QM 115B Lec 11

in thermal equilibrium every distinct state with same total is equally probable

system with one particle energies $E_1, E_2, E_3 \ldots$
degeneracies $d_1, d_2, d_3 \ldots$
configuration: occupation # $N_1, N_2, N_3 \ldots$

how many distinct states for configuration

$$Q(N_1, N_2, N_3 \ldots, E)$$

I) distinguishable
first energy bin
N choose N_1 $\qquad \binom{N}{N_1} = \frac{N!}{N_1!\,(N-N_1)!}$

arrange them $d_1^{N_1}$ ways

$$Q = \frac{N!\, d_1^{N_1}}{N_1!\,(N-N_1)!} \times \frac{(N-N_1)!\, d_2^{N_2}}{N_2!\,(N-N_1-N_2)!} \times \ldots$$

$$= N! \frac{d_1^{N_1}\, d_2^{N_2}\, d_3^{N_3}}{N_1!\, N_2!\, N_3!} \ldots = N! \prod_{n=1}^{\infty} \frac{d_n^{N_n}}{N_n!}$$

II) Identical Fermions

antisymmetrize wavefunctions

one N particle state

with given occupation #s

$\binom{d_n}{N_n}$ ways to put N_n in nth E bin

$$Q(N_1, N_2, \dots) = \prod_{n=1}^{\infty} \frac{d_n!}{N_n!(d_n - N_n)!}$$

III) Identical Bosons

symmetrize wavefunction

one N particle state with given set of occupation #'s

no restriction on how many particles in state

for nth E bin how many ways to put N_n identical particles in d_n slots?

$d_n = 5$ $N_n = 7$ • • × • × • • • × • ×

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N_n dots $d_n - 1$ crosses

distinguishable $(N_n + d_n - 1)!$ arrangements

indistinguishable $\dfrac{(N_n + d_n - 1)!}{N_n!\,(d_n - 1)!}$

$$Q(N_1, N_2, N_3, \dots) = \prod_{n=1}^{\infty} \frac{(N_n + d_n - 1)!}{N_n!\,(d_n - 1)!}$$

Most Probable Configuration

max of $Q(N_1, N_2, N_3 \ldots)$

subject to $\sum_{n=1}^{\infty} N_n = N$ $\quad \sum_{n=1}^{\infty} N_n E_n = E$

$$G = \ln Q + \alpha \underbrace{\left(N - \sum_{n=1}^{\infty} N_n\right)}_{= C_N} + \beta \underbrace{\left(E - \sum_{n=1}^{\infty} N_n E_n\right)}_{= C_E}$$

I) distinguishable

$$G = \ln N! + \sum_{n=1}^{\infty} N_n \ln d_n - \ln N_n! + \alpha C_N + \beta C_E$$

stirlings approx $\ln z! \simeq z \ln z - z$, $z >> 1$

$$G \simeq \sum_{n=1}^{\infty} \left(N_n \ln d_n - N_n \ln N_n + N_n - \alpha N_n - \beta N_n E_n \right)$$
$$+ \ln N! + \alpha N + \beta E$$

$$\frac{\partial G}{\partial N_n} = \ln d_n - \ln N_n - \alpha - \beta E_n = 0$$

$$N_n = d_n e^{-(\alpha + \beta E_n)}$$

II) Identical Fermions

$$G = \sum_{n=1}^{\infty} \ln d_n! - \ln N_n! - \ln(d_n - N_n)! + \alpha C_n + \beta C_E$$

$d_n \geq N_n >> 1$, $$G = \sum_{n=1}^{\infty} \left(\ln d_n! - N_n \ln N_n + N_n - (d_n - N_n)\ln(d_n - N_n) + (d_n - N_n) - \alpha N_n - \beta N_n E_n \right)$$
$$+ \alpha N + \beta E$$

$$\frac{\partial G}{\partial N_n} \simeq -\ln N_n + \ln(d_n - N_n) - \alpha - \beta E_n$$

$$N_n \simeq \frac{d_n}{e^{(\alpha + \beta E)} + 1}$$