115B Midterm 2 Formulae Please tear off this sheet.

$E = h\,f = h\frac{c}{\lambda} = \hbar\omega,\ p = \frac{h}{\lambda} = \hbar k, \quad E^2 = p^2c^2 + m^2c^4$

$-\frac{\hbar^2}{2m}\nabla^2\Psi(\vec{r},t) + V(\vec{r})\Psi(\vec{r},t) = i\hbar\frac{\partial\Psi(\vec{r},t)}{\partial t}$

$-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r}) + V(\vec{r})\psi(\vec{r}) = E\psi(\vec{r})$ if $\Psi(\vec{r},t) = e^{-iEt/\hbar}\,\psi(\vec{r})$

$[x,p_x] = i\hbar,\ [L_x,L_y] = i\hbar L_z,\ [L_y,L_z] = i\hbar L_x,\ [L_z,L_x] = i\hbar L_y,$

$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$L_\pm|\ell\ m\rangle = \hbar\sqrt{\ell(\ell+1) - m(m\pm1)}|\ell\ (m\pm1)\rangle$

$S_\pm|s\ m\rangle = \hbar\sqrt{s(s+1) - m(m\pm1)}|s\ (m\pm1)\rangle$

$V(x) = \begin{cases} 0 & 0<x<L \\ \infty & x<0, x>L \end{cases}, \quad \psi_n(x) = \begin{cases} \sqrt{\frac{2}{L}}\sin\frac{n\pi x}{L} & 0<x<L \\ 0 & x<0, x>L \end{cases}, \quad E_n = \frac{n^2\pi^2\hbar^2}{2mL^2}$

$H = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2x^2 = (a_+a_- + \frac{1}{2})\hbar\omega,\ E_n = (n+\frac{1}{2})\hbar\omega, \quad [a_-,a_+] = 1,$

$a_+\psi_n = \sqrt{n+1}\,\psi_{n+1}, \quad a_-\psi_n = \sqrt{n}\,\psi_{n-1},\ \psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}\,H_n(\xi)\,e^{-\xi^2/2},\ \xi = x\sqrt{\frac{m\omega}{\hbar}}$

$H_0 = 1,\ H_1 = 2\xi,\ H_2 = 4\xi^2 - 2,\ H_3 = 8\xi^3 - 12\xi,\ H_4 = 16\xi^4 - 48\xi^2 + 12$

$V(r) = -\frac{\alpha\hbar c}{r}, \quad -\frac{\hbar^2}{2m}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)R_{n\ell}(r) + \frac{\hbar^2\ell(\ell+1)}{2mr^2}R_{n\ell}(r) + V(r)R(r) = E_n\,R_{n\ell}(r)$

$a = \frac{\hbar}{\alpha mc} = 0.0529$ nm, $\quad \alpha = 1/137.04,\ E_n = -\frac{\alpha^2mc^2}{2n^2}\left[1 + \frac{\alpha^2}{n^2}\left(\frac{n}{j+1/2} - \frac{3}{4}\right)\right] \approx -\frac{13.6\text{eV}}{n^2}$

$\int_0^\infty r^2R_{n\ell}(r)^2dr = 1,\ \int_0^{2\pi}\int_0^\pi |Y_\ell^m(\theta,\phi)|^2\sin\theta\,d\theta d\phi = 1$

$\psi_{n\ell m}(\vec{r}) = R_{n\ell}(r)Y_\ell^m(\theta,\phi), \quad R_{n,n-1}(r) = A\,r^{n-1}e^{-\frac{r}{na}}$

$R_{10} = \frac{2}{a^{3/2}}\,e^{-r/a},\ R_{20} = \frac{1}{\sqrt{2a^3}}\left(1 - \frac{1}{2}\frac{r}{a}\right)\ e^{-r/2a},\ R_{21} = \frac{1}{\sqrt{24a^3}}\ \frac{r}{a}\ e^{-r/2a}$

$R_{30} = \frac{2}{\sqrt{27a^3}}\left(1 - \frac{2}{3}\frac{r}{a} + \frac{2}{27}\frac{r^2}{a^2}\right)\ e^{-r/3a},\ R_{31} = \frac{8}{27\sqrt{6a^3}}\left(1 - \frac{1}{6}\frac{r}{a}\right)\ \frac{r}{a}\,e^{-r/3a},\ R_{32} = \frac{4}{81\sqrt{30a^3}}\ \frac{r^2}{a^2}\ e^{-r/3a}$

$Y_0^0 = \frac{1}{\sqrt{4\pi}},\ Y_1^0 = \sqrt{\frac{3}{4\pi}}\cos\theta,\ Y_1^{\pm1} = \mp\sqrt{\frac{3}{8\pi}}\sin\theta\,e^{\pm i\phi}, \quad Y_2^0 = \sqrt{\frac{5}{16\pi}}\,(3\cos^2\theta - 1)$

$Y_2^{\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\,e^{\pm i\phi}, \quad Y_2^{\pm2} = \mp\sqrt{\frac{15}{32\pi}}\sin^2\theta\,e^{\pm2i\phi}$

$E_F = \frac{\hbar^2}{2m}\left(\frac{3Nq\pi^2}{V}\right)^{2/3} = \frac{\hbar^2k_F^2}{2m}, \quad dE = \frac{\hbar^2k^2}{2m}\ \frac{V}{\pi^2}\,k^2dk, \quad \rho(\omega) = \frac{\hbar\omega^3}{\pi^2c^3(e^{\hbar\omega/k_BT}-1)}$

$n(\epsilon) = \begin{cases} \frac{1}{e^{(\epsilon-\mu)/k_BT}+1} & \text{Fermi}-\text{Dirac} \\ \frac{1}{e^{(\epsilon-\mu)/k_BT}-1} & \text{Bose}-\text{Einstein} \end{cases}, \quad \int_0^\infty dx\frac{x^{\nu-1}}{e^{ax}\pm1} = a^{-\nu}\Gamma(\nu)\zeta(\nu)\begin{cases} 1-2^{1-\nu} \\ 1 \end{cases}$

$E_n^1 = \langle\psi_n^0|H'|\psi_n^0\rangle\ , \quad E_n^2 = \sum_{m\neq n}\frac{|\langle\psi_m^0|H'|\psi_n^0\rangle|^2}{E_n^0 - E_m^0}$

$\int_0^\infty x^m\,e^{-bx}\,dx = \frac{m!}{b^{m+1}}, \quad \sin^2\theta = \frac{1}{2}(1 - \cos2\theta)$

$\int\cos(a\theta)\,d\theta = \frac{1}{a}\sin(a\theta)\ , \quad \int\sin(a\theta)\cos(a\theta)\,d\theta = -\frac{\cos^2(a\theta)}{2a},\ e^{i\theta} = \cos\theta + i\sin\theta$

$\hbar = 1.05\times10^{-34}\text{J}\cdot\text{s} = 6.58\times10^{-16}\text{eV}\cdot\text{s}, \quad c = 3\times10^8\,\text{m/s}$

$m_e = 9.11\times10^{-31}\,\text{kg} = 511\,\text{keV/c}^2, \quad 1\,\text{eV} = 1.602\times10^{-19}\ \text{J}$

Name: ________________________________ ID# ________________________________

1. Fill in the blank.

The Fermi-Dirac and Bose-Einstein distributions approach the classical Maxwell-Boltzman distribution when the energy minus the chemical potential is much ____________________________ than the Boltzman constant times the temperature.

When there is only one quantum state for each energy level we can use ________________________________ perturbation theory.

When there is there is more than one quantum state for each energy level we can use non-degenerate perturbation theory if there is an hermitian operator that commutes with both the perturbed and unperturbed Hamiltonians and has distinct ________________________________ for each degenerate quantum state.

For a cold free electron gas the maximum energy of filled electron state is the ________________________________.

The magnitude of the fine structure splitting in hydrogen is proportional to the fine structure constant to the ________________________________ power.

2. Consider a one dimensional harmonic oscillator with a potential

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 = \hbar\omega\left(a_+^{(x)} a_-^{(x)} + \frac{1}{2}\right) ,$$

with a small perturbation

$$H' = b\,\hbar\omega\left(a_+^{(x)} a_+^{(x)} + a_-^{(x)} a_-^{(x)}\right) .$$

To second order in b, what is the energy of the n-th level?

3. Consider a three dimensional harmonic oscillator with a potential

$$H = -\frac{\hbar^2 \nabla^2}{2m} + \frac{1}{2} m\omega^2 (x^2 + y^2 + z^2) = \hbar\omega \left(a_+^{(x)} a_-^{(x)} + a_+^{(y)} a_-^{(y)} + a_+^{(z)} a_-^{(z)} + \frac{3}{2} \right) \ .$$

a) What are the degneracies of the ground state and the first excited state?
Now consider the perturbation

$$H' = b\,\hbar\omega \left(a_+^{(x)} a_-^{(y)} + a_+^{(y)} a_-^{(x)} \right) \ .$$

b) To first order in b, what is the energy of the ground state?
c) To first order in b, how many levels does the first excited state split into and what are their energies

4. A low temperature system of fermions is in thermal equilibrium.
a) Taking the chemical potential to be equal to the Fermi energy, what fraction of states with an energy equal to the Fermi energy should be filled?
b) At $T = 0$ what is the average energy of the fermions expressed in terms of the Fermi momentum.
b) At $T = 0$ what is the average magnitude of the momentum of the fermions expressed in terms of the Fermi momentum.

5. A new force is discovered that produces an attractive short range force between electrons and protons. The potential is given by

$$V(r) = -\frac{\hbar^2 \beta}{M r^2} \ ,$$

where M is a mass.
a) What is the first order correction to the hydrogen energy levels (work to order α^2)? You can use

$$\langle \psi_{n\ell m} | \, \frac{1}{r^2} \, | \psi_{n\ell m} \rangle = \frac{1}{(\ell + 1/2) n^3 a^2} \ .$$

b) How many levels does the second excited state split into?
c) What is the energy of the photon emitted when a transition is made from the lowest $n = 3$ level to the highest $n = 2$ level.

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