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QM 115 B Lec 25

Prob 11.9 Show the groundstate of hydrogen satisfies the integral Schrödinger eq.

$$\psi(r) = \frac{e^{-r/a}}{\sqrt{\pi a^3}} \quad V = -\frac{\hbar c \alpha}{r} \quad k = \sqrt{\frac{2mE}{\hbar}}$$

$$= \frac{i}{\hbar} \sqrt{\frac{2m \alpha^2 \hbar c^2}{2}}$$

$$= \frac{i \alpha m c}{\hbar} = \frac{i}{a}$$

$$\psi(\vec{r}) = \frac{-m}{2\pi\hbar^2} \int \frac{e^{i k |\vec{r}-\vec{r}_0|}}{|\vec{r}-\vec{r}_0|} V(\vec{r}_0) \psi(\vec{r}_0) d^3 r_0 = I$$

$$I = \frac{-m}{2\pi\hbar^2} \left(\frac{-\hbar c \alpha}{\sqrt{\pi a^3}} \right) \int \frac{e^{-|\vec{r}-\vec{r}_0|/a}}{|\vec{r}-\vec{r}_0|} \frac{1}{r_0} e^{-r_0/a} d^3 r_0$$

$$= \frac{-m \alpha c}{\hbar \sqrt{\pi a^3}} \int \frac{e^{-\sqrt{r^2+r_0^2-2rr_0\cos\theta}/a}}{\sqrt{r^2+r_0^2-2rr_0\cos\theta}} \frac{1}{r_0} e^{-r_0/a} r_0^2 dr_0 \sin\theta d\theta$$

$$= \frac{-m \alpha c}{\hbar \sqrt{\pi a^3}} \left(\frac{-a}{r r_0} \right) \left[e^{-\sqrt{r^2+r_0^2-2rr_0\cos\theta}} \right]_0^\pi e^{-r_0/a} r_0 dr_0$$

$$= \frac{+m \alpha c a}{\hbar \sqrt{\pi a^3}} \left(\int_0^\infty \int_0^\pi e^{-r_0/a} e^{-(r+r_0)/a} dr_0 - \int_0^r \int_0^\pi e^{-r_0/a} e^{-(r_0-r)/a} dr_0 \right)$$

$$= \frac{m \alpha c a}{\hbar \sqrt{\pi a^3}} \left(e^{-r/a} \left[\frac{-a}{2} e^{-2r_0/a} \right]_0^\infty - e^{-r/a} \left[r_0 \right]_0^r \right)$$

$$+ \frac{m \alpha c a}{\hbar \sqrt{\pi a^3}} \left(e^{+r/a} \left[\frac{-a}{2} e^{-2r_0/a} \right]_0^\infty \right)$$

$$= \frac{m \alpha c a}{\hbar \sqrt{\pi a^3}} \left(\frac{a}{2} e^{-r/a} - r e^{-r/a} - \frac{a}{2} e^{+r/a} \right)$$

$$I = \frac{1}{r \sqrt{\pi a^3}} r e^{-r/a} = \frac{e^{-r/a}}{\sqrt{\pi a^3}}$$

First Born Approx.

Suppose $V(r_0) \rightarrow 0$ for large $|\vec{r}_0|$
 interested in $\psi(r)$ for large $|\vec{r}|$
 integral dominated by $|\vec{r}| \gg |\vec{r}_0|$

$$\int \frac{e^{i\mathbf{k} \cdot (\vec{r} - \vec{r}_0)}}{|\vec{r} - \vec{r}_0|} V(\vec{r}_0) \psi(\vec{r}_0) d^3r_0$$

$$|\vec{r} - \vec{r}_0|^2 = r^2 + r_0^2 - 2\vec{r} \cdot \vec{r}_0 \approx r^2 \left(1 - 2 \frac{\vec{r} \cdot \vec{r}_0}{r^2} \right)$$

$$|\vec{r} - \vec{r}_0| \approx r (1 - \hat{r} \cdot \vec{r}_0)$$

$$\vec{k} = k \hat{r}$$

$$e^{i\mathbf{k} \cdot (\vec{r} - \vec{r}_0)} \approx e^{i k r - i \vec{k} \cdot \vec{r}_0}$$

$$\frac{e^{i\mathbf{k} \cdot (\vec{r} - \vec{r}_0)}}{|\vec{r} - \vec{r}_0|} \approx \frac{e^{i k r}}{r} e^{-i \vec{k} \cdot \vec{r}_0} + \mathcal{O}\left(\frac{r_0}{r}\right)$$

for scattering

incoming $\psi_0(\vec{r}) = A e^{i k z}$

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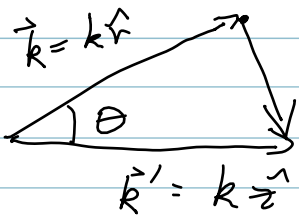
$$\Psi(\vec{r}) = A e^{i k z} - \frac{m}{2\pi\hbar^2} \frac{e^{i k r}}{r} \int e^{-i \vec{k}' \cdot \vec{r}_0} V(\vec{r}_0) \Psi(\vec{r}_0) d^3 r_0$$

$$f(\theta, \phi) = \frac{-m}{2\pi\hbar^2} \int e^{-i \vec{k}' \cdot \vec{r}_0} V(\vec{r}_0) \Psi(\vec{r}_0) d^3 r_0$$

Born Approx $\Psi(\vec{r}_0) = \Psi_0(\vec{r}_0) = A e^{i k z_0}$
 $= A e^{i \vec{k}' \cdot \vec{r}_0}$

$$\vec{k} = k \hat{z}$$

$$f(\theta, \phi) = \frac{-m}{2\pi\hbar^2} \int e^{i(\vec{k}' - \vec{k}) \cdot \vec{r}_0} V(\vec{r}_0) d^3 r_0$$



$$\vec{k}' - \vec{k} = \vec{K}$$

$$K = 2k \sin \frac{\theta}{2}$$

momentum transfer $\hbar(\vec{k}' - \vec{k})$

low energy \rightarrow small momentum \rightarrow long wavelength

$$f(\theta, \phi) = \frac{-m}{2\pi\hbar^2} \int V(\vec{r}_0) d^3 r$$

spherically symmetric potential

$$\hat{z} \propto \vec{K} \quad (\vec{k}' - \vec{k}) \cdot \vec{r}_0 = K r_0 \cos \theta$$

$$f(\theta) = \frac{-m}{2\pi\hbar^2} \int e^{i K r_0 \cos \theta} V(r_0) r_0^2 \sin \theta d\theta d\phi dr_0$$

$x = r_0 \cos \theta \quad dx = -\sin \theta d\theta$

$$= \frac{m}{\hbar^2} \int_0^\infty \int_{-1}^1 e^{i K r_0 x} dx V(r_0) r_0^2 dr_0$$

$$= -\frac{2m}{\hbar^2 K} \int_0^\infty \sin(K r_0) V(r_0) r_0 dr_0$$

Yukawa Potential

$$V(r) = \pm \beta \frac{e^{-\mu r}}{r}$$

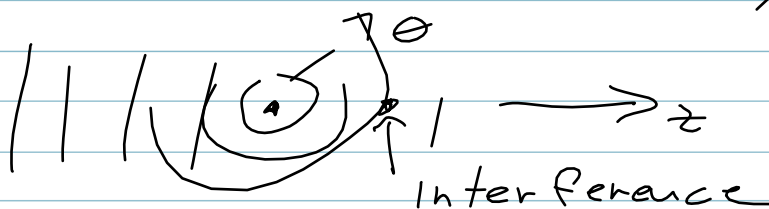
\swarrow repulsive
 \uparrow attractive

$$f(\theta) = \mp \frac{2m\beta}{\hbar k} \int_0^\infty e^{-\mu r} \sin(kr) dr$$

$$= \pm \frac{2m\beta}{\hbar^2 (\mu^2 + k^2)}$$

\swarrow rep
 \uparrow attractive

$$k = 2k \sin \frac{\theta}{2}$$



$f < 0$ repulsive destructive scatters out

$f > 0$ attractive constructive scatters in

$$\sigma = \int |f(\theta)|^2 d\Omega = \int_0^{2\pi} \int_0^\pi |f(\theta)|^2 \sin \theta d\theta d\phi$$

$$= \frac{4m^2 \beta^2}{\hbar^4 \mu^2 (\mu^2 + 4k^2)} 4\pi = \frac{16\pi m^2 \beta^2}{\hbar^4 \mu^2 (\mu^2 + 4k^2)}$$

$$k \rightarrow 0 \quad \sigma = \frac{16\pi m^2 \beta^2}{\hbar^4 \mu^4}$$

$$k \rightarrow \infty \quad \sigma = \frac{4\pi m^2 \beta^2}{\hbar^4 \mu^2 k^2} = \frac{4\pi m^2 \beta^2}{\hbar^4 \mu^2 \frac{2mE}{\hbar^2}} = \frac{2\pi m \beta^2}{\hbar^2 \mu^2 E}$$

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Coulomb: $\beta = \pm \hbar c \alpha$ $\mu = 0$

$$f(\theta) = \mp \frac{2m \hbar c \alpha}{\hbar^2 k^2}$$

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= |f|^2 = \frac{4m^2 c^2 \alpha^2}{\hbar^2 k^4} = \frac{4 \alpha^2 m^2 c^2}{\hbar^2 16 k^4 \sin^4(\theta/2)} \\ &= \frac{\hbar^2 \alpha^2 c^2}{16 E^2 \sin^4(\theta/2)} \end{aligned}$$