15

QM 115B Lec 4

Hydrogen Wave functions

$$\psi_{n\ell m}(r,\theta,\phi) = R_{n\ell}(r)\, Y_\ell^m(\theta,\phi)$$

$$R_{n\ell}(r) \propto \frac{1}{r}\, \rho^{\ell+1} e^{-\rho}\, v(\rho)$$

$\uparrow$ polynomial of degree $j_{max} = n-\ell-1$

$$c_{j+1} = c_j \left(\frac{2(j+\ell+1) - 2n}{(j+1)(j+2\ell+2)} \right)$$

ground state $n=1$ $E = -13.6\,eV$

$n = j_{max}+\ell+1$ $\ell \leq n-1$

$n=1 \Rightarrow \ell = 0$ $m=0$

$j_{max} = 0$

$$c_1 = 0$$

$$\psi_{100} = \frac{c_0}{a} e^{-r/a} Y_0^0(\theta,\phi)$$

$$\int_0^\infty |R_{10}(r)|^2 r^2 dr = \frac{|c_0|^2}{a^2} \int_0^\infty e^{-2r/a} r^2 dr$$

$$= |c_0|^2 \frac{a}{4} = 1$$

$$c_0 = \frac{2}{\sqrt{a}}$$

$$Y_0^0(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$$

$$\psi_{100}(r,\theta,\phi) = \frac{e^{-r/a}}{\sqrt{\pi a^3}}$$

1st excited states $n=2$ $\begin{cases} l=0 & m=0 \\ l=1 & m=0,\pm 1 \end{cases}$

$l=0$ $j_{max} = 2-0-1 = 1$

$$c_1 = -c_0 \qquad c_2 = 0$$

$$V(\rho) = c_0(1-\rho)$$

$$R_{20}(r) = \frac{c_0}{2a}\left(1-\frac{r}{2a}\right)e^{-r/2a}$$

$l=1$ $j_{max} = 2-1-1 = 0$

$$c_1 = 0$$

$$R_{21}(r) = \frac{c_0}{4a^2}\, r\, e^{-r/2a}$$

in general $l = 0, 1, 2, \ldots n-1$

for each l $(2l+1)$ possible ms

degeneracy of level n

$$d(n) = \sum_{l=0}^{n-1}(2l+1) = \sum_{l=0}^{n-1} 1 + 2\sum_{l=0}^{n-1} l$$

$$= n + 2\frac{(n-1)n}{2}$$

$$= n + n^2 - n = n^2$$

$$V(\rho) = L_{n-l-1}^{2l+1}(2\rho)$$

$$L_{q-p}^{p} = (-1)^p\left(\frac{d}{dx}\right)^p L_q$$ associated Laguerre poly.

17

qth Laguerre: $L_q(x) = e^{x}\left(\frac{d}{dx}\right)^q\left(e^{-x}x^q\right)$

$\int \psi^*_{nlm}\,\psi_{n'l'm'}\,r^2\sin\theta\,d\theta\,d\phi = \delta_{nn'}\,\delta_{ll'}\,\delta_{mm'}$

$R(r) = A\,r^{n-1}\,e^{-r/na}$

radial prob. $= r^2 R^2(r) = A^2 r^2\,r^{2n-2}\,e^{-\frac{2r}{na}}$

$= A^2\,r^{2n}\,e^{-2r/na}$

$\frac{d\,Prob}{dr} = A^2\left(2n\,r^{2n-1} + r^{2n}\left(\frac{-2}{na}\right)\right)e^{-\frac{2r}{na}}$

$= 0$

$2n\,r_{max}^{2n-1} = \frac{2\,r_{max}^{n}}{na}$

$n^2 a = r_{max}$

To get expectation value $\langle r \rangle$
need normalization

$$\int_0^\infty R^2 r^2 dr = 1$$

$$\int_0^\infty A^2 r^{2n-2} e^{-2r/na} r^2 dr = 1$$

$$A^2 \int_0^\infty r^{2n} e^{-2r/na} dr = 1$$

$$\int_0^\infty x^m e^{-bx} dx = \frac{m!}{b^{m+1}}$$

$m = 2n$

$b = \frac{2}{na}$

$$A^2 \frac{(2n)!}{\left(\frac{2}{na}\right)^{2n+1}} = 1$$

$$A = \sqrt{\frac{2^{2n+1}}{(na)^{2n+1}(2n)!}}$$

$$\langle r \rangle = \int_0^\infty r A^2 r^{2n} e^{-2r/na} dr$$

$$= A^2 \frac{(2n+1)!}{\left(\frac{2}{na}\right)^{2n+2}} = \frac{2^{2n+1}}{(na)^{2n+1}(2n)!} \frac{(2n+1)!}{\left(\frac{2}{na}\right)^{2n+2}}$$

$$= \frac{(2n+1)na}{2} = \left(n^2 + \frac{n}{2}\right)a$$

$\langle p^2 \rangle$

$$H = \frac{p^2}{2\mu} + V$$

$$p^2 = 2\mu(H - V)$$

$$\langle p^2 \rangle = 2\mu(E - \langle V \rangle)$$