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QM 115B Lec 17

Fine structure of Hydrogen

$$E_n^0 = -\frac{\alpha^2 mc^2}{2n^2}$$

relativistic corrections
spin-orbit $\Big\}\ \alpha^4 mc^2$

Lamb shift $\quad \alpha^5 mc^2$
hypefine splitting $\quad \frac{m}{m_p}\alpha^4 mc^2$

$$KE = \frac{p^2}{2m}$$

$$KE = \sqrt{p^2c^2+m^2c^4} - mc^2$$

$pc \ll mc^2$

$$KE \simeq mc^2\left(\sqrt{1+\frac{p^2c^2}{m^2c^4}} - 1\right)$$

$$= mc^2\left(1+\frac{p^2c^2}{2mc^2} - \frac{p^4}{8m^4c^4} + \dots - 1\right)$$

$$\simeq \frac{p^2}{2m} - \frac{p^4}{8m^3c^2}$$

$$H'_{rel} = \frac{-p^4}{8m^3c^2}$$

$[H'_{rel}, L^2] = 0 \quad [H'_{rel}, L_z] = 0$

nlm are "good" qu.
non-degen. p.t.

$$E'_{rel} = \frac{-1}{8m^3c^2}\langle \psi^0 | p^4 | \psi^0\rangle$$

$$p^2\psi = 2m(E^0 - V)\psi^0$$

$$E'_{rel} = \frac{-1}{2mc^2}\langle \psi^0 | (E^0-V)^2 | \psi^0\rangle$$

$$= \frac{-1}{2mc^2}\langle \psi^0 | E^{0\,2} - 2E^0 V + V^2 | \psi^0\rangle$$

$$V = -\frac{\hbar c\alpha}{r} \qquad E_n = -\frac{\alpha^2 mc^2}{2n^2}$$

hydrogen $E'_{rel} = \frac{-1}{2mc^2}\Big((E_n^0)^2 + 2E_n^0 \hbar c\alpha \langle \psi^0_{n\ell m} | \frac{1}{r} | \psi^0_{n\ell m}\rangle$
$+ (\hbar c\alpha)^2 \langle \psi^0_{n\ell m} | \frac{1}{r^2} | \psi^0_{n\ell m}\rangle\Big)$

$$\langle \psi^0_{n\ell m} | \frac{1}{r} | \psi^0_{n\ell m}\rangle = \frac{1}{n^2 a}$$

$$\langle \psi^0_{n\ell m} | \frac{1}{r^2} | \psi^0_{n\ell m}\rangle = \frac{1}{(\ell+1/2)n^3 a^2}$$

$$E'_{rel} = \frac{-1}{2mc^2}\left((E_n^0)^2 + \frac{2E_n^0}{n^2} \frac{\hbar c\alpha \, \alpha mc}{\hbar} + \frac{(\hbar c\alpha)^2 (\alpha mc)^2}{(\ell+1/2)n^3 \hbar^2}\right)$$

$$= \frac{-1}{2mc^2}\left(E_n^{0\,2} + E_n^0(-4E_n^0) + \frac{4n}{(\ell+1/2)} E_n^{0\,2}\right)$$

$$= -\frac{E_n^0}{2mc^2}\left(\frac{4n}{(\ell+1/2)} - 3\right)$$

$$\left|\frac{E_n^0}{mc^2}\right| = \frac{\alpha^2}{2n^2} \simeq 3\times 10^{-5}$$

for Z protons $\left|\frac{E_n^0}{mc^2}\right| \propto \frac{Z^2\alpha^2}{2n^2}$

Spin Orbit Correction

$$\vec{B} = -\frac{1}{c}\vec{v}\times\vec{E} = -\frac{1}{mc}\vec{p}\times\vec{E}$$

$$= \frac{1}{mc}\vec{E}\times\vec{p}$$

$$\vec{E} = \frac{e\,\vec{r}}{4\pi\epsilon_0 c r^3} \Rightarrow \vec{B} = \frac{e\,\vec{L}}{4\pi\epsilon_0 m c^2}$$

$$H' = -\vec{\mu}\cdot\vec{B}$$

↑ magnetic dipole

$$\vec{\mu} = -\frac{g_e e}{2m}\vec{S}$$

↙ Thomas precession

$$H'_{so} = (g_e - 1)\,\frac{e}{m}\,\frac{e}{8\pi\epsilon_0 m c^2 r^3}\,\vec{L}\cdot\vec{S}$$

$$= \frac{\alpha\hbar}{2m^2 c r^3}\,\vec{L}\cdot\vec{S}$$

$$\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c}$$

$[H'_{so}, \vec{L}] \neq 0 \qquad [H'_{so}, \vec{S}] \neq 0$

$[H'_{so}, L^2] = 0 \qquad [H'_{so}, S^2] = 0$

$\vec{J} = \vec{L} + \vec{S}$

$[H'_{so}, \vec{J}] = 0$

simultaneous q. n. E, L^2, S^2, J^2, J_z

$J^2 = (\vec{L} + \vec{S})^2 = L^2 + S^2 + 2\vec{L}\cdot\vec{S}$

$\vec{L}\cdot\vec{S} = \frac{1}{2}(J^2 - L^2 - S^2)$

eigenvalues $\frac{\hbar^2}{2}\left(j(j+1) - l(l+1) - s(s+1)\right)$

Prob 6.35(c)

$\langle \psi^0_{n,l,j,j_m} | \frac{1}{r^3} | \psi^0_{n,l,j,j_m} \rangle$

$= \frac{1}{l(l+\frac{1}{2})(l+1)n^3a^3}$

$$\langle \psi^0_{n\ell j,jm} \,|\, H'_{so} \,|\, \psi^0_{n\ell jm} \rangle$$

$$= \frac{\alpha \hbar}{2m^2c} \, \frac{\hbar^2}{2} \, \frac{\left(j(j+1) - \ell(\ell+1) - s(s+1) \right)}{\ell(\ell+\frac{1}{2})(\ell+1)\, n^3 \left(\frac{\hbar}{\alpha m c} \right)^3}$$

$$= \frac{(E_n^0)^2 n}{mc^2} \, \frac{\left(j(j+1) - \ell(\ell+1) - s(s+1) \right)}{\ell(\ell+\frac{1}{2})(\ell+1)}$$

$$E'_{fs} = \langle H'_{rel} \rangle + \langle H'_{so} \rangle$$

$$= \frac{(E_n^0)^2}{2mc^2} \left(3 - \frac{4n}{j+\frac{1}{2}} \right)$$

$$E_{nj} \cong E_n^0 + E'_{fs}$$

$$= -\frac{\alpha^2 mc^2}{2n^2} \left(1 + \frac{\alpha^2}{n^2} \left(\frac{n}{j+\frac{1}{2}} - \frac{3}{4} \right) \right)$$