Superconformal field theories

A-Maximization

classically the trace of the energy–momentum tensor for a scale-invariant theory vanishes.
trace anomaly. r:

$$T^{\mu}_{\mu} = \frac{1}{g^3}\tilde{\beta}(F^b_{\mu\nu})^2 - a(\tilde{R}_{\mu\nu\rho\sigma})^2 + \ldots \ ,$$

central charge a
$\tilde{\beta}$ is the numerator of the exact NSVZ β function
$R_{\mu\nu\rho\sigma}$ is the curvature tensor

Cardy conjectured that a satisfies:

$$a_{IR} < a_{UV}$$

A-Maximization

in SCFT a can determined by ‘t Hooft anomalies of SC R-charge

$$a = \frac{3}{32}\left(3\mathrm{Tr}R^3 - \mathrm{Tr}R\right)$$

$T_{\mu\nu}$ and the R-current ($J_{R\mu}$) in the same supermultiplet
In superspace, super-energy–momentum tensor $T_{\alpha\dot{\alpha}}(x,\theta,\bar{\theta})$ contains:
$J_{R\mu}$, $J_{\alpha\mu}$ in the θ and $\bar{\theta}$ components and $T_{\mu\nu}$ in the θ^2 component

superconformal R-charge:

$$R = R_0 + \sum_i c_i Q_i$$

A-Maximization

superconformal symmetry relates different triangle anomalies $\langle J_R J_R J_i \rangle$ related to $\langle TTJ_i \rangle$ by

$$9\,\mathrm{Tr}\,R^2 Q_i = \mathrm{Tr}\,Q_i$$

two-point function

$$\langle J_i(x) J_k(0) \rangle \propto \tau_{ik}\,\tfrac{1}{x^4}$$

Unitarity $\Rightarrow$ τ_{ik} to have positive definite eigenvalues
Superconformal symmetry $\Rightarrow$

$$\mathrm{Tr} R Q_i Q_k = -\tfrac{\tau_{ik}}{3}$$

$\Rightarrow$ $\mathrm{Tr}\,R Q_i Q_k$ is negative definite

A-Maximization

Intriligator and Wecht: correct choice of the R-charge

$$R = R_0 + \sum_i c_i Q_i$$

maximizes a-charge:

$$\begin{aligned} \frac{\partial a}{\partial c_i} &= \frac{3}{32}\left(9\,\mathrm{Tr}\,R^2 Q_i - \mathrm{Tr} Q_I\right) = 0 \\ \frac{\partial^2 a}{\partial c_i \partial c_k} &= \frac{27}{16} \mathrm{Tr}\,R Q_i Q_k < 0 \end{aligned}$$

The simplest chiral SCFT

	$SU(N)$	$SU(F)$	$SU(F+N-4)$	$U(1)_R$
Q	$\square$	$\square$	$\mathbf{1}$	$R(Q)$
$\overline{Q}$	$\overline{\square}$	$\mathbf{1}$	$\square$	$R(\overline{Q})$
A	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\mathbf{1}$	$\mathbf{1}$	$R(\mathrm{A})$

$F = 0$, breaks SUSY

$F = 1, 2$, runaway vacua

$F = 3$, quantum deformed moduli space

$F = 4$, s-confining

$F = 5$, splits into an IR free and IR fixed point sectors

$F > 5$?

Moduli Space

parameterized by mesons $M = Q\bar{Q}$, $H = \bar{Q}A\bar{Q}$ and baryons:

$$\begin{array}{cc} N \text{ even} & N \text{ odd} \\ \bar{Q}^N & \bar{Q}^N \\ A^{N/2} & QA^{N-1/2} \\ Q^2A^{(N-2)/2} & Q^3A^{(N-3)/2} \\ \vdots & \vdots \\ Q^kA^{(N-k)/2} & Q^kA^{(N-k)/2} \end{array}$$

where $k \leq min(N, F)$

R-charge

anomaly cancellation for large $F, N \Rightarrow$

$$R(\mathrm{A}) = \tfrac{F}{N}\left(2 - R(Q) - \left(\tfrac{N}{F}+1\right) R(\overline{Q})\right)$$

In general

$$\begin{array}{rcl} R(Q) & = & 2 - \frac{6}{N} + b(N-F) + c \\ R(\overline{Q}) & = & \frac{6}{N} + bF - c\frac{F}{F+N-4} \\ R(\mathrm{A}) & = & -\frac{12}{N} - 2bF \end{array}$$

$$\begin{array}{rcl} a & = & 3(N^2 - 1 + FN(R(Q)-1)^3 + N(F+N-4)(R(\overline{Q})-1)^3 \\ & & +N(N-1)/2(R(A)-1)^3) - (N^2-1) - NF(R(Q)-1) \\ & & -N(F+N-4)(R(\overline{Q})-1) - N(N-1)\frac{1}{2}(R(A)-1) \end{array}$$

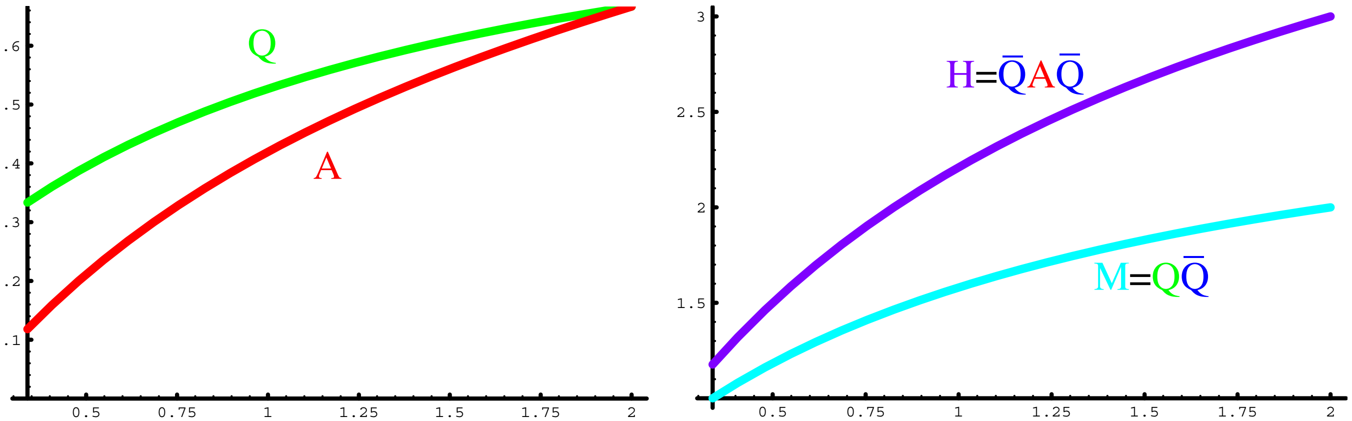
R-charge

a-maximization gives for F, N:

$$R(Q) = R(\overline{Q}) = -\frac{12-9\left(\frac{N}{F}\right)^2+\sqrt{\left(\frac{N}{F}\right)^2(-4+\frac{N}{F}(73\frac{N}{F}-4))}}{3(-4+(\frac{N}{F}-4)\frac{N}{F})}$$

even though theory is chiral

R-charge



(a) The R-charges of the fundamental fields, with $R(Q) = R(\overline{Q})$
(b) The corresponding dimensions of the meson operators.

Two Free Mesons

reduce F from the Banks–Zaks fixed point at $F \sim 2N$

meson $M = Q\bar{Q}$ goes free at

$$F = F_1 = \tfrac{9N}{4(4+\sqrt{7})} \approx 0.3386\, N$$

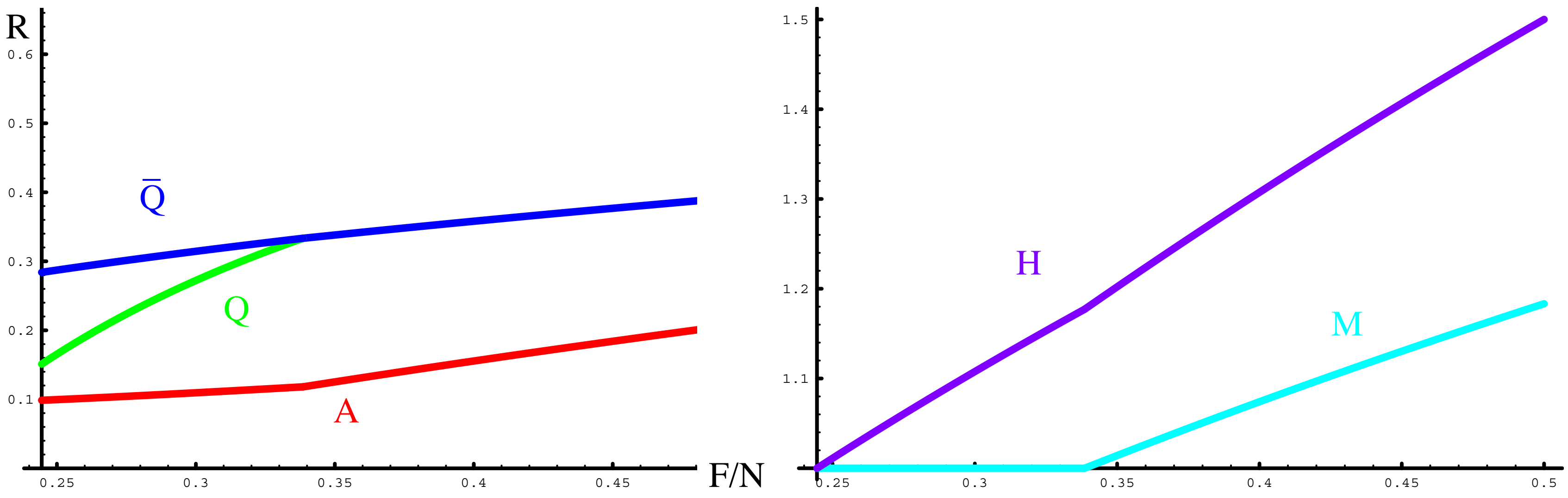
meson $H = \bar{Q}A\bar{Q}$ is still interacting

Kutasov: assume only one accidental $U(1)$ for the free meson M then

$$\begin{aligned} a_{\rm int} &= a - a(R(M)) \\ &= a - \tfrac{3}{32} F(F+N-4)\big(3(R(Q)+R(\overline{Q})-1)^3 \\ &\quad -(R(Q)+R(\overline{Q})-1)\big) \end{aligned}$$

meson $H = \bar{Q}A\bar{Q}$ goes free at $F = F_2 \approx 0.2445N$

Two Free Mesons



(a) The R-charges of the fundamental fields
(b) The corresponding dimensions of the meson operators

Dual Description

for N odd and $F \geq 5$:

	$SU(F-3)$	$Sp(2F-8)$	$SU(F)$	$SU(N+F-4)$
$\tilde{y}$	$\overline{\square}$	$\square$	$\mathbf{1}$	$\mathbf{1}$
$\overline{p}$	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
q	$\square$	$\mathbf{1}$	$\overline{\square}$	$\mathbf{1}$
a	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
l	$\mathbf{1}$	$\square$	$\mathbf{1}$	$\overline{\square}$
B_1	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\mathbf{1}$
M	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\square$
H	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$

with a superpotential

$$W = c_1 M q l \tilde{y} + c_2 H l l + B_1 q \overline{p} + a \tilde{y} \tilde{y}$$

$M = Q\bar{Q}$ and $H = \bar{Q} A \bar{Q}$ are mapped to elementary fields

Dual Description

even N and $F \geq 5$:

	$SU(F-3)$	$Sp(2F-8)$	$SU(F)$	$SU(N+F-4)$	$SU(2)$
$\tilde{y}$	$\overline{\square}$	$\square$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
$\overline{p}$	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\square$
q	$\square$	$\mathbf{1}$	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$
a	$\genfrac{}{}{0pt}{}{\square}{\square}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
l	$\mathbf{1}$	$\square$	$\mathbf{1}$	$\overline{\square}$	$\mathbf{1}$
S	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\mathbf{1}$	$\square$
B_0	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
M	$\mathbf{1}$	$\mathbf{1}$	$\square$	$\square$	$\mathbf{1}$
H	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\genfrac{}{}{0pt}{}{\square}{\square}$	$\mathbf{1}$

with a superpotential

$$W = c_1 M q l \tilde{y} + c_2 H l l + S q \overline{p} + a \tilde{y} \tilde{y} + B_0 a \overline{p}^2$$

Dual Description

M goes free at $F = F_1$, $c_1 \to 0$ chiral operator $ql\tilde{y}$ has dimension 2

H goes free at $F = F_2$, $c_2 \to 0$ chiral operator ll has dimension 2 corresponding to R-charge 4/3
l has R-charge 2/3 and dimension 1? is l free?
    self-consistent if l is a gauge-invariant operator
    recall SUSY QCD:

$$W = c\, M \phi \overline{\phi}$$

when M goes free, the coupling $c \to 0$ the chiral operator $\phi\overline{\phi}$ has dimension 2
a-maximization $\Rightarrow$ ϕ, $\overline{\phi}$ are free if we assume accidental axial symmetry for dual quarks
accidental axial symmetry only if dual gauge group is IR-free
dual β function $\Rightarrow$ dual looses asymptotic freedom for $F < 3N/2$
dual quarks are free

Dual β function

with c_1 and c_2 set to zero

$$W = B_1 q\bar{p} + a\tilde{y}\tilde{y}$$

$Sp(2F-8)$ has $\tilde{y}$ and l with gauge interactions

$$\begin{aligned}\beta(g) \quad &= \quad -\frac{g^3}{16\pi^2}\left[3(2F-6)-(F-3)(1-\gamma_{\tilde{y}|_{g=0}})-(N+F-4)\right] \\ &\quad\ +\mathcal{O}(g^5),\end{aligned}$$

nonperturbative $SU(F-3)$ and superpotential corrections through anomalous dimension $\gamma_{\tilde{y}}$

$Sp(2F-8)$ is IR free if

$$N - 4F + 11 - (F-3)\gamma_{\tilde{y}|_{g=0}} > 0$$

Dual β function

superpotential has dimension 3 $\Rightarrow$

$$\gamma_a + 2\gamma_{\tilde{y}} = 0 \qquad (*)$$

$a^{(F-3)/2}$ is a gauge-invariant operator $\Rightarrow$

$$\tfrac{F-3}{2} + \tfrac{F-3}{4}\gamma_a \geq 1 \qquad (**)$$

Combining eqns (*) and (**) we have that for large F, $\gamma_{\tilde{y}} \leq 1$
large F and large N limit with $F < N/5$:

$$\beta \propto N - 4F + 11 - (F-3)\gamma_{\tilde{y}} > 0 \ ,$$

$Sp(2F-8)$ is IR free

assuming l is free for $F < F_2$ we can check that $Sp(2F-8)$ becomes IR free at $F = F_2$

$F < F_2$: Mixed Phase

theory splits into two sectors in the IR, free magnetic sector:

	$Sp(2F-8)$	$SU(F)$	$SU(N+F-4)$
l	$\square$	$\mathbf{1}$	$\overline{\square}$
M	$\mathbf{1}$	$\square$	$\square$
H	$\mathbf{1}$	$\mathbf{1}$	$\begin{array}{c}\square\\[-4pt]\square\end{array}$

interacting superconformal sector: is

	$SU(F-3)$	$Sp(2F-8)$	$SU(F)$
$\tilde{y}$	$\overline{\square}$	$\square$	$\mathbf{1}$
$\overline{p}$	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$
q	$\square$	$\mathbf{1}$	$\overline{\square}$
a	$\begin{array}{c}\square\\[-4pt]\square\end{array}$	$\mathbf{1}$	$\mathbf{1}$
B_1	$\mathbf{1}$	$\mathbf{1}$	$\square$

$$W = B_1 q \overline{p} + a \tilde{y} \tilde{y}$$

$\mathcal{N} = 1$: Open Questioins

- how nonperturbative effects make $\gamma_Q \neq \gamma_{\bar{Q}}$ only for $F < F_1$
- mixed-phase first conjectured in theories with an adjoint, still not proven
- SO with spinors

$\mathcal{N} = 2$

$SU(N)$ with $\mathcal{N} = 2$ SUSY and F hypermultiplets in $\square$

$$\beta(g) = -\frac{g^3}{16\pi^2}\frac{(3N - N(1-2\beta(g)/g) - F)}{1 - Ng^2/8\pi^2}$$

adjoint in supermultiplet with gluon and gluino
$\Rightarrow Z = 1/g^2 \Rightarrow \gamma(g) = 2\beta(g)/g$
$\gamma_Q = 0$, non-renormalization of the superpotential $\Rightarrow$ non-renormalization of the Kähler function, both related to a prepotential
solving for $\beta(g)$

$$\beta(g) = -\frac{g^3}{16\pi^2}(2N - F)$$

exact at one-loop

$\mathcal{N} = 2$ SCFT

$$\beta(g) = -\frac{g^3}{16\pi^2}(2N - F)$$

vanishes for $F = 2N$

β vanishes independent of $g \Rightarrow$ line of fixed points

Seiberg–Witten analysis $\Rightarrow a_D^i$ have no logarithmic corrections

classical relations between a^i and a_D^j are exact

theory with $F = 2N$ hypermultiplets is nonperturbatively conformal

Argyres–Douglas fixed points

massless electrically and magnetically charged particles at the same point in the moduli space

electric charge: $g_{\mathrm{IR}} \to 0$, magnetic charge: $g_{\mathrm{IR}} \to \infty$

IR fixed point? Argyres and Douglas: yes!

$\mathcal{N} = 2$ $SU(2)$ with one flavor

adjust mass and VEV so monopole and dyon points coincide

for $m = 3\Lambda_1/4$ and $u = 3\Lambda_1^2/4$

$$y^2 = \left(x - \frac{\Lambda_1^2}{4}\right)^3$$

all three roots coincide

Seiberg–Witten analysis shows that a_D has no logarithmic corrections, theory is conformal

Argyres–Douglas fixed points

charges in $U(1)$ theories with IR fixed points do not produce long-range fields
using

$$d \geq \frac{1}{2}[C_2(\mathbf{r}) + C_2(V) - C_2(\mathbf{r}')] \ .$$

$F_{\mu\nu}$, is in a $(1,0)+(0,1)$ of $SO(4)$, has a scaling dimension $d \geq 2$
at an interacting IR fixed point generically $d > 2$
conformal symmetry and dimensional analysis $\Rightarrow$ fields fall off as $1/x^d$

Other SCFTs

have several different interactions and are superconformal
lines (or manifolds) of fixed points
if there are n interactions and only p independent β functions
then $n - p$ dimensional manifold of fixed points

moving in manifold $\leftrightarrow$ changing coupling of an exactly marginal operator
operator in $\mathcal{L}$ has scaling dimension 4, independent of couplings

can also happen in $\mathcal{N} = 1$ theories

$\mathcal{N} = 4$ SUSY gauge theory

$\mathcal{N} = 1$ SUSY gauge theory with three chiral supermultiplets in the adjoint with a particular superpotential
$\equiv \mathcal{N} = 2$ SUSY gauge theory with an adjoint hypermultiplet
In general $\mathcal{N} = 4$ theories have a global $SU(4)_R \times U(1)_R$ R-symmetry
restrictted to vector supermultiplet does not transform under the $U(1)_R$

λ, and the three adjoint fermions, ψ, transform as a **4** of the $SU(4)_R$
real adjoint scalars ϕ transform as a **6** of $SU(4)_R$
in terms of $\mathcal{N} = 1$ fields, the $SU(4)_R$ symmetry is not manifest
only $SU(3) \times U(1)$ subgroup is apparent
for canonically normalized $\mathcal{N} = 1$ superfields the superpotential is

$$W_{\mathcal{N}=4} = -i\sqrt{2}\, Y \,\mathrm{Tr}\, \Phi_1 \left[\Phi_2, \Phi_3\right] = \frac{Y}{3\sqrt{2}}\, \epsilon_{ijk} f^{abc}\, \Phi_i^c \Phi_j^a \Phi_k^b$$

where $a, \ldots, e = 1, \ldots, N^2 - 1$ are the adjoint gauge indices
$i, \ldots, m = 1, 2, 3$ are $SU(3)$ flavor indices, and $\Phi_i = T^a \Phi_i^a$
for $\mathcal{N} = 4$ SUSY, $Y = g$

$\mathcal{N}=4$ SUSY gauge theory

Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{\mathcal{N}=4} &= -\tfrac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} - i\bar{\lambda}^a\sigma^\mu D_\mu\lambda^a - i\bar{\psi}^a_i\sigma^\mu D_\mu\psi^a_i + D^\mu\phi^{\dagger a}_i D_\mu\phi^a_i \\ &-\sqrt{2}gf^{abc}(\phi^{\dagger c}_i\lambda^a\psi^b_i - \bar{\psi}^c_i\bar{\lambda}^a\phi^b_i) - \tfrac{Y}{\sqrt{2}}\epsilon_{ijk}f^{abc}(\phi^c_i\psi^a_j\psi^b_k + \bar{\psi}^c_i\bar{\psi}^a_j\phi^{\dagger\,b}_k) \\ &+\tfrac{g^2}{2}(f^{abc}\phi^b_i\phi^{\dagger\,c}_i)(f^{ade}\phi^d_j\phi^{\dagger\,e}_j) - \tfrac{Y^2}{2}\epsilon_{ijk}\epsilon_{ilm}(f^{abc}\phi^b_j\phi^c_k)(f^{ade}\phi^{\dagger d}_l\phi^{\dagger e}_m)\end{aligned}$$

$SU(N)$ gauge theory with $\mathcal{N}=2$ SUSY and A adjoint hypermultiplets has

$$\beta(g) = -\frac{g^3}{16\pi^2}(2-2A)N$$

$\Rightarrow \mathcal{N}=4$ gauge theory has $\beta = 0 \leftrightarrow$ SCFT

Quivers and Mooses

Theories with gauge groups connected by bifundamentals
called "quiver" theories and "mooses"
matter content can be represented by a quiver/moose diagram
in certain cases a quiver/moose theory can be considered as a latticization (a.k.a "deconstruction") along a discretized extra dimension

$\mathcal{N} = 4$ and orbifolds

mod-out by a discrete subgroup Γ of the gauge and global symmetries
$\longrightarrow$ "daughter" theory
$\longrightarrow$ quiver/moose

large N limit of an $SU(N)$ dominated by planar diagrams

if Γ embedded in gauge group using regular representation N times then the planar diagrams of daughter $\propto$ planar diagrams of full theory (up to a rescaling of the gauge coupling)

large N limit, daughters of the $\mathcal{N} = 4$ gauge theory are conformal

orbifolding in different ways break different amounts of SUSY:

$$\begin{array}{lll} SU(4)_R \supset \Gamma, & SU(3) \not\supset \Gamma & \Rightarrow \mathcal{N} = 0 \\ SU(3) \supset \Gamma, & SU(2) \not\supset \Gamma & \Rightarrow \mathcal{N} = 1 \\ SU(2) \supset \Gamma, & & \Rightarrow \mathcal{N} = 2 \ . \end{array}$$

unbroken SUSY $\leftrightarrow$ size of the R-symmetry subgroup invariant under Γ

$\mathcal{N} = 4$ and orbifolds

simplest case: permutation group $\Gamma = Z_k$ embedded in the gauge group regular representation of Z_k:

$$\gamma^{\mathbf{a}} = \mathrm{diag}(\omega^0, \omega^a, \omega^{2a}, \ldots, \omega^{(k-1)a})$$

where $\omega = e^{2\pi i/k}$ and $a = 0, 1, \ldots, k-1$, embed Z_k in $SU(kN)$ by defining

$$\gamma^{\mathbf{a}}_{\mathbf{N}} = \mathrm{diag}(\mathbf{1_N}, \mathbf{1_N}\omega^a, \mathbf{1_N}\omega^{2a}, \ldots, \mathbf{1_N}\omega^{(k-1)a})$$

so adjoint transforms as

$$\mathbf{Ad} \rightarrow \gamma^{\mathbf{a}}_{\mathbf{N}} \mathbf{Ad} (\gamma^{\mathbf{a}}_{\mathbf{N}})^{\dagger}$$

$\mathcal{N} = 4$ and orbifolds

parts of the $kN \times kN$ matrix of gauge fields left invariant are

$$A_{inv} = \mathrm{diag}(\mathbf{A_1}, \mathbf{A_2}, \mathbf{A_3}, \ldots, \mathbf{A_k}) \ ,$$

where $\mathbf{A_i}$ t adjoint under the ith $SU(N)$ subgroup of $SU(kN)$
orbifolded gauge group is $\Pi_{i=1}^{k} SU(N)_i$

Orbifold example

example the Z_6 orbifold where the embedding of Z_6 in the global $SU(4)$ R-symmetry is such that the four fermion fields transform as:

$$(\psi_1, \psi_2, \psi_3, \psi_4) \rightarrow (\omega^a \psi_1, \omega^{-2a} \psi_2, \omega^{3a} \psi_3, \omega^{4a} \psi_4)$$

under a global transformation
adjoint fermion ψ_3 that transforms as

$$\psi_3 \rightarrow \omega^{3a} \gamma_{\mathbf{N}}^{\mathbf{a}} \psi_3 (\gamma_{\mathbf{N}}^{\mathbf{a}})^\dagger$$

Orbifold example

invariant pieces of ψ_3:

$$\begin{matrix} 0 & 0 & 0 & \psi_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & \psi_{25} & 0 \\ 0 & 0 & 0 & 0 & 0 & \psi_{36} \\ \psi_{41} & 0 & 0 & 0 & 0 & 0 \\ 0 & \psi_{52} & 0 & 0 & 0 & 0 \\ 0 & 0 & \psi_{63} & 0 & 0 & 0 \end{matrix}$$

bifundamentals transforming as $(\square, \overline{\square})$ under $SU(N)_i \times SU(N)_j$
similar analysis for the remaining fermion and scalar fields

Orbifolds and the Hierarchy Problem

proposed that orbifold theories solve the hierarchy problem if physics was conformal above 1 TeV
exactly conformal theory has no quadratic divergences

consider the effective theory below some scale μ
calculate the one-loop β functions, set the $\beta = 0$
in daughter theories where matter fields are distinct bifundamentals, fixed points for Y, and λ_i, approach $\mathcal{N} = 4$ SUSY:
$Y = \lambda_i = g$ as $N \to \infty$

Orbifolds and the Hierarchy Problem

at fixed point the one-loop scalar mass is given by

$$m_\phi^2 = \left[N c_i \lambda_i + 3 \frac{N^2-1}{N} g^2 - 8 N Y^2 \right] \frac{\mu^2}{16\pi^2}$$

large N limit $\sum_i c_i = 5$: no quadratic divergence
leading order in $1/N$:

$$m_\phi^2 = \frac{3g^2}{N} \frac{\mu^2}{16\pi^2}$$

to get $m_\phi = 1$ TeV, with $\mu = M_{\rm Pl}$ we need $N = 10^{28}$

if scalar mass term is relevant operator in low-energy effective theory below SUSY breaking scale, a large mass is generated