113

QM 115B Lec 25

Prob 11.9 Show the groundstate of hydrogen satifies the integral Schrödinger eq.

$$\psi(r) = \frac{e^{-r/a}}{\sqrt{\pi a^3}} \qquad V = -\frac{\hbar c\alpha}{r} \qquad k = \frac{\sqrt{2mE}}{\hbar}$$

$$= \frac{i}{\hbar}\sqrt{\frac{2m\alpha^2 mc^2}{2}}$$

$$= \frac{i\alpha mc}{\hbar} = \frac{i}{a}$$

$$\psi(\vec{r}) = \frac{-m}{2\pi\hbar^2}\int \frac{e^{ik|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|} V(\vec{r}_0)\,\psi(\vec{r}_0)\,d^3r_0 = I$$

$$I = \frac{-m}{2\pi\hbar^2}\left(\frac{-\hbar c\alpha}{\sqrt{\pi a^3}}\right)\int \frac{e^{-|\vec{r}-\vec{r}_0|/a}}{|\vec{r}-\vec{r}_0|}\,\frac{1}{r_0}\,e^{-r_0/a}\,d^3r_0$$

$\hat{z} \propto \vec{r}$

$$= \frac{-m}{\hbar}\frac{\alpha c}{\sqrt{\pi a^3}}\int \frac{e^{-\sqrt{r^2+r_0^2-2rr_0\cos\theta}/a}}{\sqrt{r^2+r_0^2-2rr_0\cos\theta}}\,\frac{e^{-r_0/a}\,r_0^2}{r_0}\,dr_0\sin\theta\,d\theta$$

$$= \frac{-m}{\hbar}\frac{\alpha c}{\sqrt{\pi a^3}}\int\left(\frac{-a}{rr_0}\right)\left[e^{-\sqrt{r^2+r_0^2-2rr_0\cos\theta}}\right]_0^{\pi} e^{-r_0/a}\,r_0\,dr_0$$

$$= \frac{+m\alpha c a}{r\hbar\sqrt{\pi a^3}}\left(\int_0^\infty e^{-r_0/a}e^{-(r+r_0)/a}\,dr_0 - \int_0^r e^{-r_0/a}e^{-(r-r_0)/a}\,dr_0 - \int_r^\infty e^{-r_0/a}e^{-(r_0-r)/a}\,dr_0\right)$$

$$= \frac{m\alpha c a}{r\hbar\sqrt{\pi a^3}}\left(e^{-r/a}\left[-\frac{a}{2}e^{-2r_0/a}\right]_0^\infty - e^{-r/a}\left[r_0\right]_0^r - e^{+r/a}\left[-\frac{a}{2}e^{-2r_0/a}\right]_r^\infty\right)$$

$$= \frac{m\alpha c a}{r\hbar\sqrt{\pi a^3}}\left(\frac{a}{2}e^{-r/a} - r e^{-r/a} - \frac{a}{2}e^{-r/a}\right)$$

$$I = \frac{1}{r\sqrt{\pi a^3}} \, r e^{-r/a} = \frac{e^{-r/a}}{\sqrt{\pi a^3}}$$

First Born Approx.

Suppose $V(r_0) \to 0$ for large $|\vec{r}_0|$
interested in $\psi(r)$ for large $|\vec{r}|$
integral dominated by $|\vec{r}| \gg |\vec{r}_0|$

$$\int \frac{e^{ik|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|} V(\vec{r}_0) \psi(\vec{r}_0) \, d^3 r_0$$

$$|\vec{r}-\vec{r}_0|^2 = r^2 + r_0^2 - 2\vec{r}\cdot\vec{r}_0 \simeq r^2 \left(1 - 2\frac{\vec{r}\cdot\vec{r}_0}{r^2}\right)$$

$$|\vec{r}-\vec{r}_0| \simeq r\left(1 - \hat{r}\cdot\vec{r}_0\right)$$

$$\vec{k} = k\hat{r}$$

$$e^{ik|\vec{r}-\vec{r}_0|} \simeq e^{ikr - i\vec{k}\cdot\vec{r}_0}$$

$$\frac{e^{ik|\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|} \simeq \frac{e^{ikr}}{r} e^{-i\vec{k}\cdot\vec{r}_0} + \mathcal{O}\left(\frac{r_0}{r}\right)$$

for scattering

incoming $\psi_0(\vec{r}) = A e^{ikz}$

$$\psi(\vec{r}) = A e^{ikz} - \frac{m}{2\pi\hbar^2}\frac{e^{ikr}}{r}\int e^{-i\vec{k},\vec{r}_0} V(\vec{r}_0)\,\psi(\vec{r}_0)\,d^3r_0$$

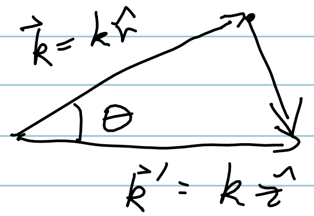
$$f(\theta,\phi) = \frac{-m}{2\pi\hbar^2}\int e^{-i\vec{k},\vec{r}_0} V(\vec{r}_0)\,\psi(\vec{r}_0)\,d^3r_0$$

Born Approx $\psi(\vec{r}_0) \simeq \psi_0(\vec{r}_0) = A e^{ik\cancel{z_0}}$

$$= A e^{i\vec{k}'\cdot\vec{r}_0}$$

$$\vec{k} = k\vec{z}$$

$$f(\theta,\phi) = \frac{-m}{2\pi\hbar^2}\int e^{i(\vec{k}'-\vec{k})\cdot\vec{r}_0} V(\vec{r}_0)\,d^3r_0$$



$\vec{k}' - \vec{k} = \vec{K}$ momentum transfer $\hbar(\vec{k}' - \vec{k})$

$K = 2k\sin\frac{\theta}{2}$

low-energy → small momentum ⇒ long-wavelength

$$f(\theta,\phi) = \frac{-m}{2\pi\hbar^2}\int V(\vec{r}_0)\,d^3r$$

spherically symmetric potential

$\hat{z} \propto \vec{K}$ $\quad (\vec{k}' - \vec{k})\cdot\vec{r}_0 = K r_0\cos\theta$

$$f(\theta) = \frac{-m}{2\pi\hbar^2}\int e^{iKr_0\cos\theta} V(r_0)\,r_0^2\sin\theta\,d\theta\,d\phi\,dr_0$$

$x = \cos\theta \qquad dx = -\sin\theta\,d\theta$

$$= \frac{m}{\hbar^2}\int_0^\infty\int_1^{-1} e^{iKr_0x}\,dx\,V(r_0)\,r_0^2\,dr_0$$

$$= -\frac{2m}{\hbar^2 K}\int_0^\infty \sin(Kr_0)\,V(r_0)\,r_0\,dr_0$$

Yukawa Potential

$$V(r) = \pm \beta \frac{e^{-\mu r}}{r}$$

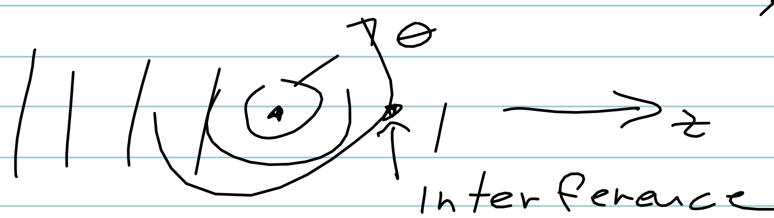
replusive (upper sign), attractive (lower sign)

$$f(\theta) = \mp \frac{2m\beta}{\hbar K} \int_0^\infty e^{-\mu r} \sin(Kr)\, dr$$

$$= \mp \frac{2m\beta}{\hbar^2(\mu^2 + K^2)}$$

rep (upper sign), attractive (lower sign)

$$K = 2k \sin \theta/2$$



$f < 0$ repulsive destructive scatters out

$f > 0$ attractive constructive scatters in

$$\sigma = \int |f(\theta)|^2 d\Omega = \int_0^{2\pi} \int_0^{\pi} |f(\theta)|^2 \sin\theta\, d\theta\, d\phi$$

$$= \frac{4m^2\beta^2 \, 4\pi}{\hbar^4 \mu^2(\mu^2 + 4k^2)} = \frac{16\pi m^2 \beta^2}{\hbar^4 \mu^2(\mu^2 + 4k^2)}$$

$k \to 0$ $\quad \sigma = \dfrac{16\pi m^2\beta^2}{\hbar^4 \mu^4}$

$k \to \infty$ $\quad \sigma = \dfrac{4\pi m^2\beta^2}{\hbar^4 \mu^2 k^2} = \dfrac{4\pi m^2 \beta^2}{\hbar^4 \mu^2 2m \frac{E}{\hbar^2}} = \dfrac{2\pi m \beta^2}{\hbar^2 \mu^2 E}$

(117)

Coulomb : $\beta = \pm \hbar c \alpha$ $\qquad \mu = 0$

$$f(\theta) = \mp \frac{2m \cancel{\hbar} c \alpha}{\hbar^{\cancel{2}} K^2}$$

$$\frac{d\sigma}{d\Omega} = |f|^2 = \frac{4m^2c^2\alpha^2}{\hbar^2 K^4} = \frac{4\alpha^2 m^2 c^2}{\hbar^2 16 k^4 \sin^4(\theta/2)}$$

$$= \frac{\hbar^2 \alpha^2 c^2}{16 E^2 \sin^4(\theta/2)}$$