

9.)

QM 115B Lec. 20
 Time Dependent Perturbation Theory
 Two level System

$$H^0 \psi_a = E_a \psi_a, \quad H^0 \psi_b = E_b \psi_b$$

$$\langle \psi_a | \psi_b \rangle = \delta_{ab}$$

$$\psi(t) = c_a \psi_a e^{-iE_a t/\hbar} + c_b \psi_b e^{-iE_b t/\hbar}$$

$$|c_a|^2 + |c_b|^2 = 1$$

time dependent perturbation $H'(t)$
 $H = H^0 + H'$

$$\psi(t) = c_a(t) \psi_a e^{-iE_a t/\hbar} + c_b(t) \psi_b e^{-iE_b t/\hbar}$$

$$H \psi = i \hbar \frac{\partial}{\partial t} \psi$$

$$\begin{aligned} & c_a E_a \psi_a e^{-iE_a t/\hbar} + c_b E_b \psi_b e^{-iE_b t/\hbar} \\ & + c_a H' \psi_a e^{-iE_a t/\hbar} + c_b H' \psi_b e^{-iE_b t/\hbar} \\ = i \hbar & \left(\dot{c}_a \psi_a e^{-iE_a t/\hbar} + \dot{c}_b \psi_b e^{-iE_b t/\hbar} \right. \\ & \left. + c_a \psi_a \left(\frac{-iE_a}{\hbar} \right) e^{-iE_a t/\hbar} + c_b \psi_b \left(\frac{-iE_b}{\hbar} \right) e^{-iE_b t/\hbar} \right) \end{aligned}$$

$$c_a H'(t) \psi_a e^{-iE_a t/\hbar} + c_b H'(t) \psi_b e^{-iE_b t/\hbar} = i \hbar \left(\dot{c}_a \psi_a e^{-iE_a t/\hbar} + \dot{c}_b \psi_b e^{-iE_b t/\hbar} \right)$$

inner prod ψ_a

$$c_a \langle \psi_a | H' | \psi_a \rangle e^{-iE_a t/\hbar} + c_b \langle \psi_a | H' | \psi_b \rangle e^{-iE_b t/\hbar} = i \hbar \dot{c}_a e^{-iE_a t/\hbar}$$

$$H'_{ij} = \langle \psi_i | H' | \psi_j \rangle$$

$$\dot{C}_a = -\frac{i}{\hbar} \left(C_a H'_{aa} + C_b H'_{ab} e^{-i(E_b - E_a)t/\hbar} \right)$$

$$\dot{C}_b = -\frac{i}{\hbar} \left(C_b H'_{bb} + C_a H'_{ba} e^{-i(E_a - E_b)t/\hbar} \right)$$

assume $H'_{aa} = H'_{bb} = 0$ $\omega_0 = \frac{E_b - E_a}{\hbar}$

$$\dot{C}_a = -\frac{i}{\hbar} H'_{ab} e^{-i\omega_0 t} C_b \quad \dot{C}_b = -\frac{i}{\hbar} H'_{ba} e^{i\omega_0 t} C_a$$

Suppose $C_a(0) = 1$ $C_b(0) = 0$

for small H' : zeroth order $C_a^{(0)}(t) = 1$ $C_b^{(0)}(t) = 0$

1st order $\frac{d}{dt} C_a^{(1)} = -\frac{i}{\hbar} H'_{ab} e^{-i\omega_0 t} C_b^{(0)} = 0 \Rightarrow C_a^{(1)}(t) = 0$

$$\frac{d}{dt} C_b^{(1)} = -\frac{i}{\hbar} H'_{ba} e^{i\omega_0 t} C_a^{(0)}$$

$$C_b^{(1)} = -\frac{i}{\hbar} \int_0^t H'_{ba}(t') e^{i\omega_0 t'} dt'$$

2nd order $\frac{d}{dt} C_a^{(2)} = -\frac{i}{\hbar} H'_{ab} e^{-i\omega_0 t} \left(-\frac{i}{\hbar} \int_0^t H'_{ba}(t') e^{i\omega_0 t'} dt' \right)$

$$C_a^{(2)} = -\frac{1}{\hbar^2} \int_0^t H'_{ab}(t') e^{-i\omega_0 t'} \int_0^{t'} H'_{ba}(t'') e^{i\omega_0 t''} dt'' dt'$$

etc.

Sinusoidal Perturbations

$$H'(\vec{r}, t) = V(\vec{r}) \cos \omega t$$

$$H'_{ab} = V_{ab} \cos \omega t$$

$$V_{ab} = \langle \psi_a | V(\vec{r}) | \psi_b \rangle \quad \text{assume } V_{aa} = 0$$

$$\begin{aligned} \text{first order } c_b(t) &\approx -\frac{i}{\hbar} V_{ba} \int_0^t \cos(\omega t') e^{i\omega_0 t'} dt' \\ &= -\frac{i V_{ba}}{2\hbar} \int_0^t \left(e^{i(\omega_0 + \omega)t'} + e^{i(\omega_0 - \omega)t'} \right) dt' \\ &= -\frac{i V_{ba}}{2\hbar} \left[\frac{e^{i(\omega_0 + \omega)t'}}{i(\omega_0 + \omega)} + \frac{e^{i(\omega_0 - \omega)t'}}{i(\omega_0 - \omega)} \right] \\ &= -\frac{V_{ba}}{2\hbar} \left[\frac{e^{i(\omega_0 + \omega)t} - 1}{\omega_0 + \omega} + \frac{e^{i(\omega_0 - \omega)t} - 1}{\omega_0 - \omega} \right] \end{aligned}$$

~~near~~ near resonance $\omega_0 + \omega \gg |\omega_0 - \omega|$

$$c_b(t) \approx -\frac{V_{ba}}{2\hbar} \frac{e^{i(\omega_0 - \omega)t/2}}{\omega_0 - \omega} \left(e^{i(\omega_0 - \omega)t/2} - e^{-i(\omega_0 - \omega)t/2} \right)$$

$$\approx -\frac{i V_{ba}}{\hbar} \frac{\sin \left[(\omega_0 - \omega)t/2 \right]}{\omega_0 - \omega} e^{i(\omega_0 - \omega)t/2}$$

$$P_{a \rightarrow b}(t) = |c_b(t)|^2 \approx \frac{|V_{ab}|^2}{\hbar^2} \frac{\sin^2 \left[(\omega_0 - \omega)t/2 \right]}{(\omega_0 - \omega)^2}$$