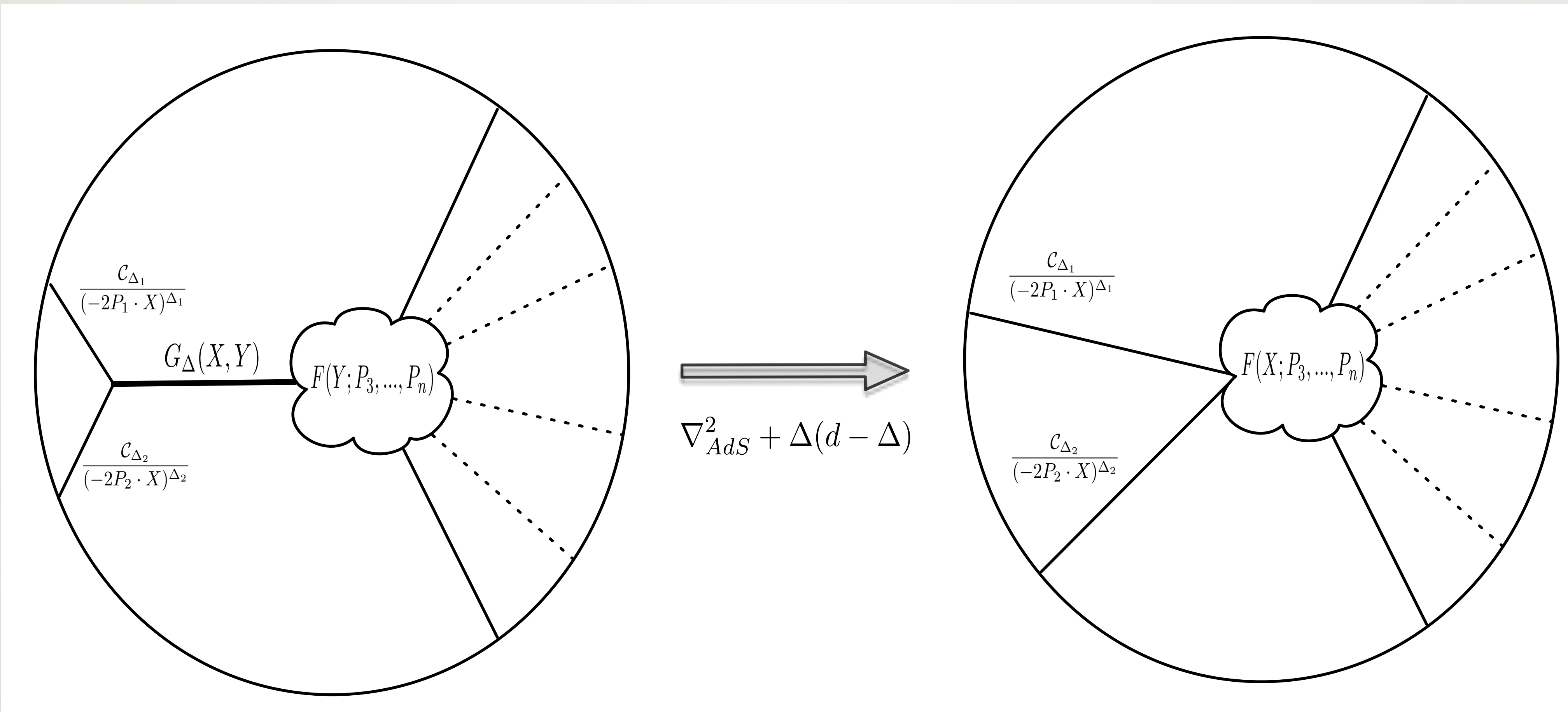
The pole prescription for the contour is





How does the Mellin Amplitude mimic Scattering Amplitudes?

In Mellin Space: The Functional Equation



Find a finite difference equation for Mellin amp:

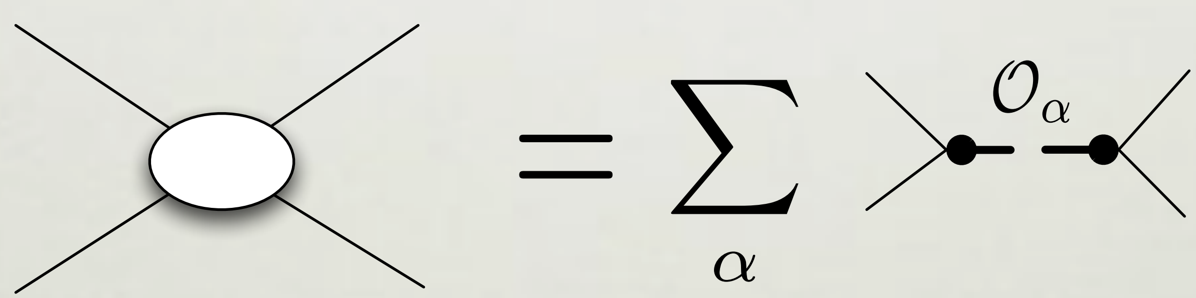
$$(\delta_{12}-a_1)(\delta_{12}-a_2)M(\delta_{12}) = (\delta_{12}-a_3)(\delta_{12}-a_4)M(\delta_{12}-1) - M_0$$

The CFT Analog of Factorization

Factorization occurs in CFTs, but is obscure in position space. Insert 1, organize with symmetry:

$$\langle \mathcal{O}_1 \mathcal{O}_2 \left(\sum_\alpha |\alpha\rangle\langle\alpha| \right) \mathcal{O}_3 \mathcal{O}_4 \rangle$$

By operator-state correspondence, this decomposition is just a sum over **exchanges of operators**:



Mellin Amp displays this **conformal block decomp** as a sum over **factorization channels**. Why?

Why Do Mellin Amplitudes Factorize?

Look at the exchange of operators more carefully:

$$A_n(x_i) \sim \sum_p \int d^d y \left\langle \prod_{i=1}^{k} \mathcal{O}_i\left(x_i\right) \mathcal{O}_p(y) \right\rangle \left\langle \widetilde{\mathcal{O}}_p(y) \prod_{i=1+k}^{n} \mathcal{O}_i\left(x_i\right) \right\rangle$$

Each $\mathcal{O}_p$ in the sum has a **definite dimension**, so each term scales as a **definite power law**.

Mellin space = the space of these powers.

Thus in Mellin space each term gives a **pole**, with a **residue** that is the product of lower correlators.

A Factorization Formula for ADS/CFT

Obtain an explicit AdS/CFT factorization formula:

$$M = \sum_{m=0}^{\infty} \frac{Res(m)}{\delta_{LR} - \Delta - 2m}$$

$$Res(m) \propto \left[L_m(\delta_{ij}) R_m(\delta_{ij})\right]_{\delta_{LR}=\Delta+2m}$$

where

$$\delta_{LR} = \sum_{i,j \leq k} \delta_{ij} = \text{“}(p_1 + \ldots + p_k)^2\text{”}$$

Mellin Amplitudes Are Meromorphic

In general, expect Mellin amplitudes must always be meromorphic functions to get an OPE.

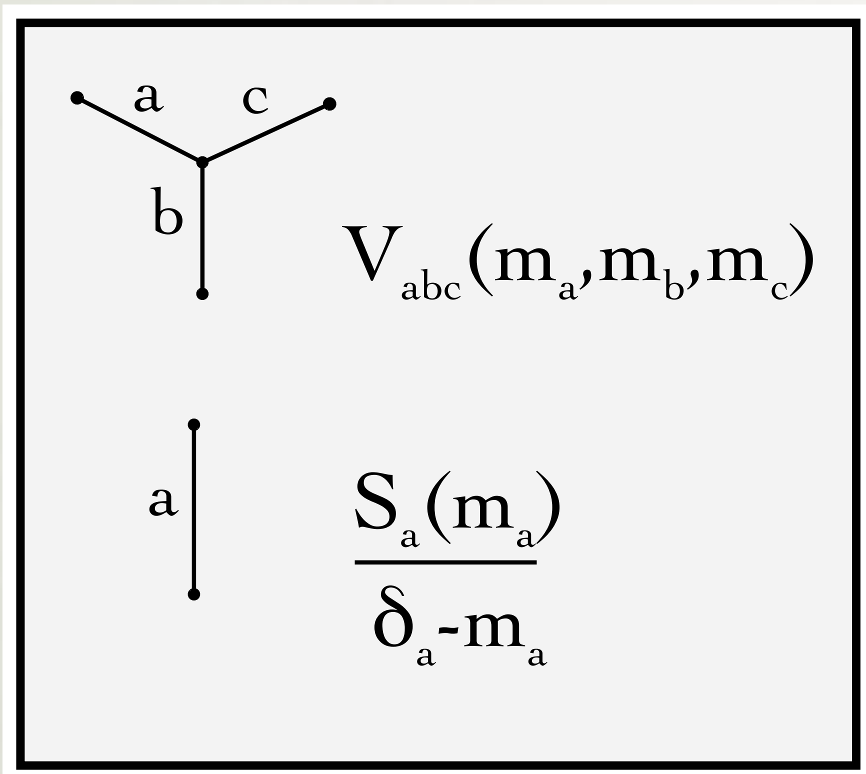
In fact, expect only simple poles, and that all poles will lie on the real axis for a unitarity CFT.

Provides a hint of analyticity for later...

Algebraic Feynman Rules?

We have a factorization formula, and we can factorize on **any propagator**, and reason to believe that Mellin amplitudes are basically just rational functions, so it would be surprising if there wasn't a constructive method for generating Mellin Amps.

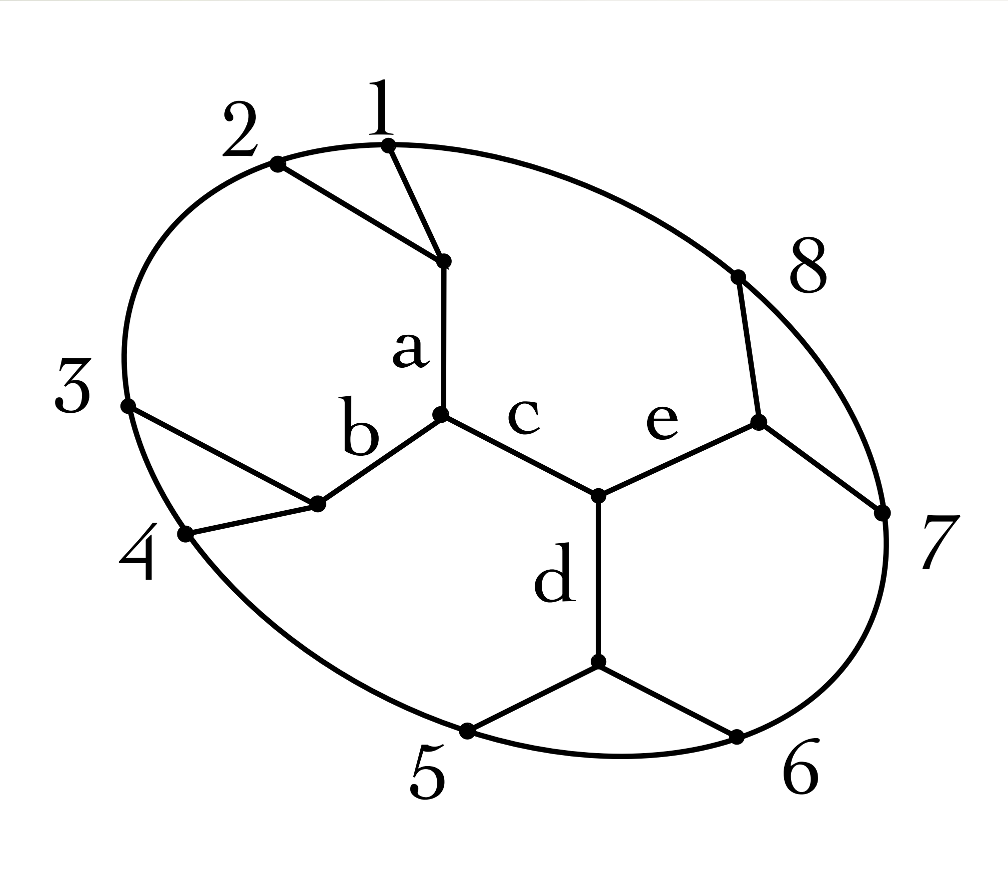
Diagrammatic Rules



Conserve fictitious
``momentum''
at all vertices.

n-pt scalar vertices are **Lauricella functions**;
proven with our finite difference equation
(nice form for vertices found by Paulos, 1107.1504).

So We Can Compute!



AdS/CFT Witten Diagrams such as this can be computed straightforwardly.

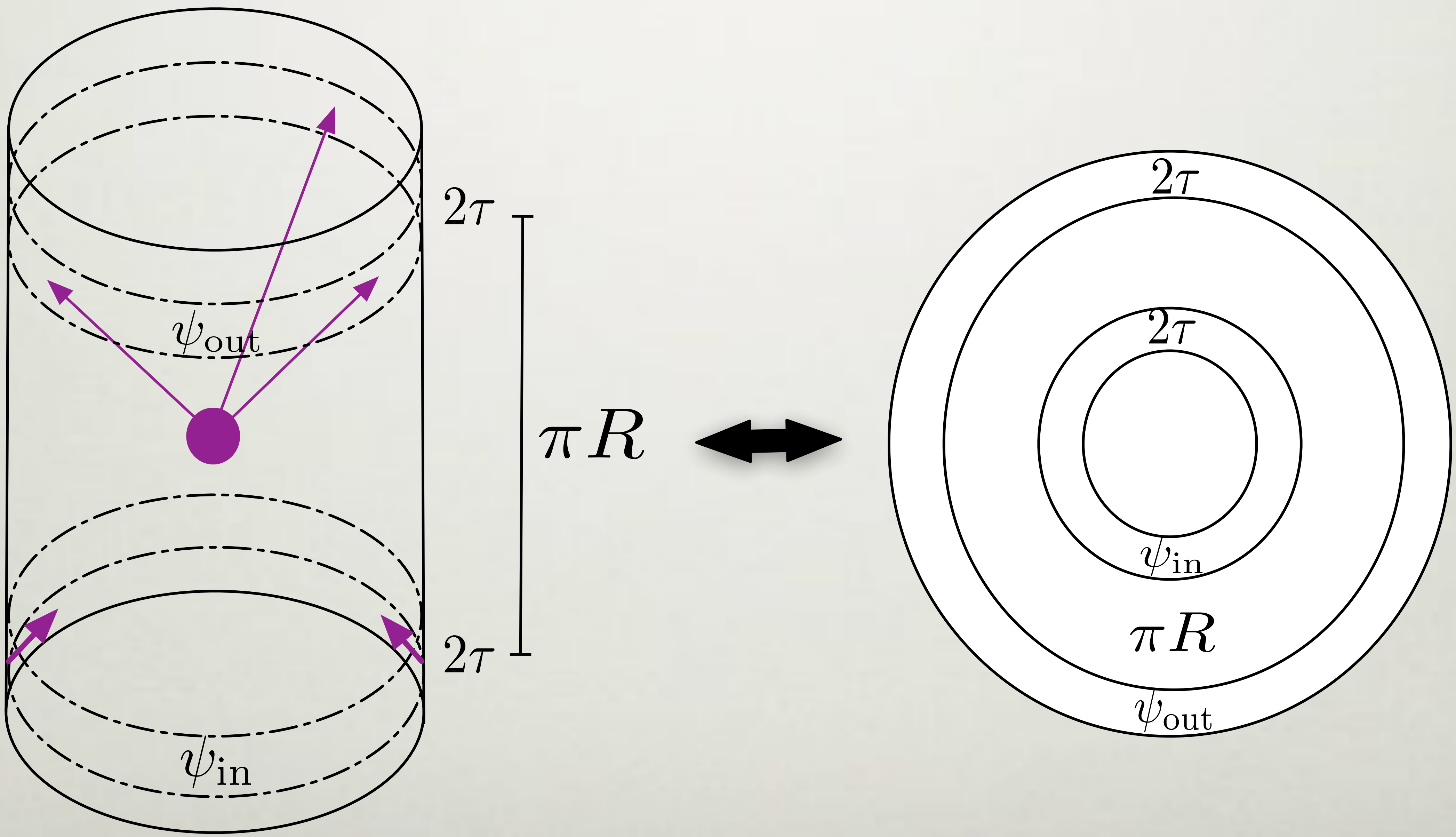
Previously, very few computations beyond 4-pt!!

Relation to Flat Space S-Matrix?

the Flat Space Limit

AdS

CFT



THE FLAT SPACE LIMIT

- Recall Bulk Energy = CFT Dimension
- Flat Space Limit requires

$$E_{\text{bulk}} R_{\text{AdS}} \to \infty$$

- This means that we must study CFT states of very **large dimension**, while

$$N^2 \propto (M_{d+1} R_{\text{AdS}})^{d-1} \to \infty$$

The Flat Space Limit

But we know that δ_{ij} ~ dimension.

Natural to guess (and Penedones did) that

$$\lim_{R\to\infty} M(\delta_{ij} = R^2 s_{ij}) \sim T(s_{ij})$$

And it works! Checked explicitly for theories of scalars at tree level for any number of particles, and some 1-loop examples. More precisely...

THE FLAT SPACE LIMIT

The exact relation for massless external states:

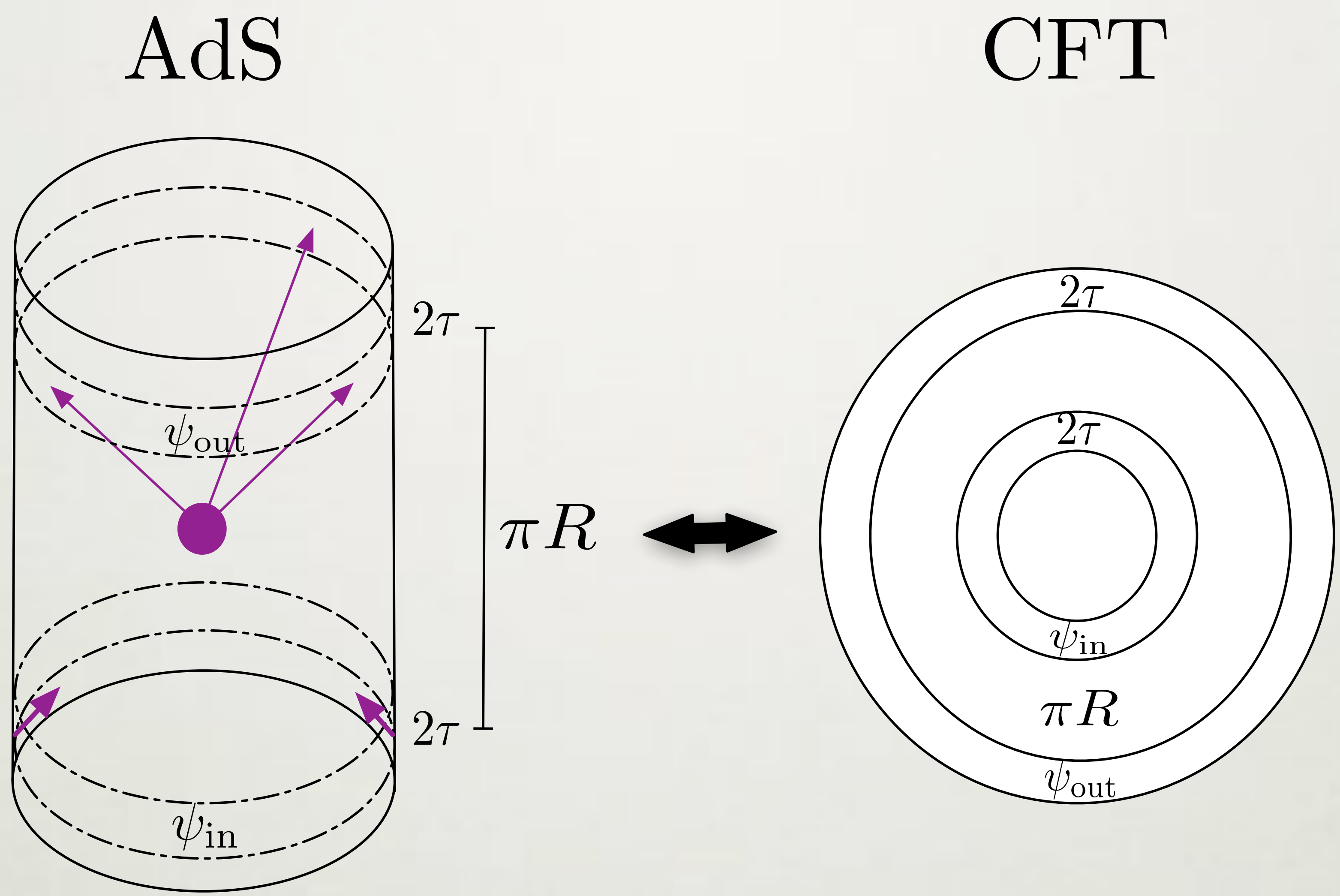
$$T(s_{ij}) = \Gamma\left(\Delta_\Sigma - h\right) \lim_{R\to\infty} \int_{-i\infty}^{i\infty} d\alpha\, e^{\alpha} \alpha^{h-\Delta_\Sigma} M\left(\delta_{ij} = \frac{R^2 s_{ij}}{2\alpha}, \Delta_a = R m_a\right)$$

A one-dimensional contour integral applied to the (meromorphic) Mellin Amplitude.

Note that as one might expect, single trace <--> single particle.

How is it derived?

Deriving the Flat Space Limit



Create in and out states by CFT operator smearing:

$$|\omega, \hat{v}\rangle = \int_{-\frac{\pi R}{2}-\tau}^{-\frac{\pi R}{2}+\tau} dt e^{i\omega t} \mathcal{O}(t, -\hat{v})|0\rangle$$

Single-trace Operator = Single Particle

Deriving the Flat Space Limit

Integrating CFT Correlator against plane waves:

$$T(s_{ij}) = \lim_{\frac{R}{\tau},\tau\to\infty} \int [d\delta] \int_{-\tau\pm\frac{\pi R}{2}}^{\tau\pm\frac{\pi R}{2}} dt_i e^{i(\omega_i-\Delta_i)t_i} M(\delta_{ij}) \prod_{i<j} \left(\cos\left(\frac{t_i - t_j}{R}\right) - \hat{p}_i \cdot \hat{p}_j \right)^{-\delta_{ij}} \Gamma(\delta_{ij})$$

Time differences small: $|t_i - t_j| \ll R$

leading to approximately Gaussian time integrals.

δ_{ij} integrals can be evaluated via stationary phase in the flat space limit of Gamma functions:

$$\int [d\epsilon] M(\delta_{ij}) \exp\left[\sum_{ij} R^2 s_{ij} \left(\frac{1}{\alpha} + \epsilon_{ij}\right) \log\left[R^2 \left(\frac{1}{\alpha} + \epsilon_{ij}\right)\right]\right]$$

The S-Matrix from the Mellin Amplitude

δ_{ij} variables align with s_{ij}, leaving us with:

$$T(s_{ij}) = \Gamma\left(\Delta_\Sigma - h\right) \lim_{R\to\infty} \int_{-i\infty}^{i\infty} d\alpha\, e^{\alpha} \alpha^{h-\Delta_\Sigma} M\left(\delta_{ij} = \frac{R^2 s_{ij}}{2\alpha}, \Delta_a = R m_a\right)$$

Our factorization formula and Feynman rules
for the Mellin amplitude reduce
to the factorization formula and Feynman rules
of the tree-level scattering amplitudes.

Analyticity and the Holographic S-Matrix

Locality = Analyticity?

Only precise notion of locality (I'm aware of) is via analyticity and boundedness of S-Matrix.

The Scattering Amplitudes are given by a simple integral transform of the Mellin Amp.

The Mellin Amplitude is a meromorphic function with only simple poles, in any CFT.

Is this how we should think of locality emerging from a CFT!?

Analyticity in the Flat Space Limit

$$T(s_{ij}) = \Gamma\left(\Delta_\Sigma - h\right) \lim_{R\to\infty} \int_{-i\infty}^{i\infty} d\alpha\, e^{\alpha} \alpha^{h-\Delta_\Sigma} M\left(\delta_{ij} = \frac{R^2 s_{ij}}{2\alpha}, \Delta_a = R m_a\right)$$

For finite R, just contour integral of meromorphic function, so obviously analytic.

Flat Space Limit just expands near infinity.
We get branch cuts and imaginary parts from the coalescence of poles.

Flat Space Limit of a Bulk Exchange

Let's see how branch cuts etc obtain, leaving a general analysis of locality for the future.

Taking flat space limit, a bulk propagator becomes:

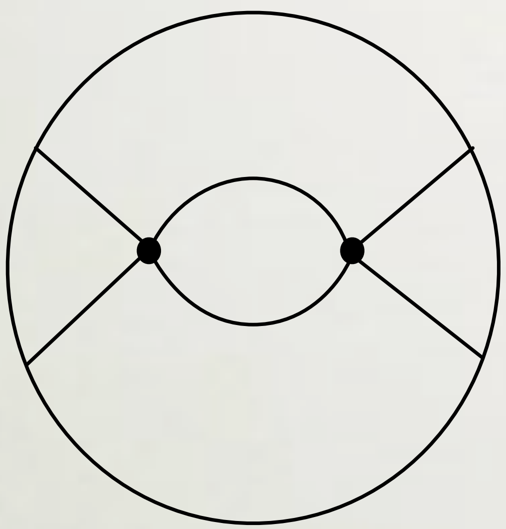
$$\sum_m \frac{R(\Delta, m)}{\delta - (\Delta + m)} \to \frac{1}{s + \Delta^2}$$

(The Mellin amplitude is dominated by poles where $m \approx \Delta^2$, when we take the flat space limit.)

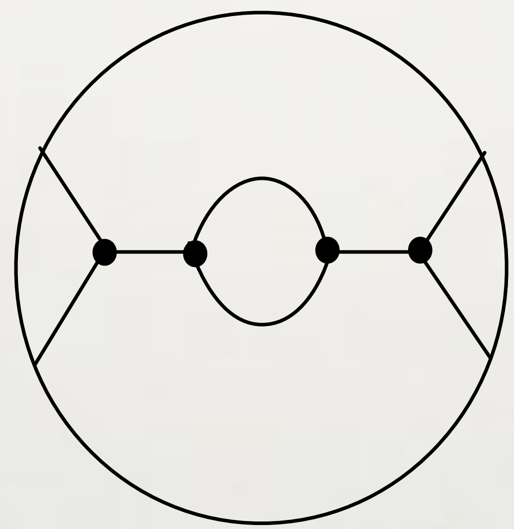
Loops?

Computing Loop Diagrams

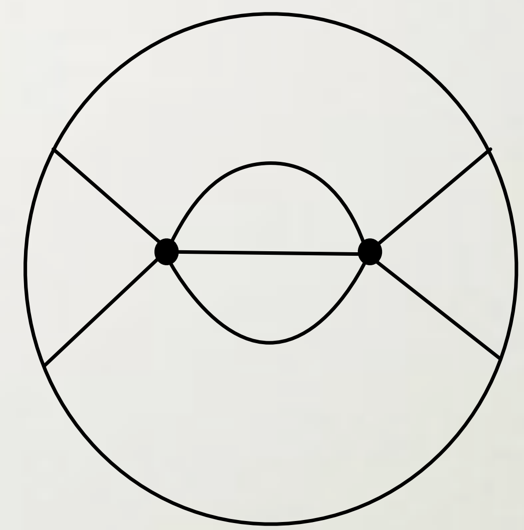
We can also compute AdS loop diagrams



$\lambda\phi^4$



$\mu\phi^2\chi$

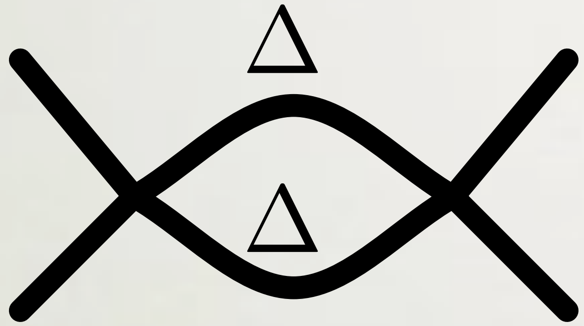


$g\phi^5$

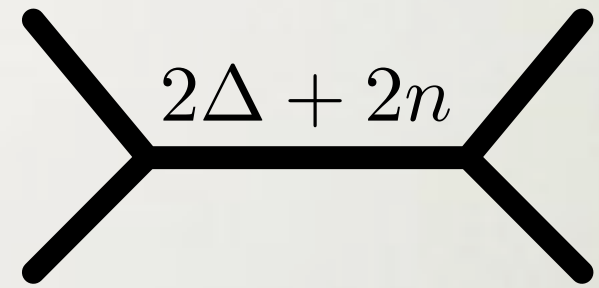
Using an AdS version of Kallen-Lehman, which makes it possible to write 2-point functions of local operators as a positive integral over free propagators.

1-Loop Computations a la Kallen-Lehman

At 1-loop, can write bubble diagram using:



$$= \sum_{n=0}^{\infty} N_{\Delta}(n)$$



or

$$G_{\Delta}(X,Y)^2 = \sum_{n=0}^{\infty} N_{\Delta}(n) G_{2\Delta+2n}(X,Y)$$

We use an inner product obeyed by the propagators to compute this decomposition.

Loop Level Mellin Amplitude

This gives a **Kallen-Lehman**-esq Mellin Amplitude:

$$M(\delta) = \sum_n N(n) \sum_m \frac{R(2\Delta + 2n, m)}{\delta - (2\Delta + 2n + m)}$$

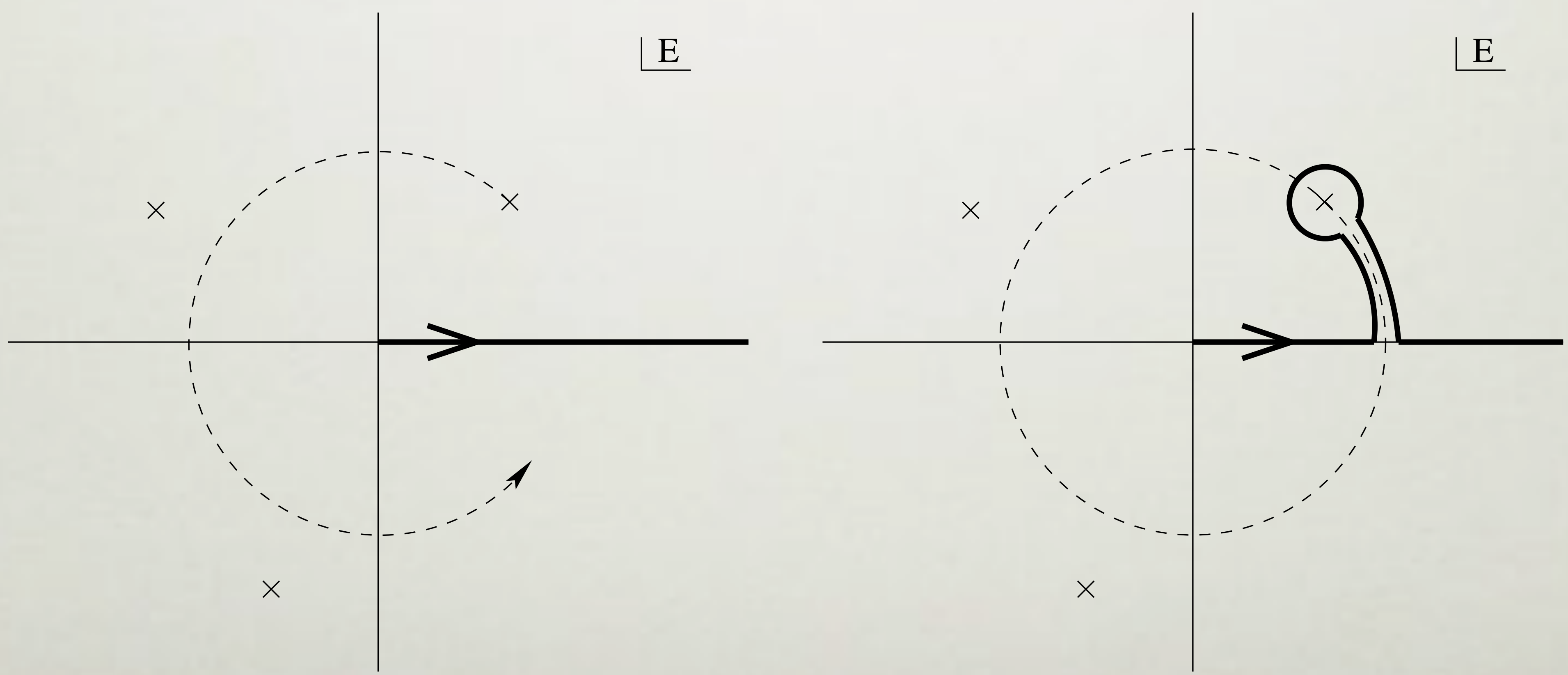
Represents the exchange of **double-trace primary** states of dimension $2\Delta + 2n$.

Branch Cuts

In the flat space limit, we find the integral:

$$M(\delta) \to \int_0^\infty dn \frac{N(n)}{s + (2\Delta + 2n)^2}$$

Circling in the complex plane gives a branch cut.



Branch Cuts from Mellin Amplitudes

$$M(\delta) \to \int_0^\infty dn \frac{N(n)}{s + (2\Delta + 2n)^2} \quad \text{with} \quad N(n) \propto n^{d-2}$$

for $\lambda\phi^4$ theory. Gives branch cut! Discontinuity:

$$\frac{N(\sqrt{s})}{\sqrt{s}} \propto \sqrt{s}^{d-3}$$

Correct for theory in d+1 dimensions.

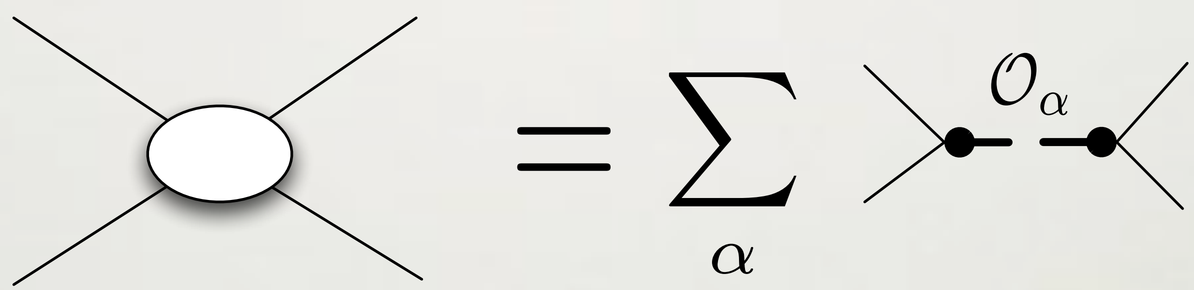
Unitarity of the Holographic S-Matrix

S-Matrix Unitarity from CFT Unitarity

The standard optical theorem with S = 1 + iT

$$-i(T - T^\dagger) = T^\dagger T$$

looks reminiscent of the **Conformal Block decomp**:



[From using conformal symmetry to organize

$$\langle \mathcal{O}_1 \mathcal{O}_2 \left(\sum_\alpha |\alpha\rangle\langle\alpha| \right) \mathcal{O}_3 \mathcal{O}_4 \rangle$$

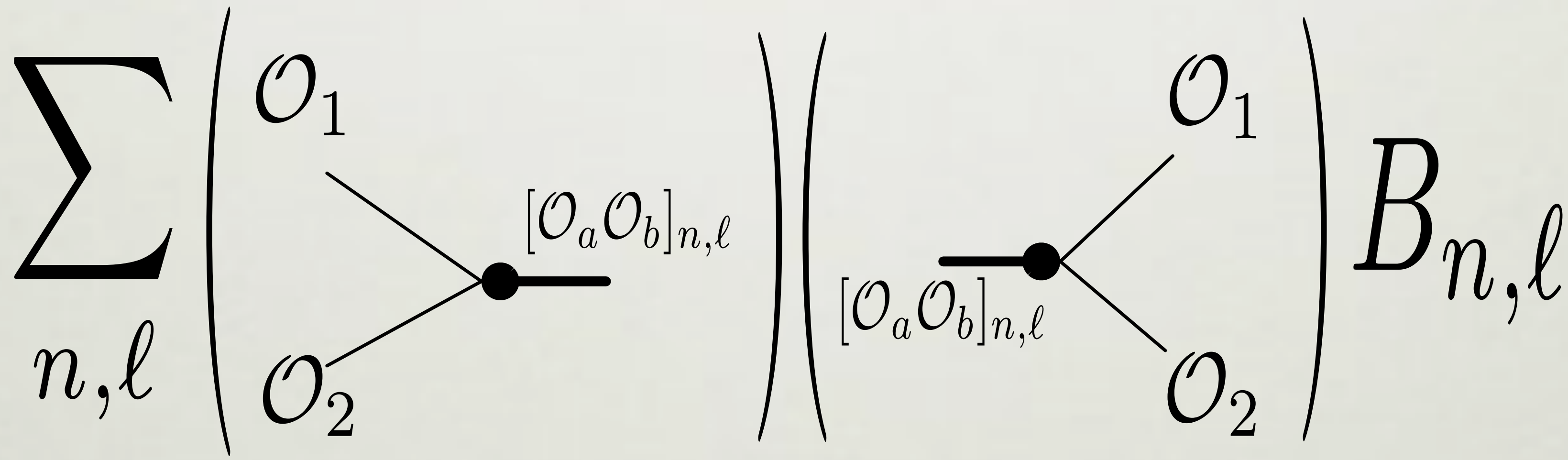
since **operators = states** in the CFT.]

Conformal Blocks and the OPE

We can apply the Operator Product Expansion

$$\mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \sum_{\Delta,\ell} c^{12}_{\Delta,\ell}\mathcal{O}_{\Delta,\ell}(x)$$

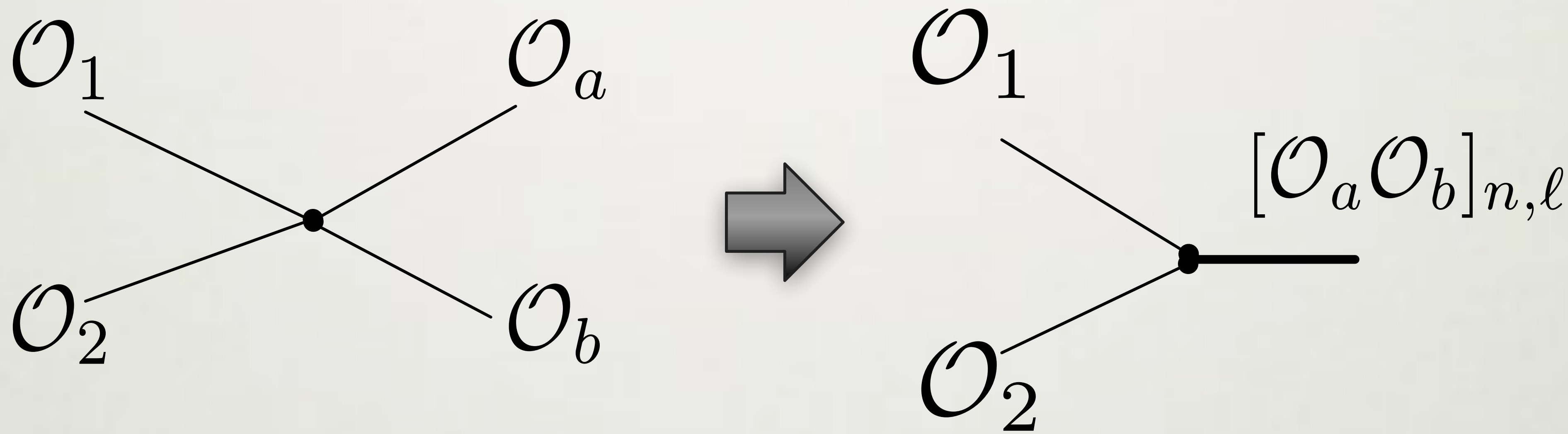
to a 4-pt correlation function to find



This is a formula for the conformal block coefficients.

Conglomerating Operators

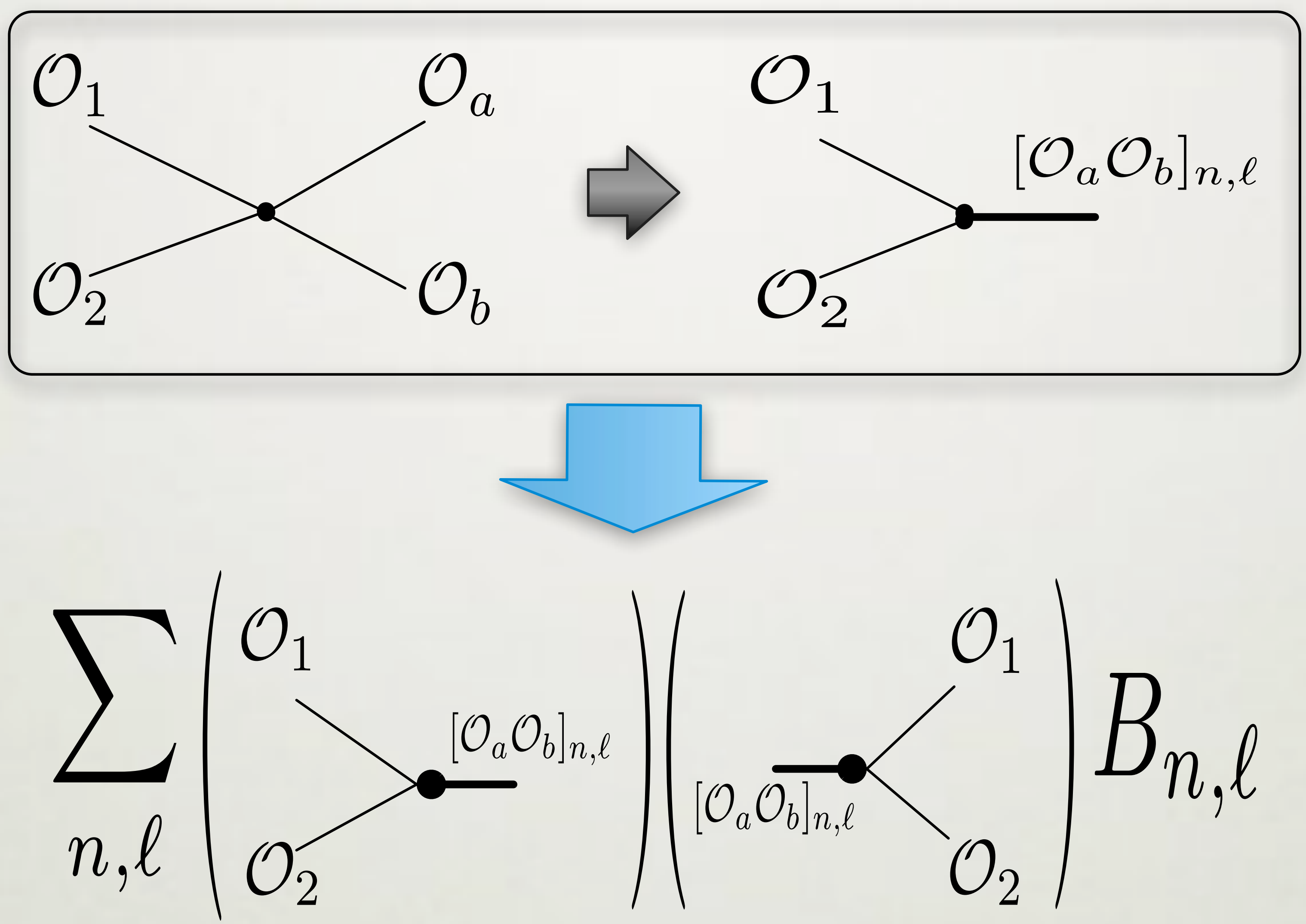
To compute need to **conglomerate** single trace operators into one multi-trace:



Can differentiate, but extremely cumbersome.

Easy in Mellin space, convolve with wavefunction.

Something like the Optical Theorem...



We can get info about next order in perturbation theory!

Let's take the flat space limit of these CFT Unitarity Operations

Bootstrap Program => S-Matrix Program

What is the flat space limit of a conformal block?

$$B_{\Delta_\alpha} \to \delta\left(s - \Delta_\alpha^2\right)$$

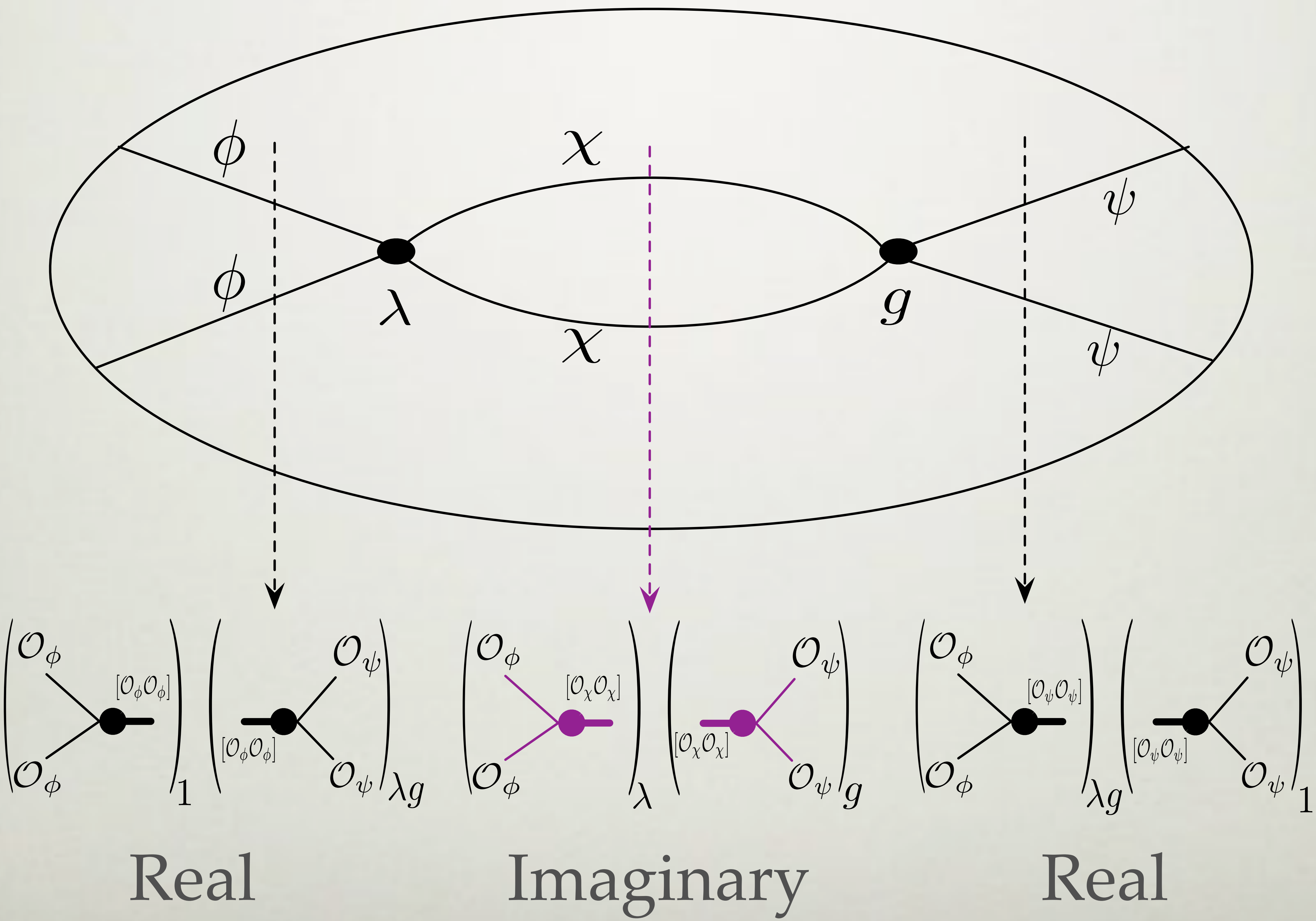
"Obvious", since blocks have definite angular momentum and definite dimension = energy.

$$M_4(\delta_{ij}) = \sum_\alpha N_B(\Delta_\alpha) B_{\Delta_\alpha}(\delta_{ij})$$

becomes (when we take the flat space limit)

$$\mathcal{M}(s,t) = N_B(s,t)$$

A 1-Loop Example



Consequences of the Holographic S-Matrix?

CFT Bootstrap VERSUS S-Matrix Program

S-Matrix program used **Unitarity** and **Analyticity**, the latter being a formalization of locality.

The Bootstrap program for CFTs instead uses **Unitarity** and **Crossing Symmetry**, along with assumptions or data about the spectrum.

The Bootstrap Program naturally allows us to relax the assumption of bulk locality!

Black Holes as Intermediate States?

$$\mathcal{S}(s,t) = N_B(s,t)$$

But on very general grounds, expect that

$$\mathcal{S}(s) \sim \exp\left[-\frac{1}{2}S_{BH}(s)\right] = \exp\left[-\frac{1}{8}\left(G_D s^{\frac{D-2}{2}}\right)^{\frac{1}{D-3}}\right]$$

This gives a concrete prediction for the OPE and the conformal block decomposition of any CFT with a gravity dual where effective field theory applies!

Some Future Directions

- Mellin diagrammatic rules for loops, higher spin particles, twistors/spinor-helicity, SUSY, compactifications, dS/CFT, beloved theories...
- bolster recent progress on CFT Bootstrap?
- broken conformal invariance (eg QCD), flows between CFTs??
- sharpen criterion for analyticity = bulk locality?
- do **all** Gravitational S-Matrices come from CFTs??
- Find a CFT description of Hawking Evaporation, or at least see its simple and robust features!?

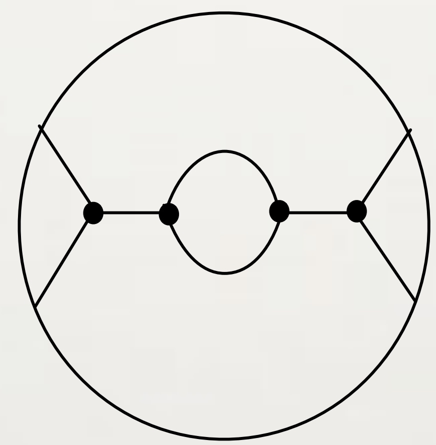
Conclusion

- Mellin Space = ``Momentum Space for CFTs", conceptually and computationally
- Mellin Amplitude -> Holographic S-Matrix
- Analyticity follows from Meromorphy
- the OPE implies Unitarity, Cutting Rules
- Expect scattering through BHs is a robust ingredient in CFT dynamics, so we should attempt to understand it!

The End

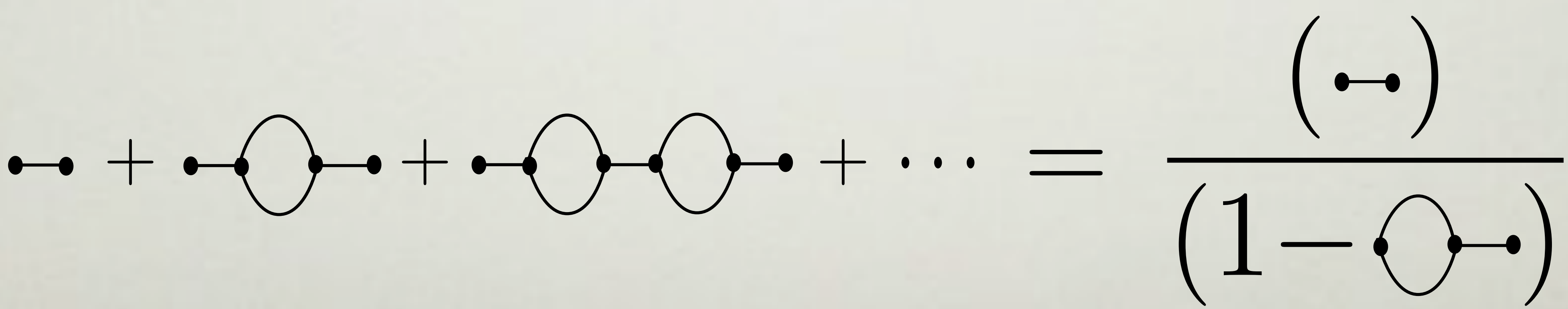
Resonances

To see how this loop diagram gives Breit-Wigner



$\mu\phi^2\chi$

we need to perform the resummation:



Resonances

With a discrete spectrum, can view as mixing

$$m_{\text{eff}}^2 = \begin{pmatrix} \Delta_\chi^2 & R^2\lambda_{\text{eff}}(0) & R^2\lambda_{\text{eff}}(1) & R^2\lambda_{\text{eff}}(2) & \cdots \\ R^2\lambda_{\text{eff}}(0) & (2\Delta_\phi)^2 & 0 & 0 & \cdots \\ R^2\lambda_{\text{eff}}(1) & 0 & (2\Delta_\phi+2)^2 & 0 & \cdots \\ R^2\lambda_{\text{eff}}(2) & 0 & 0 & (2\Delta_\phi+4)^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with

$$\lambda_{\text{eff}}(n) \equiv \lambda\sqrt{\frac{N_{2\Delta_\phi}(n)}{R^{2h-1}}}$$

This is a mixing between χ particle and the various 2ϕ states.

Resonances

By diagonalizing, one can compute the Mellin amp:

$$
\begin{aligned}
M(\delta_{ij}) &= \sum_a S_{1a} D_a(\delta_{LR}) S^T_{a1}, \\
D_a(\delta_{LR}) &= \sum_m \frac{R_m(\Delta_a)}{\delta_{LR} - (\Delta_a + 2m)}
\end{aligned}
$$

We find that roughly $\frac{\lambda_{eff} R}{m_\chi}$ 2-particle states contribute

an eigenvalue proportional to λ_{eff}, giving

$$\frac{1}{s - m_\chi^2 + i\lambda^2 m_\chi^{D-4}}$$

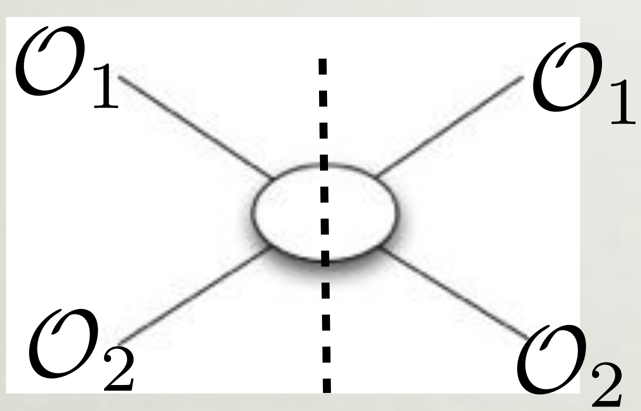
near the pole at weak coupling, as expected.

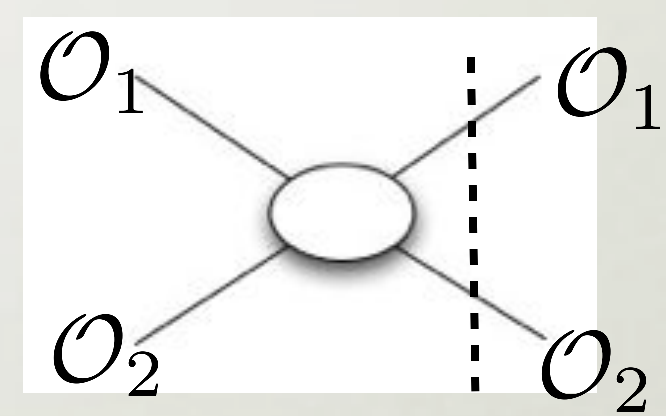
S-Matrix Unitarity from CFT Unitarity

Conformal Block Decomposition

$$\mathcal{A}(x_i) = \sum_{\Delta} c_{\Delta}^2 B_{\Delta}(x_i)$$

Cuts through diagram vs. cuts at edge:





Internal operators

$\mathcal{O}'$

Double-trace operators

$\mathcal{O}_1\mathcal{O}_2$

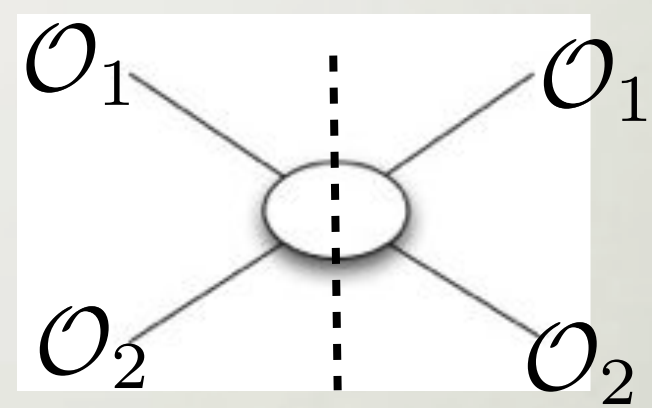
S-Matrix Unitarity from CFT Unitarity

$$\mathcal{A}(x_i) = \sum_{\Delta} c_{\Delta}^2 B_{\Delta}(x_i)$$

Flat-space limit of a conformal block is a delta function

$$B_\Delta \to N_\Delta \delta(s - \Delta^2)$$

OPE coefficients are just factorized amplitudes times phase space!

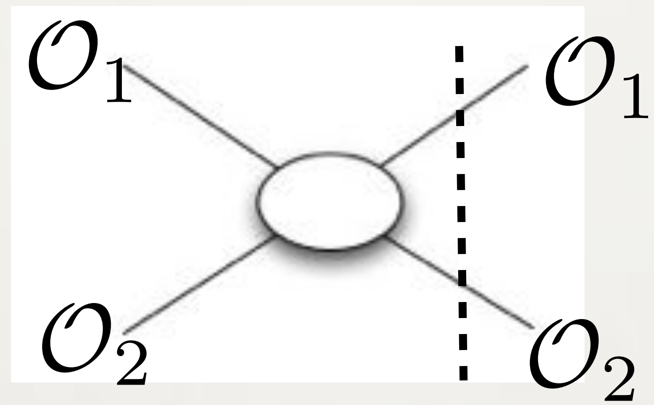


$$c_\Delta \sim \mathcal{M}_{12\to\Delta}$$

"Internal cuts" are just RHS of usual optical theorem!

$$2\mathcal{I}m(\mathcal{M}) \sim \sum_{\Delta} 2\mathcal{I}m(N_\Delta)|c_\Delta|^2 \sim \int d\text{LIPS}|\mathcal{M}_{12\to\Delta}|^2$$

What about Double-Trace Cuts?



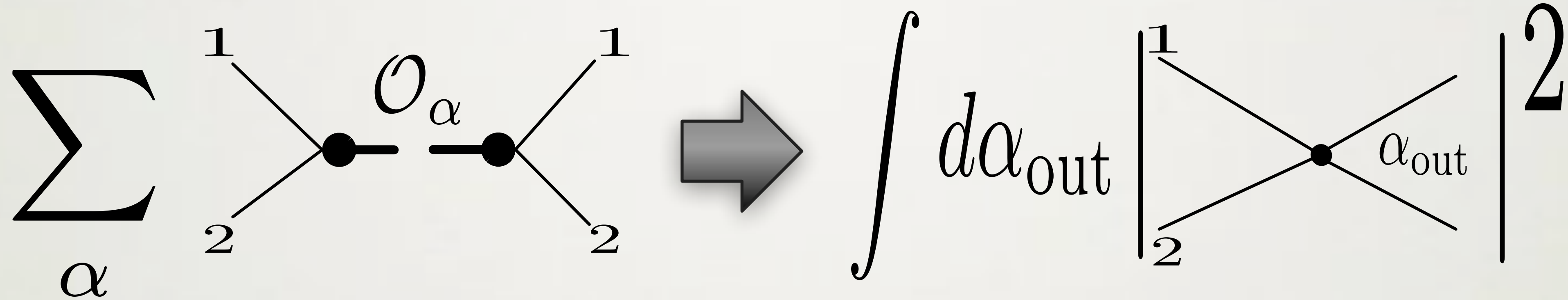
Cuts through edge of diagram are "double-traces", which contribute a total derivative

$$\mathcal{A}_{\text{d.t.}}(x_i) = \sum_n \frac{\partial}{\partial n}(c_n^2 \gamma(n) B_n(x_i))$$

Imaginary part is smooth, so in flat-space this becomes the integral of a total derivative!

$$2\mathcal{I}m(\mathcal{M}_{\text{d.t.}}) \approx \int dn \frac{\partial}{\partial n}(\dots) = 0$$

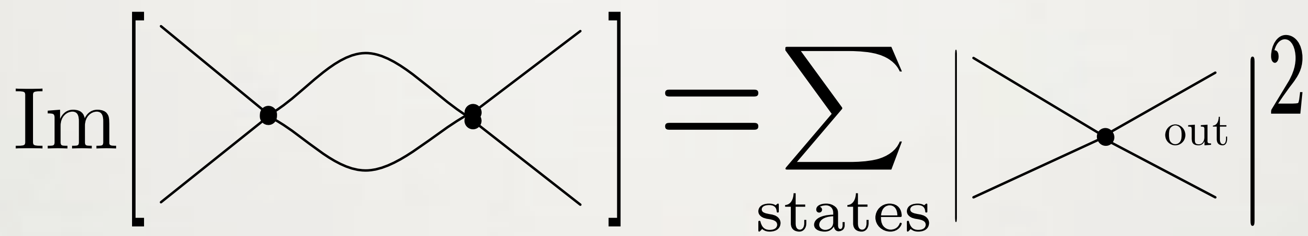
A 1-Loop Example



One can check directly that the sum over all multi-trace CFT operators at a given dimension reproduces a phase space integral in the flat space limit.

Let's Check It at 1-Loop

We need to compute both sides from the CFT.



The goal is to see that both are determined by a specific conformal block coefficient in $\lambda\phi^4$.

First let's compute the left side, using the 1-loop result we discussed.

Branch Cut Discontinuity

Recall that at 1-loop, branch cuts came from:

$$M(\delta) \to \int_0^\infty dn \frac{N_W(n)}{s + (2\Delta + 2n)^2} \implies \text{disc} = \frac{N_W(\sqrt{s})}{\sqrt{s}}$$

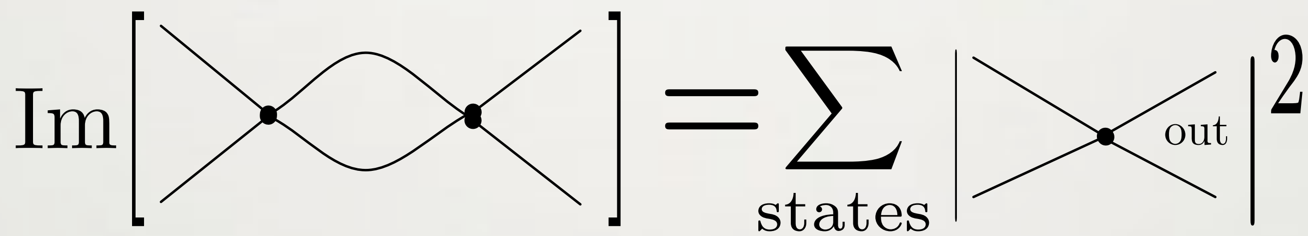
where we had defined (a la Kellan-Lehman)

$$G_\Delta(X,Y)^2 = \sum_{n=0}^{\infty} N_W(n) G_{2\Delta+2n}(X,Y)$$

But the contribution of bulk exchange implies the exchange of a primary operator in the conformal block decomposition.

Conformal Blocks and the Imaginary Piece

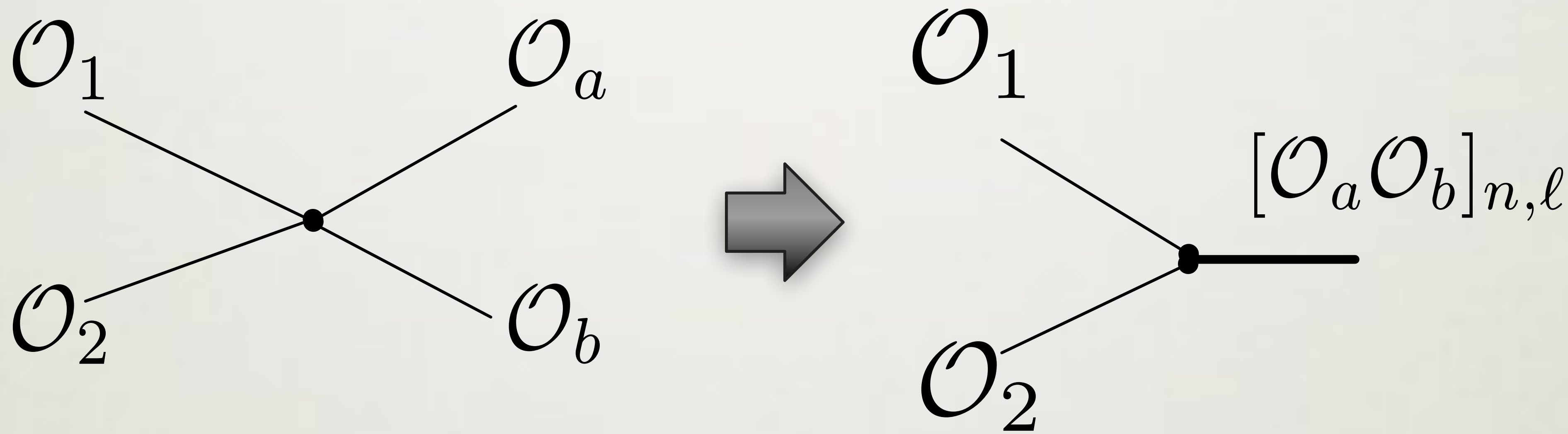
In other words, we see that the conformal block decomposition determines the left side of



Now we will compute the right side.

Conglomerating Operators

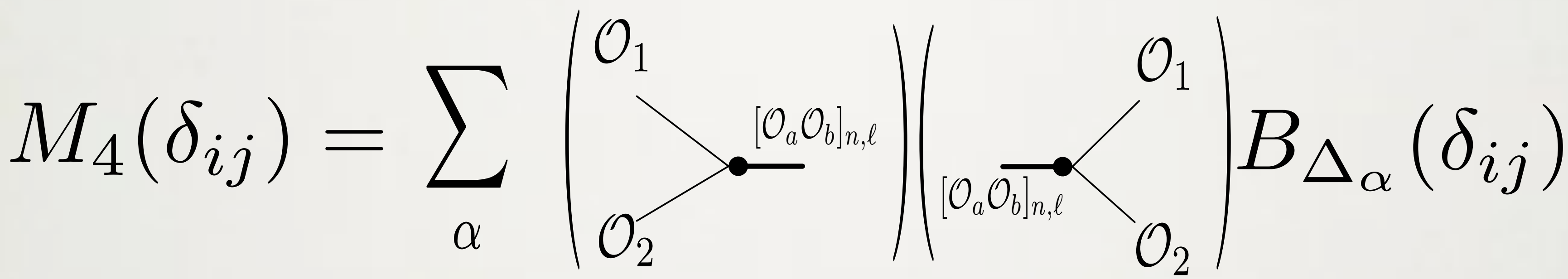
To compute need to **conglomerate** single trace operators into one multi-trace:



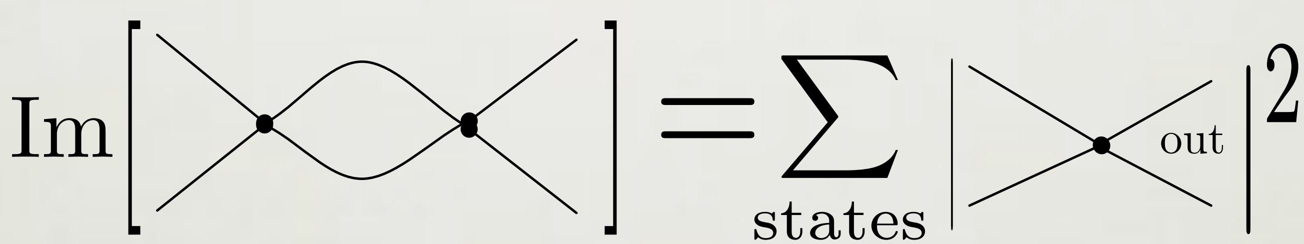
Easy in Mellin space, convolve with wavefunction.

By operator-state correspondence, this picks a state in the CFT (the state appearing in cutting rules!).

Unitarity Relation Determined by Blocks



Coeff of Block at a given Dimension/Energy
is the square of 2 --> X amp, summed over states!

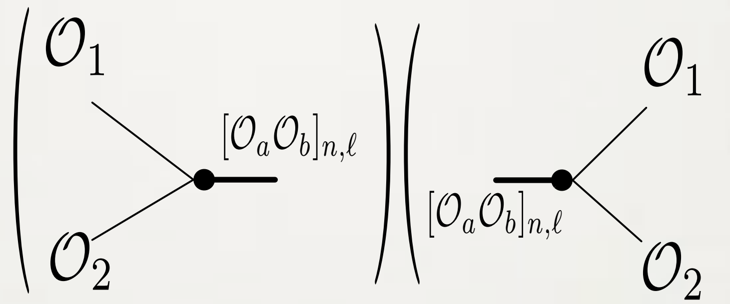


Sum on CFT states = phase space integral in Flat Limit.

Both sides compute the same Conformal Block Coeff!

Unitarity Relation Determined by Blocks

$$M_4(\delta_{ij}) = \sum_{\alpha}$$



$$B_{\Delta_\alpha}(\delta_{ij})$$

As with dispersion relations, one order in perturbation theory gives info about the next.

$$M_4(\delta_{ij}) = \sum_{\alpha} N_B(\Delta_\alpha) B_{\Delta_\alpha}(\delta_{ij})$$

Gives a distinct way to compute coefficients in the conformal block expansion.

Conformal Blocks from 3-pt Correlators

$$M_4(\delta_{ij}) = \sum_\alpha N_B(\Delta_\alpha) B_{\Delta_\alpha}(\delta_{ij})$$

Coefficients of each block come from 3-pt correlators

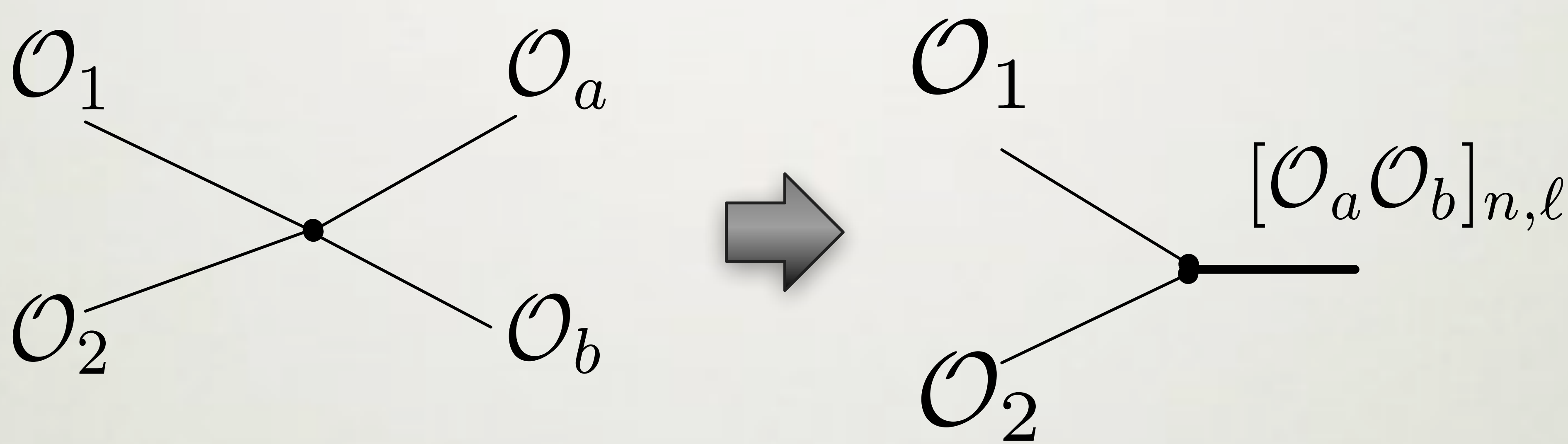
$$N_B(\Delta_\alpha) = \frac{C_3(1,2,\alpha) C_3(\alpha,3,4)}{C_2(\alpha,\alpha)}$$

Where the coefficients multiply universal functions

$$\langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_\alpha \rangle = \frac{C_3(1,2,\alpha)}{x_{12}^{\Delta_{12,\alpha}} x_{2\alpha}^{\Delta_{2\alpha,1}} x_{\alpha 1}^{\Delta_{\alpha 1,2}}}$$

Conglomerating Operators

To compute need to **conglomerate** single trace operators into one multi-trace:



Easy in Mellin space, convolve with wavefunction.

By operator-state correspondence, this picks a state in the CFT (the state appearing in cutting rules!).

Bulk Exchange Leads to Operator Exchange

$$G_\Delta(X,Y)^2 = \sum_{n=0}^{\infty} N_W(n) G_{2\Delta+2n}(X,Y)$$

implies that we must have terms in the conformal block decomposition:

$$N_B(2\Delta + 2n) = N_W(n)$$

where the decomposition is defined by

$$M_4(\delta_{ij}) = \sum_\alpha N_B(\Delta_\alpha) B_{\Delta_\alpha}(\delta_{ij})$$

Mellin Diagrams to Feynman Diagrams

δ_{ij} variables align with s_{ij}, leaving us with:

$$T(s_{ij}) = \Gamma\left(\Delta_\Sigma - h\right) \lim_{R\to\infty} \int_{-i\infty}^{i\infty} d\alpha\, e^\alpha \alpha^{h-\Delta_\Sigma} M\left(\delta_{ij} = \frac{R^2 s_{ij}}{2\alpha}, \Delta_a = R m_a\right)$$

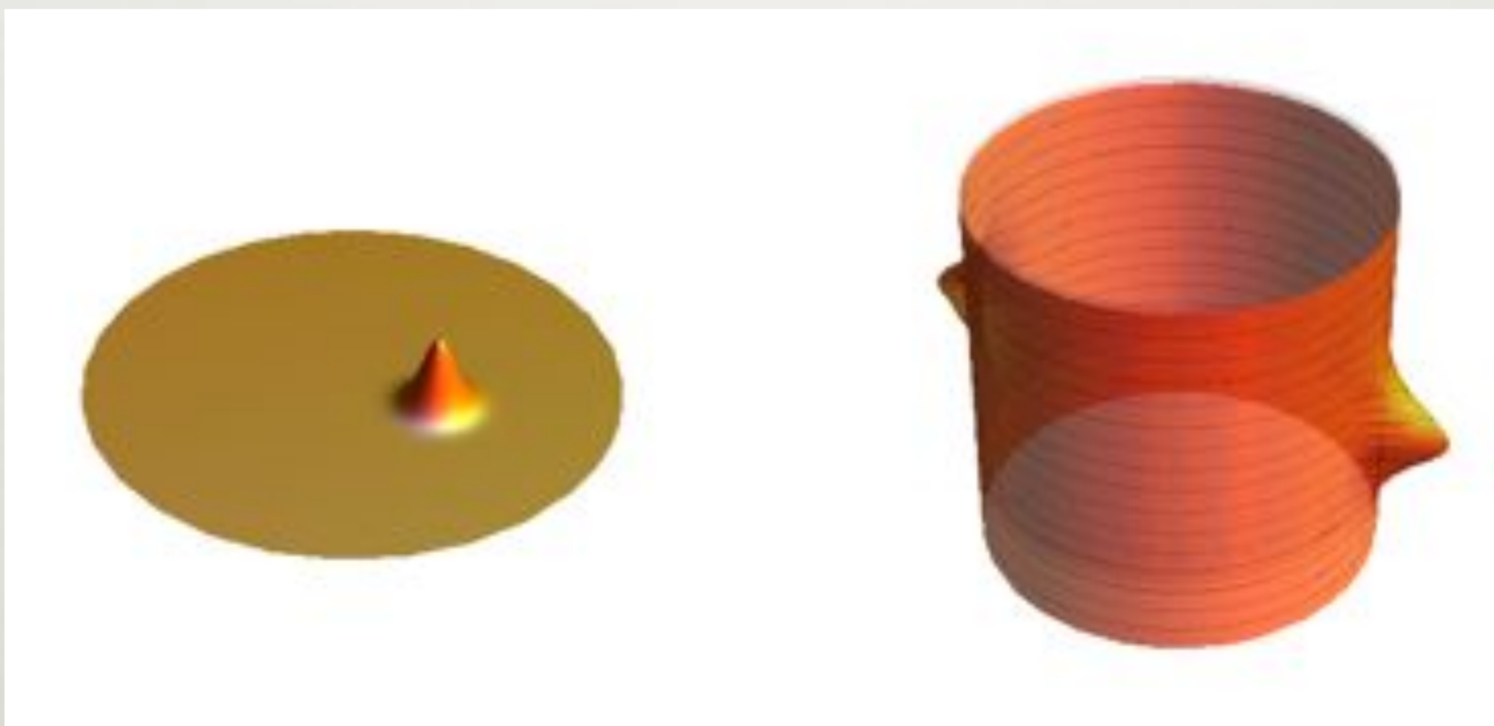
$i\epsilon$ prescription comes from CFT prescription.

We showed that our factorization formula for the Mellin amplitude reduces to factorization of the tree-level scattering amplitudes, and that our Feynman rules reduce to the flat space rules.

Deriving the Flat Space Limit

Point-source at the boundary = plane wave in the center of AdS, energy set by frequency:

$$|\omega, \hat{v}\rangle = \int_{-\frac{\pi R}{2}-\tau}^{-\frac{\pi R}{2}+\tau} dt e^{i\omega t} \mathcal{O}(t, -\hat{v})|0\rangle$$



(an example of a wave packet state)