Exercise 1

1. Construct the massive supermultiplet of $\mathcal{N}=3$ SUSY for the lowest weight state (Clifford "vacuum") having spin 0. (Use the notation where the raising operator on this state produces a state $(\square,\square)$ where the first $\square$ indicates a 3 of the $SU(3)$ R-symmetry and second $\square$ denotes a spin half doublet. You can use the following $SU(3)$ group theory results (keep in mind that $\overline{\square} = \begin{smallmatrix}\square\\ \square\end{smallmatrix} \leftrightarrow \overline{3}$)

$$\square\times\square = \begin{smallmatrix}\square\\ \square\end{smallmatrix} + \square\square \quad\leftrightarrow\quad 3\times 3 = \overline{3}_A + 6_S\ , \tag{1}$$

$$\begin{smallmatrix}\square\\ \square\end{smallmatrix}\times\square = 1 + \begin{smallmatrix}\square\square\\ \square\ \end{smallmatrix} \quad\leftrightarrow\quad \overline{3}\times 3 = 1+8\ . \tag{2}$$

Check that there are an equal number of bosonic and fermionic states in the supermultiplet. Is this state equivalent to a massive supermultiplet of $\mathcal{N}=4$?

2. Consider $\mathcal{N}=4$ SUSY with a 4×4 central charge matrix $\mathbf{Z}$. In a skew diagonal basis we can write

$$Z = \begin{pmatrix} Z_1\epsilon^{ab} & 0 \\ 0 & Z_2\epsilon^{ab} \end{pmatrix} \tag{3}$$

where $a=1,2$ and $b=1,2$. In this basis the SUSY algebra can be written as

$$\{Q^{aL}_\alpha, Q^\dagger_{\dot\alpha bN}\} = 2\sigma^\mu_{\alpha\dot\alpha}P_\mu\delta^a_b\delta^L_N\ , \tag{4}$$

$$\{Q^{aL}_\alpha, Q^{bN}_\beta\} = 2\sqrt{2}\epsilon_{\alpha\beta}\epsilon^{ab}\delta^{LN}Z_N\ , \tag{5}$$

$$\{Q^\dagger_{\dot\alpha aL}, Q^\dagger_{\dot\beta bN}\} = 2\sqrt{2}\epsilon_{\dot\alpha\dot\beta}\epsilon_{ab}\delta_{LN}Z_N\ , \tag{6}$$

where $L=1,2$; $N=1,2$; and the repeated index N is not summed over. Defining

$$A^L_\alpha = \frac{1}{2}\left[Q^{1L}_\alpha + \epsilon_{\alpha\beta}\left(Q^{2L}_\beta\right)^\dagger\right]\ , \tag{7}$$

$$B^L_\alpha = \frac{1}{2}\left[Q^{1L}_\alpha - \epsilon_{\alpha\beta}\left(Q^{2L}_\beta\right)^\dagger\right]\ , \tag{8}$$

reduces the algebra in the rest frame to

$$\{A^L_\alpha, A^\dagger_{\beta N}\} = \delta_{\alpha\beta}\delta^L_N(M+\sqrt{2}Z_L)\ , \tag{9}$$

$$\{B^L_\alpha, B^\dagger_{\beta N}\} = \delta_{\alpha\beta}\delta^L_N(M-\sqrt{2}Z_L)\ , \tag{10}$$

Consider a massive state with $M=\sqrt{2}Z_1=\sqrt{2}Z_2$ and construct the short multiplet starting with the spin 0 Clifford vacuum $|\Omega_0\rangle$. Label the elements of the multiplet with an $SU(2)_R$ representation d_R and the spin $2j+1$.