(99)

QM 115B Lec. 22

only nonzero matrix elements

$\langle 100|z|210\rangle \quad \langle 100|x|21\pm1\rangle \quad \langle 100|y|21\pm1\rangle$

$$\langle 100|z|210\rangle = \left(\frac{2^7\sqrt{2}}{3^5}\right)a$$

$$\langle 100|x|21\pm1\rangle = \mp\frac{2^7}{3^5}a$$

$$\langle 100|y|21\pm1\rangle = -i\frac{2^7}{3^5}a$$

$$\langle 100|\vec{r}|200\rangle = 0 \qquad \langle 100|\vec{r}|210\rangle = \frac{2^7\sqrt{2}}{3^5}\hat{z}$$

$$\langle 100|\vec{r}|21\pm1\rangle = \frac{2^7}{3^5}a\left(\mp\hat{x} - i\hat{y}\right)$$

$|200\rangle \to |100\rangle$: $\vec{p}^{\,2} = 0$

$$\left.\begin{array}{l}|210\rangle \to |100\rangle \\ |21\pm1\rangle \to |100\rangle\end{array}\right\} \quad |\vec{p}|^2 = |p_x|^2 + |p_y|^2 + |p_z|^2 = (ea)^2\frac{2^{15}}{3^{10}}$$

$$\omega_0 = \frac{E_2 - E_1}{\hbar} = \frac{1}{\hbar}\left(\frac{E_1}{4} - E_1\right) = -\frac{3}{4\hbar}E_1$$

$$A = \frac{\omega_0^3|p|^2}{3\pi\epsilon_0\hbar c^3} = -\frac{3^3E_1^3}{2^6\hbar^3}e^2a^2\frac{2^{15}}{3^{10}}\frac{1}{3\pi\epsilon_0\hbar c^3}$$

$$= \frac{2^9}{3^8}\left(\frac{-\hbar^2}{2ma^2}\right)\frac{E_1^2e^2a^2\,4\alpha}{c^2} = \frac{2^{10}\alpha^4m^2c^4\alpha}{\hbar\, mc^2}$$

$\alpha = \dfrac{e^2}{4\pi\epsilon_0\hbar c}$

$$A = \frac{2^{10}}{3^8} \frac{\alpha^5 mc^2}{\hbar} = .15 \frac{\alpha^4 c}{a}$$

$$= 6.27 \times 10^8 \ 1/s$$

$$\Delta E = \frac{\hbar}{\tau} = \hbar A = .15 \alpha^3 2 E_1$$
$$= 10^{-7} E_1$$

$$\tau = \frac{1}{A} = 1.6 \times 10^{-9} \ s \qquad \text{for } n=2 \ \ell=1$$

$$\tau = \infty \qquad \text{for } n=2 \ \ell=0$$

two photon emission $\tau = \frac{1}{15}$ s

Selection Rules

when is $\langle n'\ell'm' | \vec{r} | n\ell m \rangle = 0$?

$$[L_z, z] = 0$$

$$0 = \langle n'\ell'm' | L_z z - z L_z | n\ell m \rangle$$
$$= \hbar \langle n'\ell'm' | m' z - z m | n\ell m \rangle$$
$$= \hbar (m'-m) \langle n'\ell'm' | z | n\ell m \rangle$$
$$m' = m \quad \text{or} \quad \langle n'\ell'm' | z | n\ell m \rangle = 0$$

$$[L_z, x] = i\hbar y$$
$$\langle n'\ell'm' | [L_z, x] | n\ell m \rangle = \langle n'\ell'm' | L_z x - x L_z | n\ell m \rangle$$
$$i\hbar \langle n'\ell'm' | y | n\ell m \rangle = \hbar (m'-m) \langle n'\ell'm' | x | n\ell m \rangle$$

$$[L_z, y] = -i\hbar x$$
$$-i\hbar \langle n'\ell'm' | x | n\ell m \rangle = \hbar (m'-m) \langle n'\ell'm' | y | n\ell m \rangle$$

$$(m'-m)^2 = 1 \quad \text{or} \quad \langle n'\ell'm' | x | n\ell m \rangle = 0$$
$$\langle n'\ell'm' | y | n\ell m \rangle = 0$$

(1.1)

dipole

So transitions allowed only if $\Delta m = \pm 1, 0$

$$[L^2, [L^2, \vec{r}]] = 2\hbar^2 (\vec{r} L^2 + L^2 \vec{r})$$

need $\Delta l = \pm 1$

photon carries spin 1

Prob. 9.20 Magnetic Resonance

spin ½ particle gyromagnetic ratio γ

$$\vec{B} = B_{rf}\cos(\omega t)\,\hat{x} - B_{rf}\sin(\omega t)\,\hat{y} + B_0\,\hat{z}$$

radio static

Larmor prec. $\omega_0 = \gamma B_0$

$$H = -\gamma \vec{B}\cdot\vec{S} = -\frac{\gamma\hbar}{2}\left(B_x\sigma_x + B_y\sigma_y + B_z\sigma_z\right)$$

$$= -\frac{\gamma\hbar}{2}\left(B_x\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + B_y\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} + B_z\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right)$$

$$= -\frac{\gamma\hbar}{2}\begin{pmatrix}B_0 & B_{rf}(\cos\omega t + i\sin\omega t)\\ B_{rf}(\cos\omega t - i\sin\omega t) & -B_0\end{pmatrix}$$

$$= -\frac{\gamma\hbar}{2}\begin{pmatrix}B_0 & B_{rf}\,e^{i\omega t}\\ B_{rf}\,e^{-i\omega t} & -B_0\end{pmatrix}$$

at time t: $\quad \chi(t) = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix}$

$$i\hbar \dot{\chi} = H\chi$$

$$i\hbar \begin{pmatrix} \dot{a} \\ \dot{b} \end{pmatrix} = -\frac{\gamma\hbar}{2} \begin{pmatrix} B_0 & B_{rf} e^{i\omega t} \\ B_{rf} e^{-i\omega t} & -B_0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$= -\frac{\gamma\hbar}{2} \begin{pmatrix} B_0 a + B_{rf} e^{i\omega t} b \\ B_{rf} e^{-i\omega t} a - B_0 b \end{pmatrix}$$

$$\dot{a} = \frac{i\gamma}{2}\left(B_0 a + B_{rf} e^{i\omega t} b\right) = \frac{i}{2}\left(\underbrace{\Omega}_{=\gamma B_{rf}} e^{i\omega t} b + \omega_0 a\right)$$

$$\dot{b} = \frac{i}{2}\left(\Omega e^{-i\omega t} a - \omega_0 b\right)$$

$$a(t) = \left(a_0 \cos(\omega' t/2) + \frac{i}{\omega'}\left(a_0(\omega_0 - \omega) + b_0 \Omega\right) \sin\frac{\omega' t}{2}\right) e^{\frac{i\omega t}{2}}$$

$$b(t) = \left(b_0 \cos(\omega' t/2) + \frac{i}{\omega'}\left(-b_0(\omega_0 - \omega) + a_0 \Omega\right) \sin\frac{\omega' t}{2}\right) e^{-\frac{i\omega t}{2}}$$

$$\omega' = \sqrt{(\omega - \omega_0)^2 + \Omega^2} \qquad a(t=0) = a_0$$

check: $\dot{a}(t) = \left(-\frac{\omega'}{2} a_0 \sin\left(\frac{\omega' t}{2}\right) + \frac{i}{2}\left(a_0(\omega_0 - \omega) + b_0\Omega\right)\cos\frac{\omega' t}{2}\right) e^{\frac{+i\omega t}{2}}$

$$+ \frac{i\omega}{2} a(t)$$

$$= \frac{i}{2} e^{\frac{i\omega t}{2}} \Big(\omega a_0 \cos\frac{\omega' t}{2} + \frac{i\omega}{\omega'}\left(a_0(\omega_0 - \omega) + b_0\Omega\right)\sin\frac{\omega' t}{2}$$
$$+ i\omega' a_0 \sin\frac{\omega' t}{2} + \left(a_0(\omega_0 - \omega) + b_0\Omega\right)\cos\frac{\omega' t}{2}$$

$(*)$

compare with Schr. eq.

$$\dot a = \frac{i}{2}\left(\Omega e^{i\omega t} b + \omega_0 a\right)$$

$$= \frac{i}{2} e^{i\omega t/2}\Big(\Omega b_0 \cos\frac{\omega' t}{2} + i\frac{\Omega}{\omega'}\big(-b(\omega_0-\omega)+a_0\Omega\big)\sin\frac{\omega' t}{2}$$

$$+ \omega_0 a_0 \cos\frac{\omega' t}{2} + i\frac{\omega_0}{\omega'}\big(a_0(\omega_0-\omega)+b_0\Omega\big)\sin\frac{\omega' t}{2}\Big)$$

$\cos\frac{\omega' t}{2}$ terms match

from (*)

$\sin\frac{\omega' t}{2}$ terms: $\left(\frac{i\omega}{\omega'}\big(a_0(\omega_0-\omega)+b_0\Omega\big)+i\omega' a_0\right)\omega'$

$$\overset{?}{=} i\Omega\big(-b_0(\omega_0-\omega)+a_0\Omega\big)+i\omega_0\big(a_0(\omega_0-\omega)+b_0\Omega\big)$$

$$a_0\left(\omega(\omega_0-\omega)+\omega'^2-\Omega^2-\omega_0^2+\omega_0\omega\right) \overset{?}{=} b_0\left(-\Omega(\omega_0-\omega)+\omega_0\Omega-\omega\Omega\right)$$

$$a_0\left(\omega(\omega_0-\omega)+(\omega_0-\omega)^2-\omega_0^2+\omega_0\omega\right) \overset{?}{=} 0$$

$$a_0\left(2\omega\omega_0-\omega^2+\omega_0^2-2\omega_0\omega_0+\omega^2-\omega_0^2\right) = 0$$

solution works