Physics of the D=5 Chern-Simons Term

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QED in D=5

Photons propagate in bulk; Naturalness, m_e << 1/R, requires
chiral delocalization; Wilson-line mass term

Chiral delocalization requires a Chern-Simons term

Anomaly-freedom implies quantized coefficient of CS term;
"consistent anomalies" vs. "covariant anomalies"

Chern-Simons term implies new interactions amongst
bulk KK-mode photons, effective D=4 interaction,

(i) Use "Wilson-Line Gauge Transformation" to $A_5 = 0$

(ii) Large m_e limit -> Integrate out fermions
Fermionic Dirac determinant modifies
effective interaction; maintains gauge invariance

(iii) Compute K' -> K + gamma via effective interaction

Yang-Mills in D=5

"quarks" on branes; gauge theory of flavor
compactify with A_5 zero-mode -> mesons $f_{pi}=1/R$;
Wilson line mass term <-> chiral condensate

Chiral delocalization requires a Chern-Simons term;
anomaly matching, quantization

Chern-Simons term implies bulk and
holographic interactions amongst KK-modes
effective D=4 interaction,

Large m_q limit -> Fermionic Dirac determinant modifies
effective interaction; maintains gauge invariance

Obtain effective interaction: holographic part is
the full Wess-Zumino-Witten term.

Exact equivalence of the D=4 gauged Wess-Zumino-Witten term and the D=5 Yang-Mills Chern-Simons term.
Phys.Rev.D73:126009,2006

Anomalies, Chern-Simons terms and chiral delocalization in extra dimensions. Phys.Rev.D73:085001,2006

Lecture notes for massless spinor and massive spinor triangle diagrams. hep-th/0601155

Dimensional deconstruction and Wess-Zumino-Witten terms.
Christopher T. Hill (Fermilab) , Cosmas K. Zachos (Argonne) .
Phys.Rev.D71:046002,2005

Anomalies and Topology of Little Higgs Theories
Christopher T. Hill , Richard J. Hill (Fermilab).
(to appear)

AdSCFT

Holographic Duals to QCD

Low energy hadron physics in holographic QCD.

Tadakatsu Sakai, Shigeki Sugimoto

Prog.Theor.Phys.113:843-882,2005

QED Chern-Simons term

D=3: Knot Theory ⟺ "Gauss' Linking Theorem"

$$L_{CS} = \epsilon_{ijk} A^i \partial^j A^k$$

Bulk Physics: Photon Mass Term

Deser, Jackiw, Templeton, Schonfeld,
Siegel; Niemi, Semenoff, Y.S. Wu

D=5: $$L_{CS} = \epsilon_{ABCDE} A^A \partial^B A^C \partial^D A^E$$

Bulk Physics: ⟺ New interactions amongst KK-modes

D=5 Yang-Mills

Topological object: “instantonic soliton”

Deser’s Theorem
Ramond and CTH

Conserved Topological Currents:

Singlet: $$J_A = \epsilon_{ABCDE}\,\mathrm{Tr}(G^{BC}G^{DE})$$

Adjoint: $$J^a_A = \epsilon_{ABCDE}\,\mathrm{Tr}(\frac{\lambda^a}{2}\{G^{BC},G^{DE}\})$$

These currents come from a “completion” of the Lagrangian

Adjoint current - 2nd Chern character:

$$c\epsilon^{ABCDE}\,\mathrm{Tr}(A_A\partial_B A_C\partial_D A_E - \frac{3i}{2}A_A A_B A_C\partial_D A_E - \frac{3}{5}A_A A_B A_C A_D A_E)$$

Singlet currents - auxiliary characters:

$$c'\epsilon_{ABCDE}V^A\,\mathrm{Tr}(G^{BC}G^{DE})$$

Topology of the D=5 pure Yang Mills theory can be directly matched to D=4

Chiral Lagrangian theory obtainable via deconstruction Bianchi ID's, etc.:

CTH, CTH & Zachos

Mathematically exact matchings:

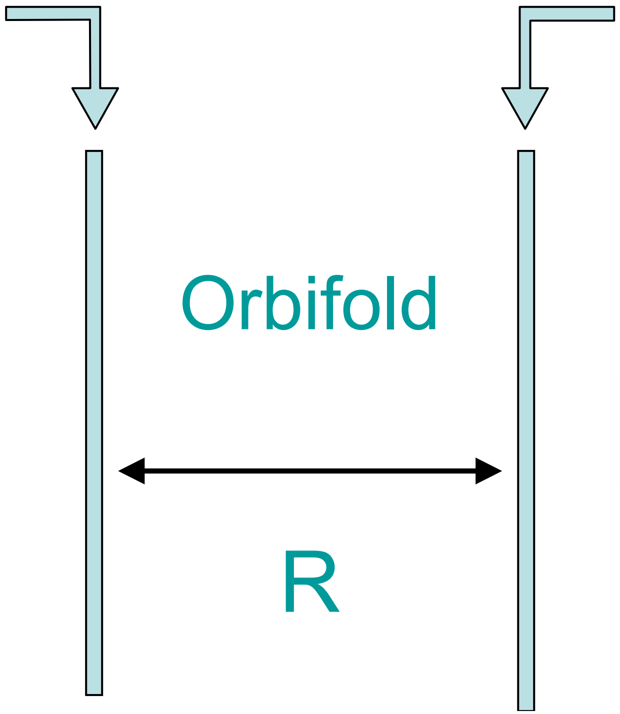
Instantonic Soliton	⟺	Skyrmion
Gauge currents	⟺	Chiral currents
Chern-Simons term + boundary term	⟺	WZW term

I. Technically natural QED in D=5:

Bulk: $D_A = \partial_A - iA_A$, $F_{AB} = i[D_A, D_B]$, $L_0 = -\frac{1}{4\tilde{e}^2} F_{AB} F^{AB}$

$$\int_I d^4x\, \overline{\psi}_L i \not{D}_L \psi_L$$

$$\int_{II} d^4x\, \overline{\psi}_R i \not{D}_R \psi_R$$



$$D_{L\mu} = \partial_\mu - iA_\mu(x_\mu, 0)$$

$$D_{R\mu} = \partial_\mu - iA_\mu(x_\mu, R)$$

$$m\overline{\psi}_L(x_\mu, 0) W \psi_R(x_\mu, R) + h.c.$$

$$W = \exp(i \int_0^R - A_5(x_\mu, x_5) dx^5)$$

Orbifold Boundary Conditions:

Horava-Witten = Magnetic Josephson Junction

Spectrum: (a) A_μ zero mode and KK tower
(b) No A_5 zero mode
(c) All A_5 modes eaten -> longitudinal dof's

Flipped Orbifold Boundary Conditions:

parity reversed Horava-Witten = Josephson Junction

Spectrum: (a) A_5 zero mode
(b) No A_{mu} zero mode
(c) All other A_5 modes eaten -> longitudinal dof's

Gauge transformation in D=5:

$$A_A(x_\mu, y) \to A_A(x_\mu, y) + \partial_A \theta(x_\mu, y)$$

$$\psi_L(x_\mu) \;\to\; \exp(i\theta(0, x_\mu))\psi_L(x_\mu)$$

$$\psi_R(x_\mu) \;\to\; \exp(i\theta(R, x_\mu))\psi_R(x_\mu)$$

$$S_{branes} \;\to\; S_{branes} + \int_I d^4x\, \overline{\psi}_L \gamma_\mu \partial^\mu \theta \psi_L(x_\mu, 0) + \int_{II} d^4x\, \overline{\psi}_R \gamma_\mu \partial^\mu \psi_R(x_\mu, R)$$

$$\to\; S_{branes} - \int_I d^4x\, \theta(x_\mu, 0)\partial_\mu J^\mu_L - \int_{II} d^4x\, \theta(x_\mu, R)\partial_\mu J^\mu_R$$

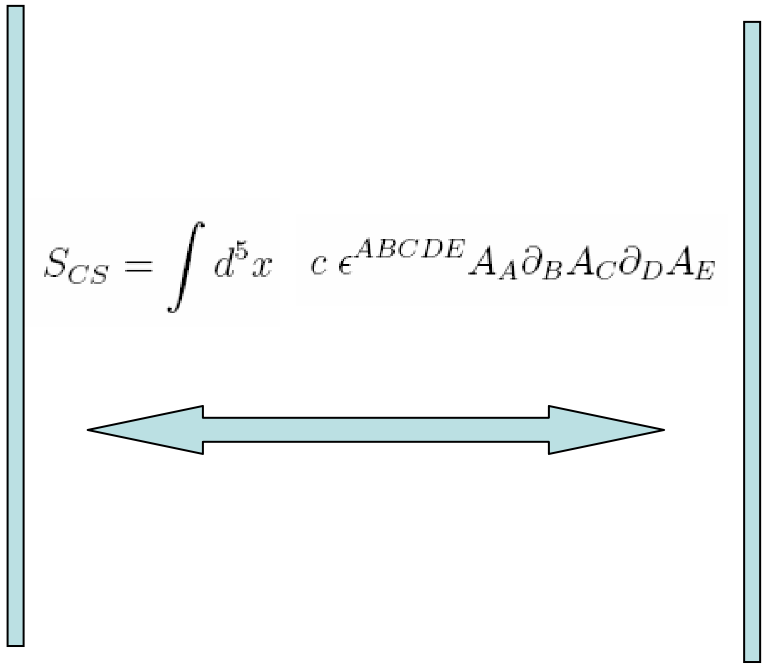
$$\partial_\mu J^\mu_L \;=\; -\frac{1}{48\pi^2} F^{\mu\nu}(0)\widetilde{F}_{\mu\nu}(0)$$

$$\partial_\mu J^\mu_R \;=\; \frac{1}{48\pi^2} F^{\mu\nu}(R)\widetilde{F}_{\mu\nu}(R)$$

↑ "Consistent" Anomalies ↑

QED in D=5 requires Chern-Simons term:

$$L_{CS} \;=\; c\,\epsilon^{ABCDE} A_A \partial_B A_C \partial_D A_E = \frac{c}{4}\epsilon^{ABCDE} A_A F_{BC} F_{DE}$$



$$S_{CS} = \int d^5x \quad c\,\epsilon^{ABCDE} A_A \partial_B A_C \partial_D A_E$$

$$S_{CS} \;\rightarrow\; S_{CS} + \frac{c}{4}\int_{II} d^4x\;\theta(R)\;\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}(R) - \frac{c}{4}\int_{I} d^4x\;\theta(0)\;\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}(0)\;.$$

Anomaly Cancellation Condition:

$$S_{branes} \rightarrow S_{branes} + \frac{1}{48\pi^2}\int_I d^4x\, \theta(x_\mu, 0) F^{\mu\nu}\widetilde{F}_{\mu\nu}(0) \;-\; \frac{1}{48\pi^2}\int_{II} d^4x\, \theta(x_\mu, R) F^{\mu\nu}\widetilde{F}_{\mu\nu}(R)$$

$$S_{CS} \rightarrow S_{CS} - \frac{c}{2}\int_I d^4x\, \theta(x_\mu, 0) F^{\mu\nu}\widetilde{F}_{\mu\nu} + \frac{c}{2}\int_{II} d^4x\, \theta(x_\mu, R) F^{\mu\nu}\widetilde{F}_{\mu\nu}$$

Consistent Anomalies: $c = \frac{1}{24\pi^2}$

C-S coefficient obtainable in any odd D from gauss linking for a generalized Dirac monopole solenoid $(dA)^{(D-1)/2}$

Summary of Anomalies: W. A. Bardeen, PR 184, 1848 (199)

Consistent Anomalies:

Consistent $L = V - A$ and $R = V + A$ Forms:

(1) Pure Massless Weyl Spinors ($p_i \cdot p_j >> m^2$):

$$\partial^\mu \overline{\psi}\gamma_\mu \psi_L = -\frac{1}{48\pi^2} F_{L\mu\nu}\tilde{F}_L^{\mu\nu}$$

$$\partial^\mu \overline{\psi}\gamma_\mu \psi_R = \frac{1}{48\pi^2} F_{R\mu\nu}\tilde{F}_R^{\mu\nu}$$

$$\partial^\mu \overline{\psi}\gamma_\mu \psi = \frac{1}{12\pi^2} F_{V\mu\nu}\tilde{F}_A^{\mu\nu}$$

$$\partial^\mu \overline{\psi}\gamma_\mu \gamma^5 \psi = \frac{1}{24\pi^2}(F_{V\mu\nu}\tilde{F}_V^{\mu\nu} + F_{A\mu\nu}\tilde{F}_A^{\mu\nu})$$

(2) Heavy Massive Weyl Spinors ($p_i \cdot p_j << m^2$):

$$\partial^\mu \overline{\psi}\gamma_\mu \psi_L + im(\overline{\psi}_L\psi_R - \overline{\psi}_R\psi_L) = -\frac{1}{48\pi^2} F_{L\mu\nu}\tilde{F}_L^{\mu\nu}$$

$$\partial^\mu \overline{\psi}_R\gamma_\mu \psi_R + im(\overline{\psi}_R\psi_L - \overline{\psi}_L\psi_R) = \frac{1}{48\pi^2} F_{R\mu\nu}\tilde{F}_R^{\mu\nu}$$

$$\partial^\mu \overline{\psi}\gamma_\mu \psi = \frac{1}{12\pi^2} F_{V\mu\nu}\tilde{F}_A^{\mu\nu}$$

$$\partial^\mu \overline{\psi}\gamma_\mu \gamma^5 \psi - 2im\overline{\psi}\gamma^5\psi = \frac{1}{24\pi^2}(F_{V\mu\nu}\tilde{F}_V^{\mu\nu} + F_{A\mu\nu}\tilde{F}_A^{\mu\nu})$$

$$\partial^\mu \overline{\psi}\gamma_\mu \psi_L = \frac{1}{48\pi^2}(F_{L\mu\nu}\tilde{F}_R^{\mu\nu} + F_{R\mu\nu}\tilde{F}_R^{\mu\nu})$$

$$\partial^\mu \overline{\psi}\gamma_\mu \psi_R = -\frac{1}{48\pi^2}(F_{L\mu\nu}\tilde{F}_R^{\mu\nu} + F_{L\mu\nu}\tilde{F}_L^{\mu\nu})$$

$$\partial^\mu \overline{\psi}\gamma_\mu \psi = \frac{1}{12\pi^2} F_{V\mu\nu}\tilde{F}_A^{\mu\nu}$$

$$\partial^\mu \overline{\psi}\gamma_\mu \gamma^5 \psi = -\frac{1}{12\pi^2}(F_{V\mu\nu}\tilde{F}_V^{\mu\nu})$$

$$im\overline{\psi}\gamma^5\psi \to -\frac{1}{48\pi^2}[F_{L\mu\nu}\tilde{F}_L^{\mu\nu} + F_{R\mu\nu}\tilde{F}_R^{\mu\nu} + F_{L\mu\nu}\tilde{F}_R^{\mu\nu}]$$

Summary of Anomalies (cont'd): see CTH [HEP-TH 0601155]

Covariant Forms:

Add a term to the lagrangian of the form $(1/6\pi^2)\epsilon_{\mu\nu\rho\sigma}A^\mu V^\nu \partial^\rho V^\sigma$. The currents are now modified to $\tilde{J} = J + \delta J$ and $\tilde{J}^5 = J^5 + \delta J^5$

$$\frac{\delta S'}{\delta V_\mu} = \delta J^\mu = -\frac{1}{3\pi^2}\epsilon_{\mu\nu\rho\sigma}A^\nu \partial^\rho V^\sigma + \frac{1}{6\pi^2}\epsilon_{\mu\nu\rho\sigma}V^\nu \partial^\rho A^\sigma$$

$$\frac{\delta S'}{\delta A_\mu} = \delta J^{5\mu} = \frac{1}{6\pi^2}\epsilon_{\mu\nu\rho\sigma}V^\nu \partial^\rho V^\sigma$$

(1) Pure Massless Weyl Spinors ($p_i \cdot p_j >> m^2$):

$$\partial^\mu \tilde{J}_\mu = 0$$

$$\partial^\mu \tilde{J}^5_\mu = \frac{1}{8\pi^2}(F_{V\mu\nu}\tilde{F}^{\mu\nu}_V + \frac{1}{3}F_{A\mu\nu}\tilde{F}^{\mu\nu}_A)$$

(2) Heavy Massive Weyl Spinors ($p_i \cdot p_j << m^2$):

$$\partial^\mu \tilde{J}_\mu = 0$$

$$\partial^\mu \tilde{J}^5_\mu - 2im\bar{\psi}\gamma^5\psi = \frac{1}{8\pi^2}(F_{V\mu\nu}\tilde{F}^{\mu\nu}_V + \frac{1}{3}F_{A\mu\nu}\tilde{F}^{\mu\nu}_A)$$

Dirac determinant =
(-1) X Bardeen's counterterm

$$\partial^\mu \tilde{J}_\mu = 0$$

$$\partial^\mu \tilde{J}^5_\mu = 0$$

Summary: Technically natural QED in D=5

Bulk: $D_A = \partial_A - iA_A \,, \qquad F_{AB} = i[D_A, D_B] \,, \qquad L_0 = -\frac{1}{4\tilde{e}^2} F_{AB} F^{AB}$

$\int_I d^4x \, \overline{\psi}_L i \not{D}_L \psi_L$

$\int_{II} d^4x \, \overline{\psi}_R i \not{D}_R \psi_R$

Orbifold

$D_{L\mu} = \partial_\mu - iA_\mu(x_\mu, 0)$

$D_{R\mu} = \partial_\mu - iA_\mu(x_\mu, R)$

$$S_{CS} = \int d^5x \, \frac{1}{24\pi^2} \quad \epsilon^{ABCDE} A_A \partial_B A_C \partial_D A_E$$

$$m\overline{\psi}_L(x_\mu, 0) W \psi_R(x_\mu, R) + h.c.$$

$$W = \exp(i \int_0^R A_5(x_\mu, x_5) dx^5)$$

Pass to $A_5 = 0$ Gauge

$$L_{CS} = \frac{3c}{4}\epsilon^{\mu\nu\rho\sigma}A_5F_{\mu\nu}F_{\rho\sigma} + c\,\epsilon^{\mu\nu\rho\sigma}(\partial_5A_\mu)A_\nu F_{\rho\sigma}\ .$$

Consider a Wilson line that emanates from, *e.g.*, brane I, $x^5 = 0$, toward an arbitrary point in the bulk, $x^5 = y$:

$$U(y) = \exp\left(i\int_0^y dx^5 A_5(x^5)\right) \qquad\qquad \partial_y U = iA_5(y)U$$

Using the Wilson line as a gauge transformation, we have:

$$A_A \to A_A + iU^\dagger\partial_A U \qquad A_5 \to A_5(y) + iU^\dagger\partial_y U = A_5(y) - \partial_y\int_0^y dx^5 A_5(x^5) = 0$$

$$B_\mu = A_\mu - \partial_\mu\int_0^y A_5 dx^5; \qquad F_{B\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu.$$

$$S_{CS} = c\,\epsilon^{\mu\nu\rho\sigma}\int d^4x\int_0^R dy\,(\partial_y B_\mu)B_\nu F_{B\rho\sigma}.$$

B_μ field will lead to gauge invariant

("Stueckelberg") combinations for each massive KK-mode in the compactified theory

The orbifold mode expansion

$$A^0_\mu(x,y) = \sqrt{\frac{1}{R}}\tilde{e}A^0_\mu(x)$$

$$A_\mu(x,y) = \sum_{n=1}^{\infty}(-1)^n\sqrt{\frac{2}{R}}\tilde{e}\cos(n\pi y/R)A^n_\mu(x)$$

$$A_5(x,y) = \sum_{n=1}^{\infty}(-1)^{n+1}\sqrt{\frac{2}{R}}\tilde{e}\sin(n\pi y/R)A^n_5(x)$$

$$S_1 = -\frac{1}{4\tilde{e}^2}\int_0^R dy\int d^4x\, F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\sum_n\int d^4x\, F^n_{\mu\nu}F^{n\mu\nu}$$

$$S_2 = \frac{1}{2\tilde{e}^2}\int_0^R dy\int d^4x\, F_{\mu 5}F^{\mu 5} = \frac{1}{2}\sum_{n=1} M_n^2\int d^4x\, B^n_\mu B^{n\mu}$$

$$M_n = n\pi/R\,; \qquad B^n_\mu = A^n_\mu + \frac{1}{M_n}\partial_\mu A^n_5\,; \qquad F^n_{\mu\nu} \equiv \partial_\mu B^n_\nu - \partial_\nu B^n_\mu .$$

"Stueckelberg fields."

$$e = \tilde{e}/\sqrt{R} \equiv e_0 \qquad\qquad e' = \sqrt{2}\tilde{e}/\sqrt{R} = \sqrt{2}e \equiv e_n \quad (n \neq 0)$$

$$S_{CS} = \frac{1}{24\pi^2}\int_0^R dy \int d^4x\, \epsilon^{\mu\nu\rho\sigma}(\partial_y B_\mu) B_\nu F_{\rho\sigma}$$

$$\equiv \frac{1}{12\pi^2}\sum_{nmk}\int d^4x\, (e_n e_m e_k) c_{nmk} (B^n_\mu B^m_\nu \widetilde{F}^{k\mu\nu})$$

$$c_{nmk} = (-1)^{(k+n+m)}\int_0^1 dz\, \partial_z [\cos(n\pi z)] \cos(m\pi z)\cos(k\pi z)$$

$$= \frac{n^2(k^2+m^2-n^2)\left[(-1)^{(k+n+m)}-1\right]}{(n+m+k)(n+m-k)(n-k-m)(n-m+k)}$$

$$\begin{aligned} c_{nm0} &= c_{n0m} = -\frac{n^2}{n^2-m^2}\left[(-1)^{n+m}-1\right] \\ c_{0nm} &= c_{000} = 0 \\ c_{n00} &= \left[1-(-1)^n\right]. \end{aligned}$$

D=4 Effective Theory

$$S_{full} = \int d^4x \, [\, \overline{\psi}(i\partial\!\!\!/ + V\!\!\!/ + A\!\!\!/ \gamma^5 - m)\psi + \frac{1}{12\pi^2} \sum_{nmk} c_{nmk} B^n_\mu B^m_\nu \widetilde{F}^{k\mu\nu}$$

$$- \frac{1}{4e^2} F^0_{\mu\nu} F^{0\mu\nu} - \frac{1}{4e'^2} \sum_{n\geq 1} F^n_{\mu\nu} F^{n\mu\nu} + \sum_{n\geq 1} \frac{1}{2e_n^2} M_n^2 B^n_\mu B^{n\mu}]$$

$$V_\mu = \sum_{n\ even} B^n_\mu, \qquad \mathcal{A}_\mu = \sum_{n\ odd} B^n_\mu$$

if we truncate the theory on the zero mode B^0 and first KK-mode, B^1,

$$\frac{1}{12\pi^2} c_{100} B^1_\mu B^0_\nu \widetilde{F}^{0\mu\nu} = \frac{1}{6\pi^2} \epsilon^{\mu\nu\rho\sigma} A_\mu V_\nu \partial_\rho V_\sigma$$

D=4 Effective Theory Current Algebra

$$\tilde{J}_\mu^n = \frac{\delta S}{\delta B^{n\mu}} = \bar{\psi}\gamma_\mu\psi|_{n\ even} + \bar{\psi}\gamma_\mu\gamma^5\psi|_{n\ odd} + J_\mu^{n\ CS}$$

$$J_{mu}^{n\ CS} = \frac{\epsilon_{\mu\nu\rho\sigma}}{12\pi^2}\sum_{mk}\left[(c_{nmk} - c_{mnk} + c_{kmn} - c_{mkn})B^{m\nu}\partial^\rho B^{k\sigma}\right]$$

$$\partial^\mu J_\mu^n = \frac{1}{48\pi^2}\sum_{mk}\left(1-(-1)^{n+m+k}\right)F_{\mu\nu}^m\widetilde{F}^{k\mu\nu}$$

$$\partial^\mu J_\mu^{n\ CS} = \frac{1}{48\pi^2}\sum_{m,k}(c_{nmk} - c_{mnk} + c_{nkm} - c_{knm})F_{\mu\nu}^m\widetilde{F}^{k\mu\nu}$$

KK-mode anomalies:

$$\partial^\mu \tilde{J}_\mu^n = \frac{1}{24\pi^2}\sum_{m,k} d_{nmk}F_{\mu\nu}^m\widetilde{F}^{k\mu\nu}$$

$$\begin{aligned}
d_{nmk} &= \frac{1}{2}\left[(1-(-1)^{n+m+k}) + (c_{nmk} - c_{mnk} + c_{nkm} - c_{knm})\right] \\
&= \frac{3}{2}[(-1)^{n+m+k} - 1]\frac{n^2(k^2+m^2-n^2)}{(k+m-n)(k+m+n)(k+n-m)(k-m-n)} \\
&= \frac{3}{2}c_{nmk}
\end{aligned}$$

Anomaly coefficient

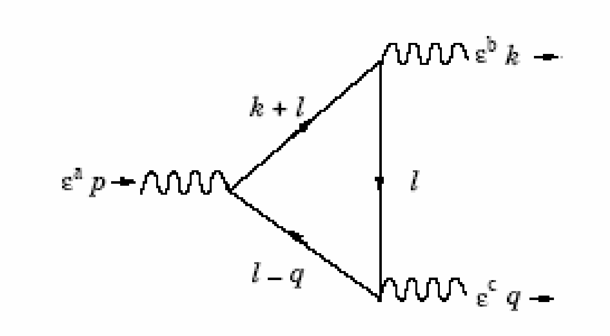
$$c_{0mk} = 0$$

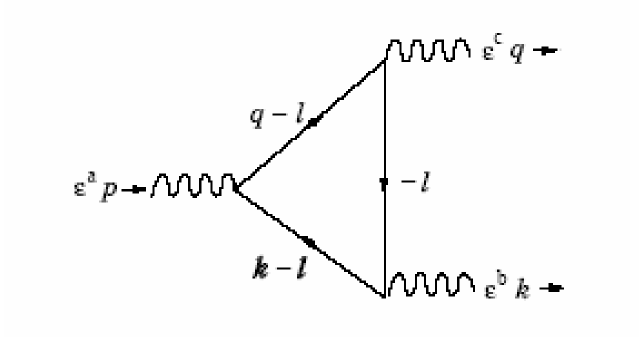
$$\partial^\mu J_\mu^5 = \frac{1}{16\pi^2}\left(c_{100}F_{\gamma\ \mu\nu}\widetilde{F}_\gamma^{k\mu\nu} + c_{111}F_{B\ \mu\nu}\widetilde{F}_B^{k\mu\nu}\right) = \frac{1}{8\pi^2}F_{\gamma\ \mu\nu}\widetilde{F}_\gamma^{\mu\nu} + \frac{1}{24\pi^2}F_{B\ \mu\nu}\widetilde{F}_B^{\mu\nu}$$

D=4 Effective Theory in large m limit

$$S_{tree} = \int d^4x \, [\frac{1}{12\pi^2} \sum_{nmk} \bar{c}_{nmk} B^n_\mu B^m_\nu \widetilde{F}^{k\mu\nu} - \frac{1}{4e^2} F^0_{\mu\nu} F^{0\mu\nu} - \frac{1}{4e'^2} \sum_{n\geq 1} F^n_{\mu\nu} F^{n\mu\nu} + \sum_{n=0} \frac{1}{2e_n^2} M_n^2 B^n_\mu B^{n\mu}]$$

Integrate out the Fermions: Dirac Determinant effective interactions





Dirac Determinant effective interaction:

(integrate out the fermions)

$$\mathcal{O}_3 = -\frac{1}{12\pi^2}\epsilon^{\mu\nu\rho\sigma}\sum_{nmk}(e_n e_m e_k)a_{nmk}B_\mu^n B_\nu^m \partial_\rho B_\sigma^k$$

$$a_{nmk} = \frac{1}{2}(1-(-1)^{n+m+k})(-1)^{m+k}$$

This operator is equivalent to $(-1/6\pi^2)\epsilon_{\mu\nu\rho\sigma}A^\mu V^\nu \partial^\rho V^\sigma$

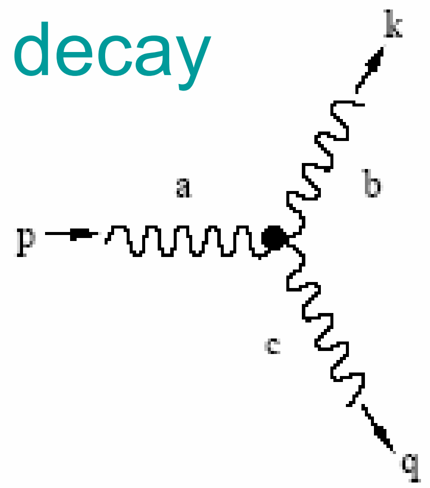
Dirac Determinant effective interaction is equivalent to (-1)x Bardeen’s counterterm: consistent -> covariant

Full Effective Theory

$$\bar{c}_{nmk} = c_{nmk} - a_{nmk} \qquad \text{(massive spinors)}$$

$$\bar{c}_{nmk} = \left[(-1)^{(k+n+m)} - 1\right]\left(\frac{n^2(k^2+m^2-n^2)}{(n+m+k)(n+m-k)(n-k-m)(n-m+k)} + \frac{1}{2}(-1)^{m+k}\right)$$

Two Examples: Compute KK-Mode decay



Feynman rule for a vertex $B^a \to B^b + B^c$:

$$T_{CS} = -\frac{ee'^2}{12\pi^2}\left[\left(-\bar{c}_{abc} + \bar{c}_{bac} + \bar{c}_{bca} - \bar{c}_{cba}\right)[B] + \left(\bar{c}_{acb} - \bar{c}_{cab} + \bar{c}_{bca} - \bar{c}_{cba}\right)[A]\right]$$

$$[A] = \epsilon^{\mu\nu\rho\sigma}\epsilon^a_\mu \epsilon^b_\nu \epsilon^\gamma_\rho k^\sigma \qquad [B] = \epsilon^{\mu\nu\rho\sigma}\epsilon^a_\mu \epsilon^b_\nu \epsilon^\gamma_\rho q^\sigma$$

A. Decay of KK-mode to KK-mode plus γ

c = photon

$$T_{CS} = -\frac{ee'^2}{12\pi^2}\left[\left(-\bar{c}_{ab0} + \bar{c}_{ba0} + \bar{c}_{b0a} - \bar{c}_{0ba}\right)[B] + \left(\bar{c}_{a0b} - \bar{c}_{0ab} + \bar{c}_{b0a} - \bar{c}_{0ba}\right)[A]\right]$$

$$= \frac{ee'^2}{2\pi^2}\left(\frac{M_b^2}{M_a^2 - M_b^2} - \frac{1}{2}((-1)^b - 1)\right)[B].$$

Gauge invariant in photon

$$\left.\begin{aligned}\Gamma_{1-\to 1+\gamma} &= \frac{2\alpha^3}{3\pi^3}\left(\frac{M_a^3}{M_b^2}\right) \\ \Gamma_{1+\to 1-\gamma} &= \frac{2\alpha^3}{3\pi^3}M_b\end{aligned}\right\} \qquad M_a^2 >> M_b^2$$

$$\Gamma_{1\pm\to 1\mp\gamma} = \frac{2\alpha^3}{3\pi^3}\Delta M \qquad \Delta M = M_a - M_b << M_a$$

$$\Gamma_{1-\to 1+\gamma} = \frac{\alpha(0)\alpha'^2(M_a)}{6\pi^3}\left(\frac{M_a}{R^2 M_b^2}\right) \qquad \Gamma_{1+\to 1-\gamma} = \frac{\alpha(0)\alpha'^2(M_a)}{6\pi^3}\left(\frac{M_b}{R^2 M_a^2}\right)$$

B. Zero Mode + Zero Mode → KK-Mode Vanishes

$$T_{CS} = -\frac{ee'^2}{12\pi^2}\left[(-\bar{c}_{a00} + \bar{c}_{0a0} + \bar{c}_{00a} - \bar{c}_{00a})[B] + (\bar{c}_{a00} - \bar{c}_{0a0} + \bar{c}_{00a} - \bar{c}_{00a})[A]\right]$$

= 0 gauge invariance (Landau-Yang theorem)

Summary of D=5 QED :

The KK-mode parity is locked to the parity of spacetime:

T-parity is no longer an independent symmetry

Lightest KK-modes are not stable

Destabilized dark matter candidates

Topological interactions are a general feature of extra dimensions, containing both holographic (boundary only) and bulk effects.

Yang-Mills gauge theory of quark flavor in D=5:

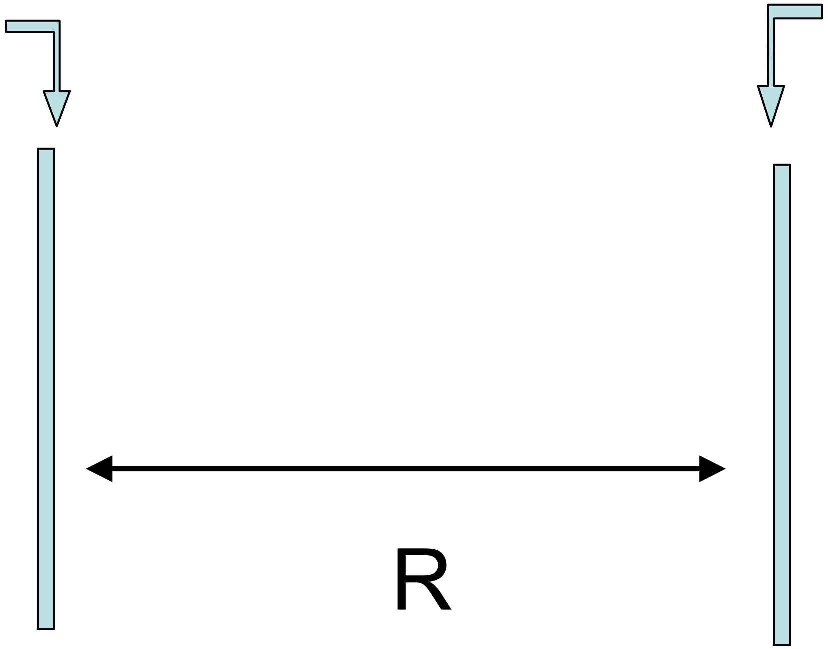
Bulk:

$$D_A = \partial_A - iB_A \qquad B_A = B_A^a T^a,$$

$$G_{AB} = i[D_A, D_B] = \partial_A B_B - \partial_B B_A - i[B_A, B_B].$$

$$\int_I d^4x\, \overline{\psi}_L i \not{D}_L \psi_L \qquad \int_{II} d^4x\, \overline{\psi}_R i \not{D}_R \psi_R$$

$$D_{L\mu} = \partial_\mu - iB_\mu(x_\mu, 0) \qquad D_{R\mu} = \partial_\mu - iB_\mu(x_\mu, R)$$

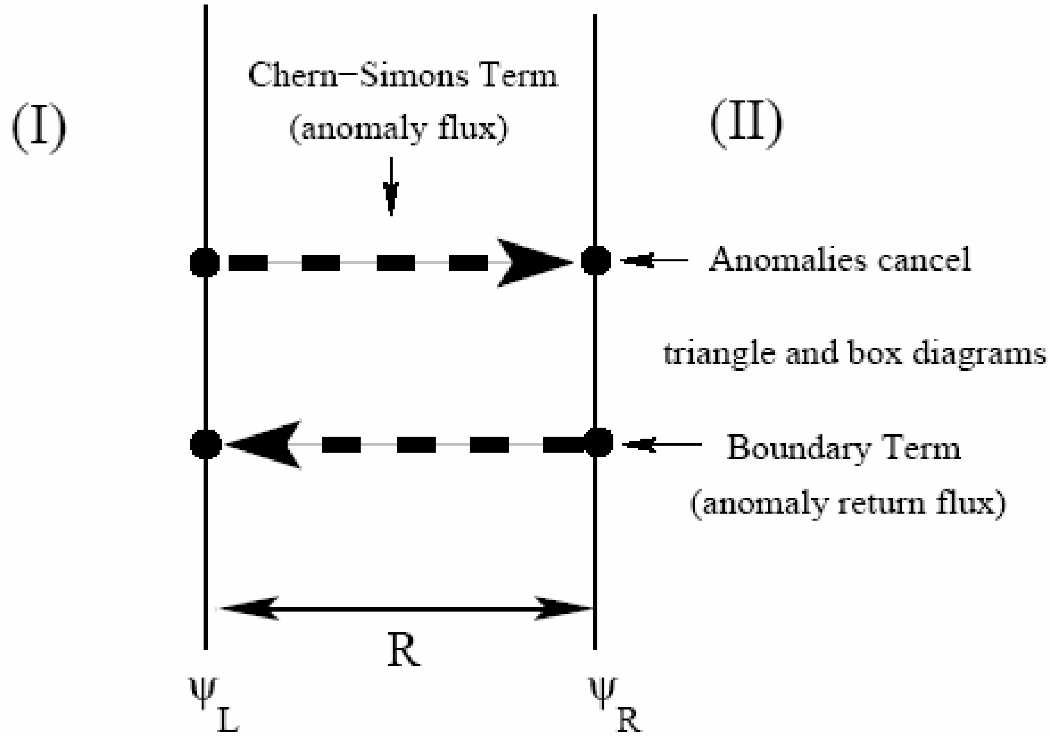


$$m\overline{\psi}_L(x_\mu, 0) W \psi_R(x_\mu, R) + h.c.,$$

$$W = P\exp\left(-i\int_0^R dx^5 B_5\right) = \exp(2i\tilde{\pi}/f_\pi)$$

Generic compactification; include B_5 zero-mode

Derivation of the full Wess-Zumino-Witten term directly from the Yang-Mills theory



Theory Requires Chern-Simons Term:

$$\mathcal{L}_{CS} = c\epsilon^{ABCDE}\,\mathrm{Tr}\Big(A_A\partial_B A_C\partial_D A_E - \frac{3i}{2}A_A A_B A_C\partial_D A_E - \frac{3}{5}A_A A_B A_C A_D A_E\Big)$$

$$= \frac{c}{4}\epsilon^{ABCDE}\,\mathrm{Tr}\big(A_A G_{BC} G_{DE} + iA_A A_B A_C G_{DE} - \frac{2}{5}A_A A_B A_C A_D A_E\big)\ .$$

Gauge transformation: $A_A \to V(A_A + i\partial_A)V^\dagger \qquad \text{where:} \qquad V = \exp(i\theta^a T^a)$

$$\delta S_{CS} = c\epsilon^{\mu\nu\rho\sigma}\theta^a\,\mathrm{Tr}[T^a(\partial_\mu A_\nu\partial_\rho A_\sigma - \frac{i}{2}(\partial_\mu A_\nu A_\rho A_\sigma - A_\mu\partial_\nu A_\rho A_\sigma + A_\mu A_\nu\partial_\rho A_\sigma)]\Big|_0^R$$

Consistent Anomaly; To cancel against fermion anomalies: $c = \dfrac{N_c}{24\pi^2}\ .$

Transforming to Axial Gauge, $B_5 \to 0$

$$V(x^\mu, y) = P\exp\left(-i\int_0^y dx^5\; B_5^0(x^\mu, x^5)\right)$$

$$\psi'_L = \psi_L, \qquad \psi'_R = V(R)\psi_R$$

$$\tilde{B}_\mu(x^\mu, y) = V(B_\mu + i\partial_\mu)V^\dagger \qquad \tilde{B}_5(x^\mu, y) = V(B_5 + i\partial_y)V^\dagger = 0$$

$$\overline{\psi}(i\partial\!\!\!/ + B\!\!\!/\,)\psi = \overline{\psi}'(i\partial\!\!\!/ + \tilde{B}\!\!\!/\,)\psi' \qquad \overline{\psi}_L W \psi_R = \overline{\psi}'_L \psi'_R \qquad W \to V(0)WV^\dagger(R) = 1.$$

B is now a tower of vector mesons comingled
with the spin-0 mesons; must extract the physical mesons:

Redefinition: $$\tilde{B}_\mu(x^\mu, y) = \tilde{U}(y)(A_\mu(x^\mu, y) + i\partial_\mu)\tilde{U}^\dagger(y) \qquad \tilde{U} = \exp(2i\tilde{\pi}y/f_\pi)$$

Compactification Decomposition of Chern-Simons Term:

$$S_{CS} = c\int d^5x\, \epsilon^{ABCDE}\,\mathrm{Tr}\Big(B_A\partial_B B_C\partial_D B_E - \frac{3i}{2}B_A B_B B_C \partial_D B_E - \frac{3}{5}B_A B_B B_C B_D B_E)\Big)$$

$$= \frac{c}{2}\,\mathrm{Tr}\int d^4x \int_0^R dy\,\Big[(\partial_5 B_\mu)K^\mu + \frac{3}{2}\epsilon^{\mu\nu\rho\sigma}\,\mathrm{Tr}(B_5 G_{\mu\nu}G_{\rho\sigma})\Big],$$

$$K^\mu \equiv \epsilon^{\mu\nu\rho\sigma}\left(iB_\nu B_\rho B_\sigma + G_{\nu\rho}B_\sigma + B_\nu G_{\rho\sigma}\right).$$

CTH and Zachos

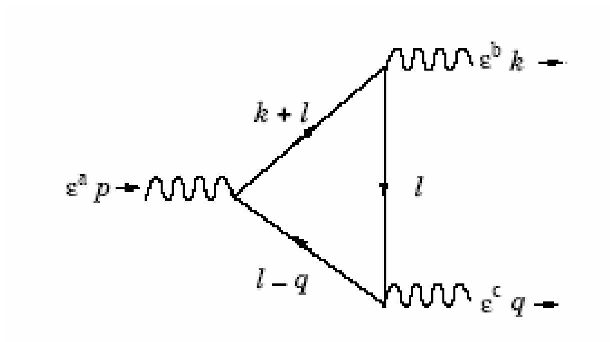
Axial Gauge:

$$S_{CS} = \frac{c}{2}\epsilon^{\mu\nu\rho\sigma}\int d^4x \int_0^R dy\ \mathrm{Tr}\left[\partial_y \tilde{B}_\mu \left(i\tilde{B}_\nu \tilde{B}_\rho \tilde{B}_\sigma + G(\tilde{B})_{\nu\rho}\tilde{B}_\sigma + \tilde{B}_\nu G(\tilde{B})_{\rho\sigma}\right)\right]$$

$$= \frac{c}{2}\,\mathrm{Tr}\int d^4x \int_0^R dy\,(\partial_y \tilde{B})(2d\tilde{B}\tilde{B} + 2\tilde{B}d\tilde{B} - 3i\tilde{B}^3)$$

Form notation: $G(\tilde{B}) = 2d\tilde{B} - 2i\tilde{B}^2.$

Integrate out the Fermions:
Dirac Determinant effective interactions



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Dirac Determinant effective interaction is (-1) x Bardeen Counterterm as in QED:

$$S_{boundary} = -\frac{c}{2}\int \mathrm{Tr}\,\Big(\frac{1}{2}(G_R\tilde{B}_R + \tilde{B}_R G_R)\tilde{B}_L - \frac{1}{2}(G_L\tilde{B}_L + \tilde{B}_L G_L)\tilde{B}_R + i\tilde{B}_R^3\tilde{B}_L - i\tilde{B}_L^3\tilde{B}_R - \frac{i}{2}(\tilde{B}_R\tilde{B}_L)^2\Big)$$

"Boundary term"

Notation:

$$\tilde{B}_\mu = \tilde{A}_\mu - i\alpha_\mu \qquad \tilde{A}_\mu = \tilde{U} A_\mu \tilde{U}^\dagger$$

$$\tilde{A}_{L\mu} = A_{L\mu} = A_\mu(x^\mu, 0) \qquad \tilde{A}_{R\mu} = U A_\mu(x^\mu, R) U^\dagger$$

$$\alpha_\mu = -\tilde{U}\partial_\mu \tilde{U}^\dagger \qquad \beta_\mu = U^\dagger \partial_\mu U = U^\dagger \alpha_\mu U \qquad \tilde{U} = \exp(2i\tilde{\pi}y/f_\pi)$$

$$S_{CS} \Rightarrow \frac{c}{2}\,\mathrm{Tr}\int d^4x\, dy\, [-i(\partial_y \alpha) + (\partial_y \tilde{A})]$$
$$\times(2d\tilde{A}\tilde{A} - 2i\alpha^2\tilde{A} - 2id\tilde{A}\alpha - 4\alpha^3 + 2\tilde{A}d\tilde{A} - 2i\tilde{A}\alpha^2 - 2i\alpha d\tilde{A}$$
$$-3i\tilde{A}^3 - 3\alpha\tilde{A}^2 - 3\tilde{A}\alpha\tilde{A} - 3\tilde{A}^2\alpha + 3i\alpha^2\tilde{A} + 3i\alpha\tilde{A}\alpha + 3i\tilde{A}\alpha^2 + 3\alpha^3)$$

$$S_{boundary} \Rightarrow \frac{c}{2}\int \mathrm{Tr}\,[(dA_L A_L + A_L dA_L)UA_R U^\dagger - (dA_R A_R + A_R dA_R)U^\dagger A_L U$$
$$-i(dA_L A_L + A_L dA_L)\alpha - A_L^3\alpha - A_L\alpha^3 + iA_R^3 U^\dagger A_L U - iA_L^3 U A_R U^\dagger$$
$$-i(dA_R dU^\dagger A_L U - dA_L dU A_R U^\dagger) - (A_R U^\dagger A_L U A_R \beta + A_L U A_R U^\dagger A_L \alpha)$$
$$+\frac{i}{2}A_L\alpha A_L\alpha + \frac{i}{2}U A_R U^\dagger A_L U A_R U^\dagger A_L - i(A_L U A_R U^\dagger \alpha^2 - A_R U^\dagger A_L U \beta^2)]$$

We first isolate the term:

$$S_{CS0} \;=\; i\frac{c}{2}\,\mathrm{Tr}\int(\partial_y\alpha)\alpha^3$$

$$\partial_y\alpha = \partial_y\tilde{U}d\tilde{U}^\dagger = \frac{2i}{Rf_\pi}\tilde{U}d\tilde{\pi}\tilde{U}^\dagger \qquad\qquad \alpha \approx \frac{2iy}{f_\pi}d\tilde{\pi} - \frac{2y^2}{f_\pi^2}[\tilde{\pi}, d\tilde{\pi}] + ...$$

$$\begin{aligned} S_{CS0} \;&=\; -\frac{2N_c}{3\pi^2 f_\pi^5}\int d^4x\; dy y^4\,\mathrm{Tr}(\tilde{\pi}d\tilde{\pi}d\tilde{\pi}d\tilde{\pi}d\tilde{\pi}) + ... \\ &=\; -\frac{2N_c}{15\pi^2 f_\pi^5}\int d^4x\,\mathrm{Tr}(\tilde{\pi}d\tilde{\pi}d\tilde{\pi}d\tilde{\pi}d\tilde{\pi}) + ... \end{aligned}$$

$$S_{CS\ \alpha^3\tilde{A}} = -i\frac{c}{2}\,\mathrm{Tr}\int(\partial_y\alpha)(-2id\tilde{A}\alpha - 2i\alpha d\tilde{A} - 2i\alpha^2\tilde{A} - 2i\tilde{A}\alpha^2 + 3i(\alpha^2\tilde{A} + \alpha\tilde{A}\alpha + \tilde{A}\alpha^2))$$
$$-\frac{c}{2}\,\mathrm{Tr}\int(\partial_y\tilde{A})[\alpha^3] \qquad (42)$$

Note that, upon integrating in $D = 4$ by parts:

$$\mathrm{Tr}\int(\partial_y\alpha)(d\tilde{A}\alpha + \alpha d\tilde{A}) = 2\,\mathrm{Tr}\int(\partial_y\alpha)(\alpha\tilde{A}\alpha)$$

Thus, we can immediately write:

$$S_{CS\ \alpha^3\tilde{A}} = -i\frac{c}{2}\,\mathrm{Tr}\int(\partial_y\alpha)(i\alpha^2\tilde{A} + i\tilde{A}\alpha^2 - i\alpha\tilde{A}\alpha) - \frac{c}{2}\,\mathrm{Tr}\int d^4x dy(\partial_y\tilde{A})[\alpha^3]$$
$$= \frac{c}{2}\,\mathrm{Tr}\int d^4x\int_0^1 dy\ \partial_y(\alpha^3\tilde{A})$$

If we now explicitly perform this integral we obtain:

$$S_{CS\ \alpha^3\tilde{A}} = -\frac{c}{2}\,\mathrm{Tr}(A_R\beta^3)$$

where use has been made $\mathrm{Tr}(\alpha^3\tilde{A}_R) = \mathrm{Tr}(\alpha^3 U A_R U^\dagger) = \mathrm{Tr}(U^\dagger\alpha^3 U A_R) = \mathrm{Tr}(\beta^3 A_R) = -\mathrm{Tr}(A_R\beta^3)$. We see the operational parity asymmetry of our gauge tranformation leads to the absence of a corresponding parity conjugate term, $-\mathrm{Tr}(A_L\alpha^3)$. As mentioned above, this term will come from the boundary term, and the overall final result will be parity symmetric.

Obtain the full Wess-Zumino-Witten Term

$$\tilde{S} = S_{CS} + S_{boundary} \quad = S_{WZW} + S_{bulk}$$

$$\begin{aligned}
S_{WZW} \;=\; & S_{CS0} + \frac{N_c}{48\pi^2}\,\mathrm{Tr}\int d^4x[-(A_L\alpha^3 + A_R\beta^3) - (A_L^3\alpha + A_R^3\beta) \\
& -i((dA_LA_L + A_LdA_L)\alpha + dA_RA_R + A_RdA_R)\beta) + \frac{i}{2}[(A_L\alpha)^2 - (A_R\beta)^2] \\
& -i(A_L^3UA_RU^\dagger - A_R^3U^\dagger A_LU) \\
& +(dA_LA_L + A_LdA_L)UA_RU^\dagger - (dA_RA_R + A_RdA_R)U^\dagger A_LU \\
& -i(dA_RdU^\dagger A_LU - dA_LdUA_RU^\dagger) - (A_LUA_RU^\dagger A_L\alpha + A_RU^\dagger A_LUA_R\beta) \\
& +\frac{i}{2}UA_RU^\dagger A_LUA_RU^\dagger A_L - i(A_LUA_RU^\dagger\alpha^2 - A_RU^\dagger A_LU\beta^2)]
\end{aligned}$$

$$\tilde{S}_{CS0} \;=\; -\frac{2N_c}{15\pi^2 f_\pi^5}\int d^4x\,\mathrm{Tr}(\tilde{\pi}d\tilde{\pi}d\tilde{\pi}d\tilde{\pi}d\tilde{\pi}) + \ldots$$

in complete agreement with Kaymakcalan, Rajeev and Schechter

Effective brane (holographic) interaction

Normal Derivation of WZW term:

- promote full theory of mesons to D=5.

- In D=5, a certain manifestly chirally invariant and topologically interesting Chern-Simons term occurs, which is included into the theory.

- Compactify the fifth dimension with the Chern-Simons term, back into to D=4, resulting in the Wess-Zumino term.

- Perform gauge transformations upon the resulting object, and infer how to ``integrate in'' the gauge fields by brute force and some guess work.

$$
\begin{aligned}
S_{bulk} &= -i\frac{c}{2}\,\mathrm{Tr}\int(\partial_y\alpha)(\tilde{U}(3dAA+3AdA-4iA^3)\tilde{U}^\dagger) \\
&\quad +\frac{c}{2}\,\mathrm{Tr}\int(\partial_y\tilde{A})[\tilde{U}(2dAA+2AdA-3iA^3)\tilde{U}^\dagger]
\end{aligned}
$$

$$
\partial_y\alpha = \frac{2i}{f_\pi}\tilde{U}(d\tilde{\pi})\tilde{U}^\dagger \qquad\qquad \partial_y\tilde{A} = \partial_y\tilde{U}A\tilde{U}^\dagger = \frac{2i}{f_\pi}\tilde{U}([\tilde{\pi},A])\tilde{U}^\dagger
$$

$$
S_{bulk} = -\frac{3c}{2f_\pi}\int d^4x\int_0^1 dy\,\mathrm{Tr}(\tilde{\pi}GG)+\frac{c}{2}\int d^4x\int_0^1 dy\,\mathrm{Tr}(\partial_yA)(2dAA+2AdA-3iA^3)) \quad (6
$$

Effective bulk interaction

Suppose we don't integrate out the quarks?

Parity symmetric redefinition field: $\tilde{U}(y) = \exp\left(\frac{2i\tilde{\pi}(y-1/2)}{f_\pi}\right)$

$$\tilde{B}_L = \xi A_L \xi^\dagger - j_L \qquad \tilde{B}_R = \xi^\dagger A_L \xi - j_R$$

chiral currents $j_L = i\xi d\xi^\dagger \qquad j_R = -i\xi^\dagger d\xi$

$$\begin{aligned} S &= S_{CS0} + S'_{WZW} + S_{bulk} \\ &+ \int_I d^4x\, \overline{\psi}_L (i\partial\!\!\!/ + \xi A\!\!\!/_L \xi^\dagger - j_L)\psi_L + \int_{II} d^4x\, \overline{\psi}_R (i\partial\!\!\!/ + \xi^\dagger A\!\!\!/_R \xi - j_R)\psi_R \end{aligned}$$

$$\begin{aligned} S'_{WZW} &= -\frac{c}{2}\mathrm{Tr}(A_R j_R^3 + A_L j_L^3) - \frac{c}{2}\mathrm{Tr}(A_R^3 j_R + A_L^3 j_L) - i\frac{c}{4}\mathrm{Tr}(A_R j_R A_R j_R - A_L j_L A_L j_L) \\ &-i\frac{c}{2}\mathrm{Tr}[(dA_R A_R + A_R dA_R) j_R + (dA_L A_L + A_L dA_L) j_L] \qquad (67 \end{aligned}$$

Effective theory with unintegrated massless fermions

Summary:

(1)Wess-Zumino-Witten term
derived from D=5 Chern-Simons Term
+ Dirac Determinant (= -1 x Bardeen c.t.)

(2) Exact matching of D=5 Y-M to D=4 Chiral L

(3) D=5 C-S term yields new bulk interactions

(4) Will be present in most models of e.d.'s

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Some Envisioned applications:

(1) Little Higgs Theories. (CTH & Richard Hill to appear)

(2) RS Models

(3) A WZW Term for the Goldstone-Wilczek Current

(4) Skyrme/instanton baryogenesis/b+L violation
in extra dimensional theories

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