

# 2+1 Dimensional QED and a Novel Phase Transition

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## Abstract

We investigate the chiral phase transition in 2+1 dimensional QED. Previous studies of symmetry breaking have found that symmetry breaking ceases to occur when the number of fermion flavors exceeds a critical value. Here we focus on the order of the transition. We find that there are no light scalar degrees of freedom present as the critical number of flavors is approached from above (in the symmetric phase). Thus the phase transition is not second order. However, the order parameter vanishes continuously in the broken phase, so the transition is also unlike a conventional first order phase transition.

The study of dynamical mass generation in 2+1 dimensional gauge theories has attracted a good deal of attention recently. There are a number of reasons for this interest. These theories can describe the high temperature thermodynamics of 4-dimensional relativistic systems, as well as the statistical mechanics of certain planar condensed matter systems. Also, they have enough structure to be non-trivial, yet are sufficiently simple to admit analytic solutions, often unlike their 4-dimensional counterparts. Thus they can serve as theoretical laboratories for investigating aspects of dynamical symmetry breaking.

Mass terms for fermion and gauge fields in 2+1 dimensions serve as order parameters for discrete as well as continuous symmetry breaking. Dirac mass terms for 2-component complex fermions violate parity (P) and time reversal symmetry (T) in 2+1 dimensions. Also, gauge fields admit P and T violating Chern-Simons mass terms [1]. Combining two 2-component fermion fields it is possible to write down a parity invariant fermion mass term resembling a 4-dimensional Dirac mass. If this mass is dynamically generated, the continuous flavor symmetry of the system is spontaneously broken from  $U(2)$  to  $U(1) \otimes U(1)$ .

The dynamical generation of such a parity invariant mass term in 2+1 dimensional QED was investigated by Pisarski [2] using the Schwinger-Dyson (SD) gap equation. With  $N$  four-component Dirac spinors, he used a  $1/N$  expansion with

$$\alpha = \frac{e^2 N}{8} \tag{1}$$

fixed. Here,  $e$  is the dimensionful gauge coupling. If the fermions are massless, this theory possesses a  $U(2N)$  global flavor symmetry. If all the four-component fermions acquire the same mass, then the mass term by itself has an  $U(N) \otimes U(N)$  symmetry. Pisarski's analysis led him to the conclusion that there is dynamical mass generation for large  $N$ .

A more refined analysis of the gap equation for QED3 [3, 4] using the  $1/N$  expansion concluded that there is a phase transition at a critical number of fermion flavors  $N_c$  ( $3 < N_c < 4$ ) below which dynamical mass generation takes place but above which the fermions remain massless. This can be understood as follows. To leading order in the large  $N$  limit, the amplitude for the interaction of two fermions by the exchange of a photon is given by  $e^2/k^2(1 + \alpha/k)$ . This can be thought of

as a photon propagator of the form  $1/k$  multiplied by the effective dimensionless coupling  $e^2/(k + \alpha)$ . In the infrared limit,  $k \ll \alpha$  the effective coupling approaches the infrared fixed point  $8/N$  [5]. This infrared coupling weakens like  $1/N$  as  $N$  increases due to the screening effect of the fermions. The dimensionful parameter  $\alpha$  plays only the role of an ultraviolet cutoff for this infrared theory. For  $N$  greater than  $N_c$ , it was then shown that the effective infrared coupling is too weak to cause fermion condensation [3]. This is analogous to what happens in 3+1 dimensional gauge theories where the gauge coupling must exceed a critical value for chiral symmetry breaking to occur.

For  $N < N_c$  this analysis [3] found nonzero solutions of the gap equation (dynamical masses) which scale as

$$\Sigma_n(p \approx 0) = \alpha e^{\delta+2} \exp\left(\frac{-2n\pi}{\sqrt{N_c/N - 1}}\right), \quad (2)$$

in the limit  $N \rightarrow N_c$ . Here,  $n = 1, 2, 3, \dots$ , and  $\delta$  is a function of  $N$  that is non-singular as  $N \rightarrow N_c$ . For  $N > N_c$  no solutions of the gap equation are found. It was shown in ref. [3] that the  $n = 1$  solution minimizes the effective potential, and hence is the ground state solution. Subsequent lattice Monte-Carlo analysis [6] supported this critical behavior in QED3. Further studies [7] have analysed the finite volume dependence, and additional lattice studies are needed to verify the exponential behavior of equation (2). Also, estimates of the higher order terms in the  $1/N$  expansion indicated that  $N_c$  receives a correction of no more than 25% [4].

The gap equation analysis relies on the fact that the effective infrared coupling is proportional to  $1/N$  up to small constant corrections. The analysis could break down if, in higher orders in the  $1/N$  expansion, the effective coupling received

corrections proportional to powers of  $\ln(k/\alpha)$  for  $k \ll \alpha$  [8]. It has been shown, however, that this does not happen to any order in the  $1/N$  expansion [9], that is, that the infrared fixed point persists to all orders and the effective infrared theory is scale invariant to all orders.

Recently the question of a phase transition at  $N_c$  was re-analyzed [10] using renormalization group (RG) methods. A renormalizable effective Lagrangian for the scalar field order parameter for flavor symmetry breaking was constructed respecting the original  $U(2N)$  symmetry of the gauge theory. It was written in terms of a set of fields  $\phi$  transforming as an  $SU(2N)$  adjoint. The coupling of this field to the fermions, and hence to the gauge field, was neglected. The RG flows of the effective scalar self couplings was computed and it was argued that, when the quadratic term in the scalar potential is tuned to zero, radiative corrections generate a non-zero vacuum expectation value for the scalar field provided  $N$  is greater than  $\sqrt{5}/2$ . It was pointed out that this symmetry breaking is analogous to the Coleman-Weinberg mechanism in the Abelian Higgs model [12] in 3+1 dimensions. The universality hypothesis of Wilson was then invoked to conclude that in the original gauge theory, flavor symmetry breaking accompanied by fermion mass generation will occur for an arbitrarily large number of fermion flavors. This conclusion contradicts the gap equation analysis and the lattice Monte-Carlo results.

The problem with this RG analysis is that universality only relates theories with identical massless degrees of freedom. In particular, we will argue that for  $N > N_c$  (the symmetric phase), there are no light scalar degrees of freedom. For this purpose one could study the effective potential of this theory, but this would yield direct information only about the zero-momentum mass, not the physical mass of

the relevant scalar field. We will use an alternative approach based on the fact that light composite scalars would show up as poles in the fermion-antifermion scattering amplitude. This is the classic method employed by Nambu and Jona-Lasinio [11] (NJL).

We next solve the SD equation for this scattering amplitude in the symmetric phase. We restrict attention to the ladder approximation which was used in the solution to the gap equation. This approximation should be as reliable here as it was in the case of the gap equation [4]. We set the (Euclidian) momentum of the initial fermion and antifermion to  $q/2$ , but keep a non-zero momentum transfer by assigning momenta  $q/2 \pm p$  for the final fermion and antifermion. If the theory contains a light scalar resonance, the scattering amplitude should display a pole in (Minkowsky)  $q^2$ .

If the Dirac indices of the initial fermion and antifermion are  $\lambda$  and  $\rho$ , and the those of the final state fermion and antifermion are  $\sigma$  and  $\tau$ , then the scattering amplitude can be written as:

$$T_{\lambda\rho\sigma\tau}(p, q) = \delta_{\lambda\rho}\delta_{\sigma\tau}T(p, q) + \dots , \quad (3)$$

where the ... indicates pseudoscalar, vector, axial-vector, and tensor components. We contract Dirac indices so that we obtain the SD equation for the the scalar s-channel scattering amplitude,  $T(p, q)$ , containing only t-channel photon exchanges. If  $p \gg q$ , then  $q$  will simply act as an infrared cutoff in the loop integrations. The SD equation in the scalar channel is:

$$T(p, q) = \frac{16\alpha}{3Np^2(1 + \frac{\alpha}{p})} + \frac{16\alpha}{3\pi^2Np} \int_q^\infty \frac{dk T(k, q)}{k} \ln \left( \frac{k + p + \alpha}{|k - p| + \alpha} \right) . \quad (4)$$

For convenience, we have used the gauge  $\xi = 2/3$  where the fermion wavefunction

renormalization vanishes [13]. The demonstration of the gauge invariance of our results parallels that of refs. [4] and [14], and will not be presented here. Note that the first term in equation (4) is simply one photon exchange in the large  $N$  limit. It is this large  $N$  propagator that is used as the kernel in deriving equation (4). The integral in equation (4) is rapidly damped for  $k > \alpha$ .

For  $p \ll \alpha$  we use the approximation:

$$T(p, q) = \frac{16}{3Np} + \frac{16}{3\pi^2 Np} \int_q^\alpha \frac{dk T(k, q)}{k} (k + p - |k - p|) . \quad (5)$$

For momenta  $p > q$ , equation (5) can be converted to a differential equation:

$$p \frac{d^2}{dp^2} (pT) = \frac{-32 T}{3\pi^2 N} , \quad (6)$$

with appropriate boundary conditions determined from equation (5). The solutions of equation (6) have the form.

$$T(p, q) = \frac{A(q)}{\alpha} \left(\frac{p}{\alpha}\right)^{-\frac{1}{2} + \frac{1}{2}\eta} + \frac{B(q)}{\alpha} \left(\frac{p}{\alpha}\right)^{-\frac{1}{2} - \frac{1}{2}\eta} , \quad (7)$$

where

$$\eta = \sqrt{1 - N_c/N} , \quad (8)$$

and  $N_c \equiv 128/3\pi^2$ . The coefficients,  $A$  and  $B$ , can be determined by substituting the solution back into equation (5). This gives:

$$A = \frac{-\left(\frac{1}{2} - \frac{1}{2}\eta\right)^2 \pi^2 \left(\frac{q}{\alpha}\right)^{-\frac{1}{2} + \frac{1}{2}\eta}}{2\left(\frac{1}{2} + \frac{1}{2}\eta\right) \left(1 - \left(\frac{1-\eta}{1+\eta}\right)^2 \left(\frac{q}{\alpha}\right)^\eta\right)} , \quad (9)$$

and

$$B = \frac{\left(\frac{1}{2} - \frac{1}{2}\eta\right) \pi^2 \left(\frac{q}{\alpha}\right)^{-\frac{1}{2} + \frac{1}{2}\eta}}{2\left(1 - \left(\frac{1-\eta}{1+\eta}\right)^2 \left(\frac{q}{\alpha}\right)^\eta\right)} . \quad (10)$$

Note that there is an infrared divergence in the limit  $q \rightarrow 0$  in the numerators of both (9) and (10). That this is an infrared divergence rather than a pole corresponding to a bound state can be seen from the fact that the divergence exists for arbitrarily weak coupling ( $1/N$ ). In fact, this infrared divergence can already be seen at order  $1/N^2$  in the one-loop (two photon exchange) diagram. The strength of this divergence results from the  $1/k$  behavior of the (large  $N$  resummed) photon propagator. As required by the KLN theorem [15], this infrared divergence will be cancelled in a physical scattering process by the emission of soft quanta. We have verified [16] this cancellation to order  $1/N^3$ .

If we denote the location of the poles of the functions  $A$  and  $B$  in the complex  $q$  plane by  $q_0$ , we then have

$$|q_0| = \alpha \left( \frac{1 + \eta}{1 - \eta} \right)^{\frac{2}{\eta}} . \quad (11)$$

In the limit that  $\eta \rightarrow 0$ , we have

$$|q_0| \rightarrow \alpha \exp(4) . \quad (12)$$

Thus we see there are no poles in the complex  $q_0$ -plane within the domain of validity of our approximations (i.e.  $|q_0| \ll \alpha$ ), as  $\eta \rightarrow 0$ . In particular there is no pole which approaches zero momentum as  $\eta \rightarrow 0$ . This is the main result of this letter. The non-appearance of a scalar whose physical mass approaches zero at the the critical coupling [17],  $N_c$ , demonstrates that the phase transition is not second order.

This behavior can be contrasted with that of the gauged Nambu-Jona-Lasinio model, where there is a light scalar resonance near the critical line separating the symmetric and broken phases [18]. For weak gauge coupling, as the critical line is

approached, the width and mass of the scalar tend to zero, implying that there are long range correlations and therefore a second order transition, in that model.

How does this result match on to the gap-equation analysis in the broken phase and the lattice Monte-Carlo calculations? Our result may seem to contradict these previous studies, since equation (2) shows that the dynamical mass  $\Sigma_n$  (which can serve as an order parameter for the chiral phase transition) vanishes continuously as  $N \rightarrow N_c$  in the broken phase ( $N < N_c$ ). Note, however, that the order parameter,  $\phi$ , of a transition with a finite correlation length (i.e. *not* second order) may vanish continuously if the effective potential is not analytic at  $\phi = 0$ . In quantum field theories with long range forces, effective potentials are generally not analytic at  $\phi = 0$ .

Recall that there are an infinite number of solutions (see equation (2)) for the dynamical mass (in the broken phase) labeled by the integer  $n$ . All of these solutions must correspond to extrema of the effective potential. Consider the effective potential  $V(m)$  as a function of the dynamical mass,  $m$ , of the fermion at zero momentum. Even without an explicit calculation of  $V(m)$ , we can infer some of its properties from the knowledge we already have about its extrema. When  $m = \Sigma_n(p = 0)$ , the mass is at a local minimum or maximum of  $V(m)$ . Since the global minimum is at  $m = \Sigma_1(p = 0)$ , we can infer that there is a local maximum at  $m = \Sigma_2(p = 0)$ , and a local minimum at  $m = \Sigma_3(p = 0)$ , and so on. Note that  $\Sigma_1(p = 0) > \Sigma_2(p = 0) > \Sigma_3(p = 0) > \dots$ , so there are an infinite number of local minima (metastable states) between the symmetric (false vacuum) state at the origin ( $m = 0$ ) and the true ground state at  $m = \Sigma_1(p = 0)$ . Thus there are an infinite number of energy barriers between the symmetric state and the true



ground state. This is suggestive of a first order transition. However, we note that we can take  $\Sigma_1(p = 0)$  to be an order parameter for the phase transition, and as we approach the critical coupling,  $N_c$ , the order parameter approaches zero, and we expect a scalar bound state with a mass of the order of  $2\Sigma_1(p = 0)$ . This latter behavior is more typical of a second order transition. However the spectrum is not continuous as we go through the critical value  $N_c$ , since there is no light scalar in the symmetric phase. This unusual critical behavior can be attributed to the fact that the effective potential is not analytic at  $m = 0$ .

To conclude, we have re-examined the critical behavior of QED3 as a function of the number of fermions,  $N$ , by solving a Schwinger-Dyson equation for the fermion-antifermion scattering amplitude in the symmetric phase ( $N > N_c$ ). We have argued that no light scalar degrees of freedom appear in the symmetric phase, and hence that the chiral symmetry breaking phase transition is not second order. Since the order parameter varies continuously from the broken to the symmetric phase, this does not look like a conventional first order phase transition. This unusual behavior can be attributed to the presence of long range forces [19] in this model. We have also described the likely structure of the effective potential in the broken phase, and noted the unusual features of the critical behavior arising from the non-analyticity of the effective potential. The fact that there are no light scalar degrees of freedom in the symmetric phase indicates that the analysis of ref. [10] is not relevant to the chiral phase transition in QED3. Another way to say this is that the  $SU(2N)$  scalar field theory analysed there is not in a universality class with QED3. (Indeed, since the chiral phase transition in QED3 is not second order, QED3 does not have a universality class.) It is not surprising that QED3 and

the  $SU(2N)$  scalar theory have different phase transitions, since universality arguments are only expected to be applicable to theories with finite-range interactions, a property that gauge theories do not enjoy.

The behavior of this phase transition should be confirmed by lattice Monte-Carlo studies. Such simulations should check that, in the broken phase, the order parameter vanishes continuously as  $N \rightarrow N_c$ , and that in the symmetric phase there is no evidence of a scalar resonance whose mass approaches zero as  $N \rightarrow N_c$ .

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