Conventions for spinor helicity formalism

Elvang pre-SUSY summer school lectures August 2015

The conventions of these notes follow those in Srednicki's QFT book.

1 Metric and γ-matrix conventions

We use a "mostly-plus" metric, $\eta_{\mu\nu} = \mathrm{diag}(-1,+1,+1,+1)$ and define

$$(\sigma^\mu)_{a\dot{b}} = (1,\sigma^i)_{a\dot{b}}\,, \qquad (\bar{\sigma}^\mu)^{\dot{a}b} = (1,-\sigma^i)^{\dot{a}b} \tag{1.1}$$

with Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.2}$$

Two-index spinor indices are raised/lowered using

$$\varepsilon^{ab} \;=\; \varepsilon^{\dot{a}\dot{b}} \;=\; \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \;=\; -\,\varepsilon_{ab} \;=\; -\,\varepsilon_{\dot{a}\dot{b}}\,, \tag{1.3}$$

which obey $\varepsilon_{ab}\varepsilon^{bc} = \delta_a{}^c$.

We list the following properties

$$\begin{aligned}
(\bar{\sigma}^\mu)^{\dot{a}a} &= \varepsilon^{ab}\varepsilon^{\dot{a}\dot{b}}(\sigma^\mu)_{a\dot{b}}\,, & (1.4)\\
(\sigma^\mu)_{a\dot{a}}(\sigma_\mu)_{b\dot{b}} &= -2\varepsilon_{ab}\varepsilon_{\dot{a}\dot{b}}\,, & (1.5)\\
\left(\sigma^\mu\bar{\sigma}^\nu + \sigma^\nu\bar{\sigma}^\mu\right)_a{}^c &= -2\eta^{\mu\nu}\delta_a{}^b\,, & (1.6)\\
\mathrm{Tr}\,(\sigma^\mu\bar{\sigma}^\nu) &= \mathrm{Tr}\,(\bar{\sigma}^\mu\sigma^\nu) \;=\; -\,2\eta^{\mu\nu}\,. & (1.7)
\end{aligned}$$

Define γ-matrices:

$$\gamma^\mu = \begin{pmatrix} 0 & (\sigma^\mu)_{a\dot{b}} \\ (\bar{\sigma}^\mu)^{\dot{a}b} & 0 \end{pmatrix}, \qquad \{\gamma^\mu,\gamma^\nu\} = -2\eta^{\mu\nu}\,, \tag{1.8}$$

and

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad L = \frac{1}{2}(1-\gamma_5)\,, \qquad R = \frac{1}{2}(1+\gamma_5)\,. \tag{1.9}$$

For a momentum 4-vector $p^\mu = (p^0, p^i) = (E, p^i)$ with $p^\mu p_\mu = -m^2$, we define momentum bi-spinors

$$p_{a\dot{b}} \;\equiv\; p_\mu\,(\sigma^\mu)_{a\dot{b}}\,, \qquad p^{\dot{a}b} \;\equiv\; p_\mu\,(\bar{\sigma}^\mu)^{\dot{a}b}\,. \tag{1.10}$$

For example,

$$p_{a\dot{b}} = \begin{pmatrix} -p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^0 - p^3 \end{pmatrix} \tag{1.11}$$

Taking the determinant of this 2×2 matrices gives

$$\det p = -p^\mu p_\mu = m^2\,. \tag{1.12}$$

2 Selected Feynman rules

Here we list a few Feynman rules that will be used in the first lecture:

- External scalar: 1.
- External outgoing fermion: $\bar{u}_s(p)$.
- External outgoing anti-fermion: $v_s(p)$.
- Gluon propagator (Feynman or Gervais-Neveu gauge): $\dfrac{\delta^{ab}\eta_{\mu\nu}}{P^2}$.
- gluon-quark-antiquark vertex: $\bar{q}\not{D}q \quad\longrightarrow\quad \dfrac{ig}{\sqrt{2}}\gamma^\mu T^a_{ij}\,.$
- gluon-squark-antisquark vertex: $|D\tilde{q}|^2 \;\supset\; \dfrac{ig}{\sqrt{2}}(p_j - p_i)^\mu T^a_{ij}\;.$

Here the $1/\sqrt{2}$ is included to compensate for our choice of normalization of the gauge group generators, which is $\mathrm{Tr}(T^aT^b) = \delta^{ab}$.

3 Massless particles

3.1 Spin-1/2: Angle and square spinors

In momentum space, the Dirac equation for a massless 4-component spinor is

$$\not{p}\, v_\pm(p) = 0\,, \qquad \bar{u}_\pm(p)\,\not{p} = 0\,. \tag{3.1}$$

The Feynman rules for a massless external *outgoing* fermion is $\overline{u}_\pm$ (and antifermions $v_\pm$). The subscript $\pm$ indicates the helicity $h = \pm 1/2$. Crossing gives $u_\pm = v_\mp$ and $\overline{v}_\pm = \overline{u}_\mp$.

We write the two independent solutions to the Dirac equation (3.1) as

$$v_+(p) = \begin{pmatrix} |p]_a \\ 0 \end{pmatrix}, \qquad v_-(p) = \begin{pmatrix} 0 \\ |p\rangle^{\dot{a}} \end{pmatrix}, \tag{3.2}$$

and

$$\overline{u}_-(p) \;=\; \big(\,0\,,\,\langle p|_{\dot{a}}\big)\,, \qquad \overline{u}_+(p) \;=\; \big(\,[p|^a\,,\,0\,\big)\,. \tag{3.3}$$

The angle- and square spinors are 2-component commuting spinors (think 2-component vectors) which satisfy the massless Weyl equation,

$$p^{\dot{a}b}|p]_b = 0\,, \qquad p_{a\dot{b}}|p\rangle^{\dot{b}} = 0\,, \qquad [p|^b\,p_{b\dot{a}} = 0\,, \qquad \langle p|_{\dot{b}}\,p^{\dot{b}a} = 0\,. \tag{3.4}$$

Raising and lowering their indices is business as usually:

$$[p|^a = \epsilon^{ab}|p]_b \qquad |p\rangle^{\dot{a}} = \epsilon^{\dot{a}\dot{b}}\langle p|_{\dot{b}}\,. \tag{3.5}$$

For *complex-valued momenta* p^μ, we regard the angle and square spinor solutions to be independent. However, for *real momenta*, the solutions are related via the Dirac conjugate $\overline{\psi}$, defined as:

$$\overline{\psi} \;\equiv\; \psi^\dagger\gamma^0\,. \tag{3.6}$$

For *real* momenta this gives

$$[p|^a = (|p\rangle^{\dot a})^* \qquad \text{and} \qquad \langle p|_{\dot a} = (|p]_a)^* . \tag{3.7}$$

Note that $u_s(p)$ are eigenstates of the L- and R-projections:

$$L\,u_-(p) = u_-(p)\,, \qquad R\,u_-(p) = 0\,, \qquad R\,u_+(p) = u_+(p)\,, \qquad L\,u_+(p) = 0\,. \tag{3.8}$$

The spin sum completeness relation with $m = 0$ reads $u_-\overline{u}_- + u_+\overline{u}_+ = -\not p$. In spinor helicity notation this is

$$-\not p \;=\; |p\rangle[p| + |p]\langle p|\,. \tag{3.9}$$

One can read off from this relation that

$$p_{a\dot b} \;=\; -\,|p]_a\,\langle p|_{\dot b}\,, \qquad p^{\dot a b} \;=\; -\,|p\rangle^{\dot a}\,[p|^b\,, \tag{3.10}$$

This should not shock you. After all, you learned in your algebra class that if a 2×2 matrix has vanishing determinant, it can be written as a product of two 2-component vectors: $p_{a\dot b} \;=\; -\,\lambda_a\,\tilde\lambda_{\dot b}$. In fact, this is often the starting point of introductions to the spinor helicity formalism.

3.2 Spinor brackets

For two light-like vectors p^μ and q^μ, we define spinor brackets

$$\langle p\,q\rangle \;=\; \langle p|_{\dot a}\,|q\rangle^{\dot a}\,, \qquad [p\,q] \;=\; [p|^a\,|q]_a\,. \tag{3.11}$$

Since indices are raised/lowered with the anti-symmetric Levi-Civitas (1.3), cf. (3.5), these products are antisymmetric:

$$\langle p\,q\rangle = -\langle q\,p\rangle\,, \qquad [p\,q] = -[q\,p]\,. \tag{3.12}$$

All other "inner products" vanish, e.g. $\langle p|q] = 0$.

For real momenta, the spinor products satisfy $[p\,q]^* = \langle q\,p\rangle$.

Using (1.7) one finds

$$\langle p\,q\rangle\,[p\,q] \;=\; 2\,p\cdot q \;=\; (p+q)^2 \;=\; -\,s_{pq}. \tag{3.13}$$

In the last equality, we defined the Mandelstam variable s_{pq}.

It is also useful to note the following properties:

$$[q|\gamma^\mu|p\rangle \;=\; \langle p|\gamma^\mu|q]\,, \tag{3.14}$$

$$[q|\gamma^\mu|p\rangle^* \;=\; [p|\gamma^\mu|q\rangle \qquad (\text{for real momenta})\,, \tag{3.15}$$

$$\langle p_1|\gamma^\mu|p_2]\langle p_3|\gamma_\mu|p_4] \;=\; 2\langle p_1 p_3\rangle[p_2 p_4]\,. \tag{3.16}$$

We often use $\langle p|P|q] \equiv P_\mu\langle p|\gamma^\mu|q]$ and obvious generalizations thereof. Note that for $P^2 \neq 0$ we may write $\langle q|P|q] \;=\; 2\,P\cdot q$.

In amplitude calculations, ***momentum conservation*** is imposed on n-particles as $\sum_{i=1}^{n} p_i^\mu = 0$ (consider all particles outgoing). This is encoded in the spinor helicity formalism as

$$\sum_{i=1}^{n} |i\rangle[i| \;=\; 0 \qquad \text{or} \qquad \sum_{i=1}^{n} \langle qi\rangle[ik] \;=\; 0 \tag{3.17}$$

for any light-like vectors q and k. For example, you can (and should) show that for $n = 4$ momentum conservation implies $\langle 12\rangle[23] = -\langle 14\rangle[43]$ and $\langle 12\rangle[12] = \langle 34\rangle[34]$.

The ***Schouten identity*** is a fancy name for a rather trivial fact: three vectors in a plane cannot be linearly independent. So if we have three 2-component vectors $|i\rangle$, $|j\rangle$, and $|k\rangle$, you can write one of them as a linear combination of the two others:

$$|k\rangle = a|i\rangle + b|j\rangle \qquad \text{for some } a \text{ and } b. \tag{3.18}$$

Dot in spinors $\langle\cdot|$ and form antisymmetric angle brackets to solve for the coefficients a and b. Show that (3.18) can then be cast in the form

$$|i\rangle\langle jk\rangle + |j\rangle\langle ki\rangle + |k\rangle\langle ij\rangle = 0\,. \tag{3.19}$$

This is the Schouten identity. It is often written with a fourth spinor $\langle q|$ "dotted-in":

$$\langle qi\rangle\langle jk\rangle + \langle qj\rangle\langle ki\rangle + \langle qk\rangle\langle ij\rangle = 0\,. \tag{3.20}$$

A similar Schouten identity holds for the square spinors.

3.3 Spin-1: Polarizations

In the spinor helicity formalism we write polarization vectors as

$$\epsilon_-^\mu(p;q) = -\frac{\langle p|\gamma^\mu|q]}{\sqrt{2}\,[p\,q]}\,, \qquad \epsilon_+^\mu(p;q) = -\frac{\langle q|\gamma^\mu|p]}{\sqrt{2}\,\langle p\,q\rangle}\,, \tag{3.21}$$

where $q \neq p$ denotes an arbitrary reference spinors.

The choice of q simply encodes the gauge freedom of shifting the polarization vector by any number times the momentum of the particle:

$$\epsilon^\mu(p) = \tilde{\epsilon}^\mu(p) + C\,p^\mu\,. \tag{3.22}$$

Now we have the basic spinor helicity tools needed to calculated scattering amplitudes of massless particles in 4d.