EFT of Dark Matter Direct Detection With Collective Excitations

Zhengkang "Kevin" Zhang (Caltech)

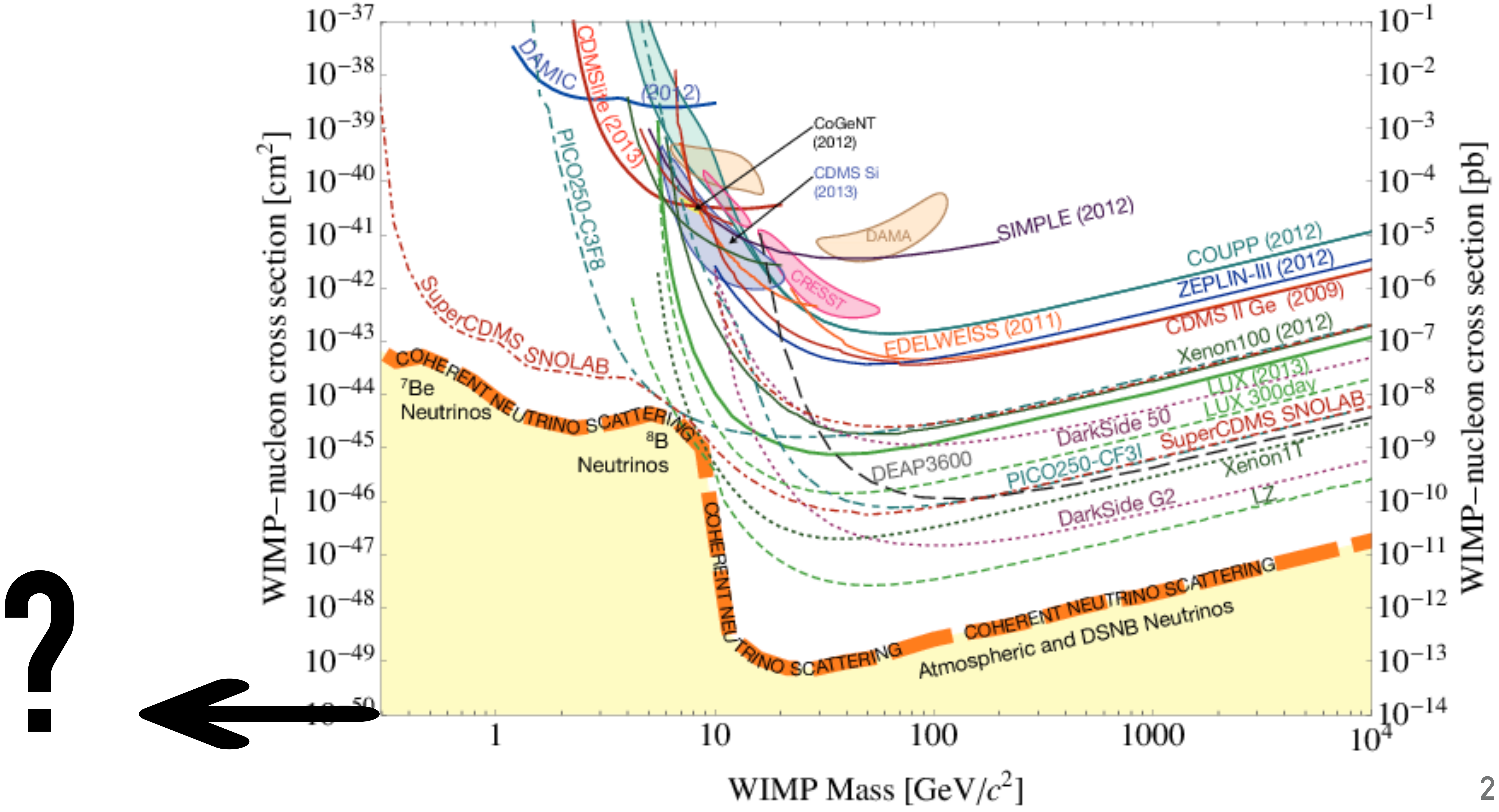
Based on 2009.13534 (w/ Tanner Trickle, Kathryn Zurek)

BURKE INSTITUTE

Caltech

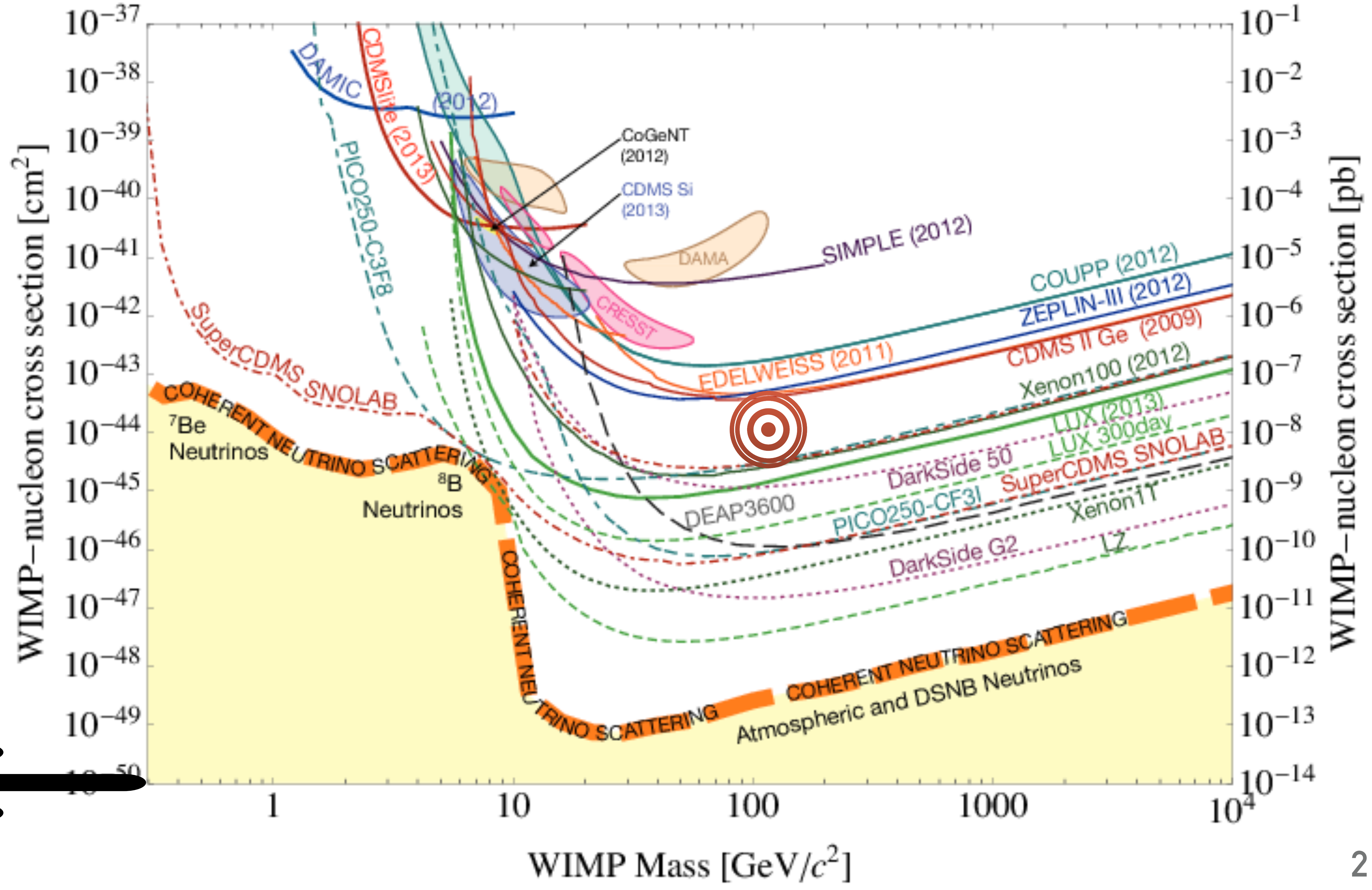
Toward lighter DM in direct detection

- Conventional WIMP searches.
    - Nuclear recoils.
    - Lose sensitivity below DM mass ~GeV.



Toward lighter DM in direct detection

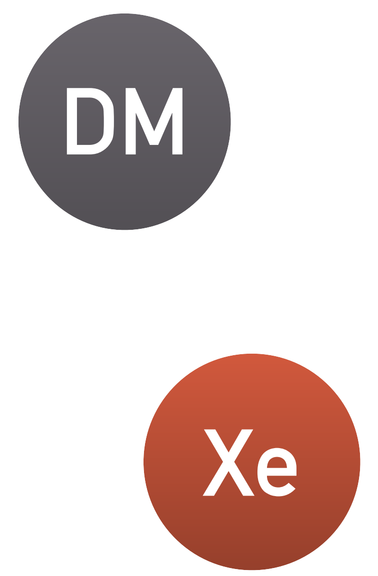
- Conventional WIMP searches.
  - Nuclear recoils.
  - Lose sensitivity below DM mass ~GeV.



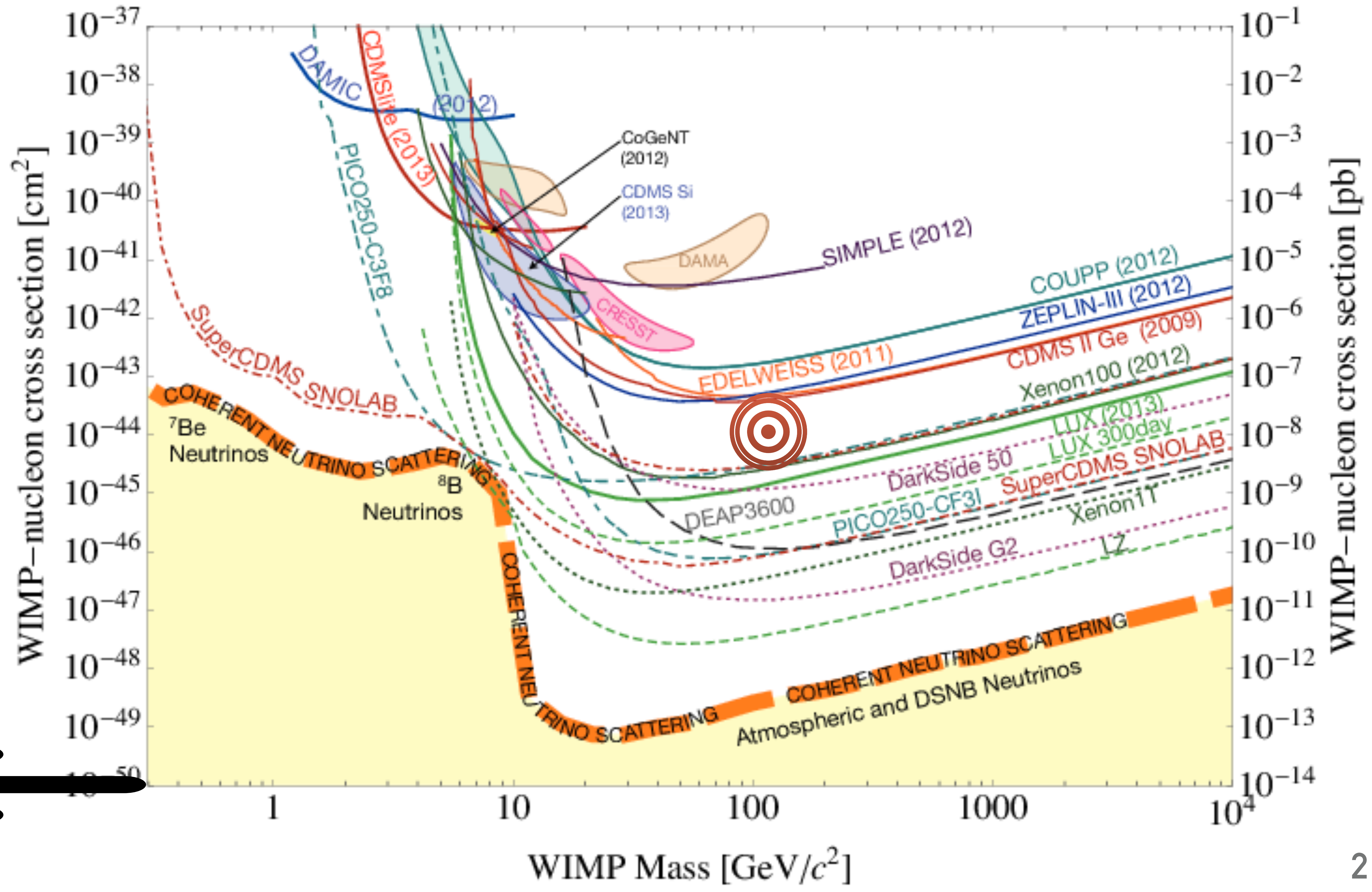
?

Toward lighter DM in direct detection

- Conventional WIMP searches.
  - Nuclear recoils.
  - Lose sensitivity below DM mass ~GeV.



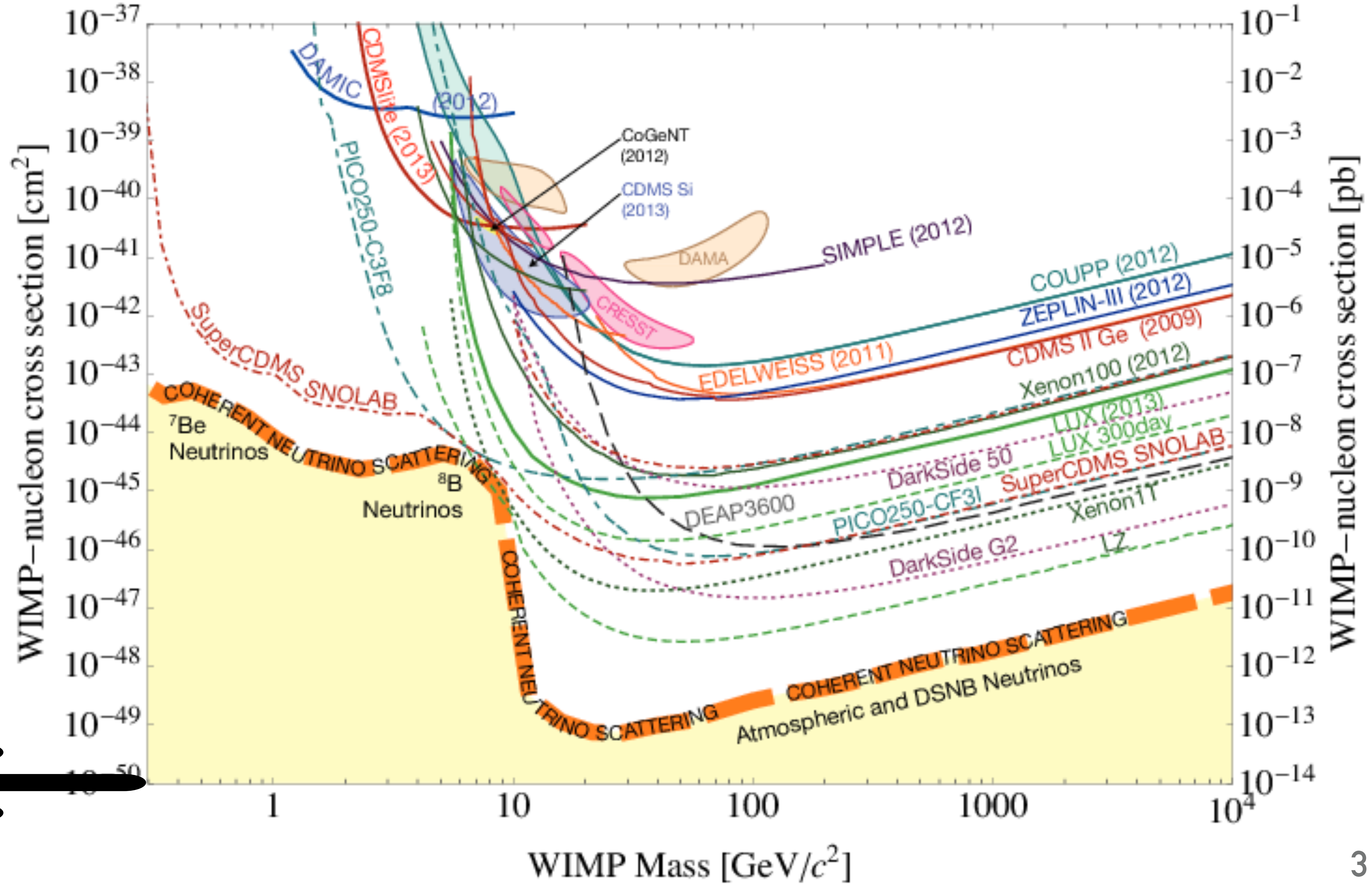
?



Toward lighter DM in direct detection

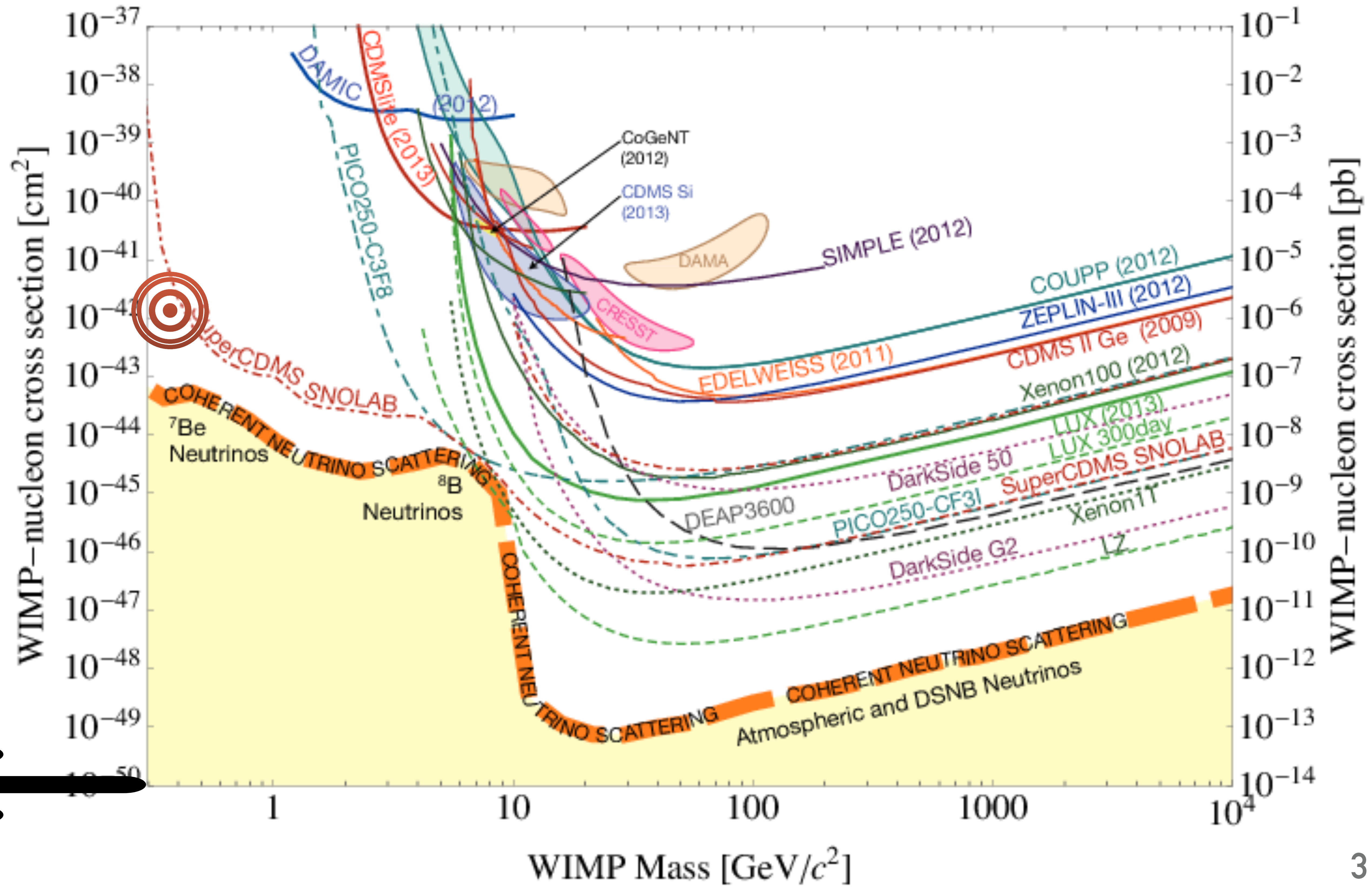
- Conventional WIMP searches.
  - Nuclear recoils.
  - Lose sensitivity below DM mass ~GeV.

?



Toward lighter DM in direct detection

- Conventional WIMP searches.
  - Nuclear recoils.
  - Lose sensitivity below DM mass ~GeV.

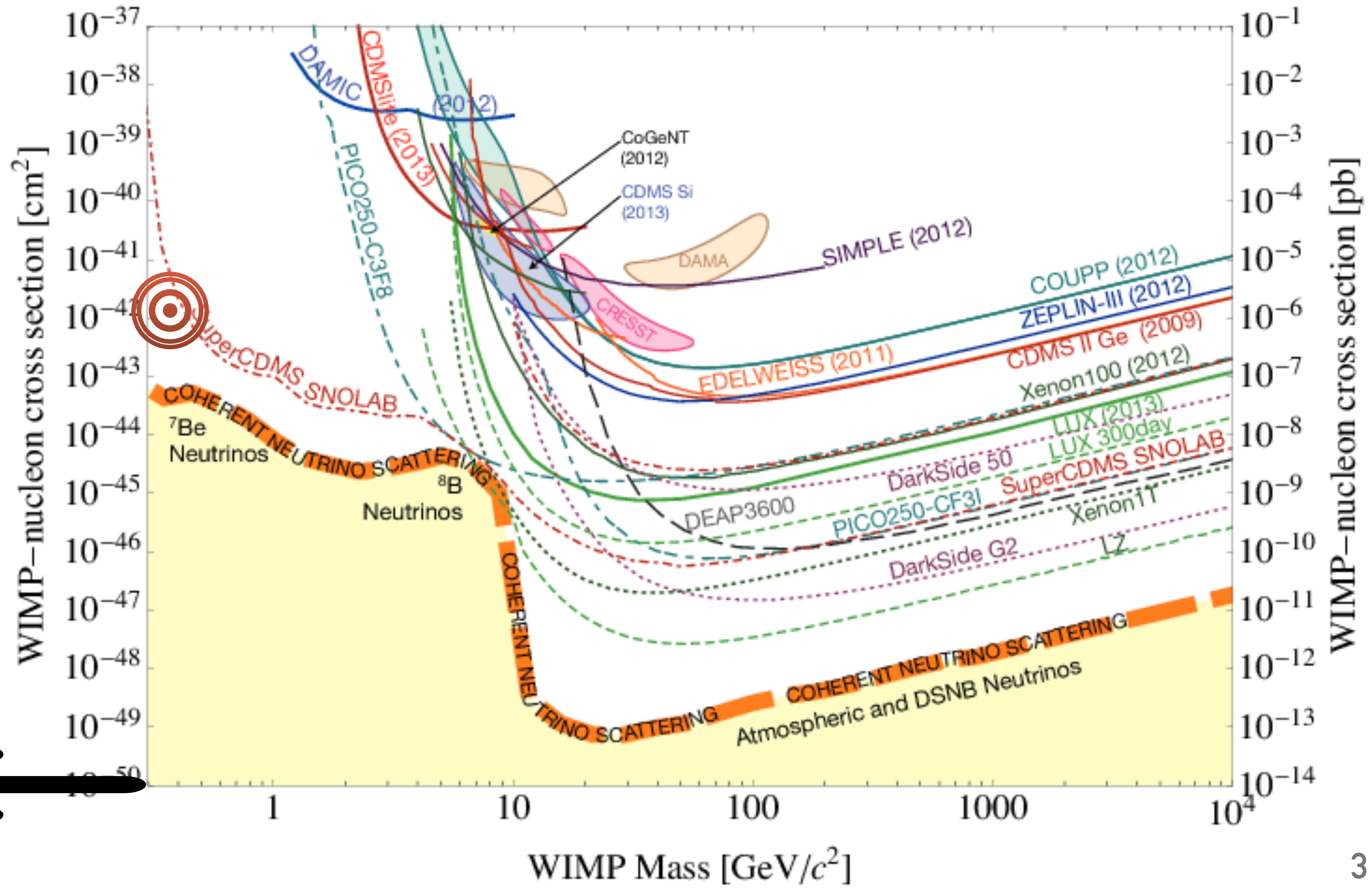


?

Toward lighter DM in direct detection

- Conventional WIMP searches.
  - Nuclear recoils.
  - Lose sensitivity below DM mass ~GeV.

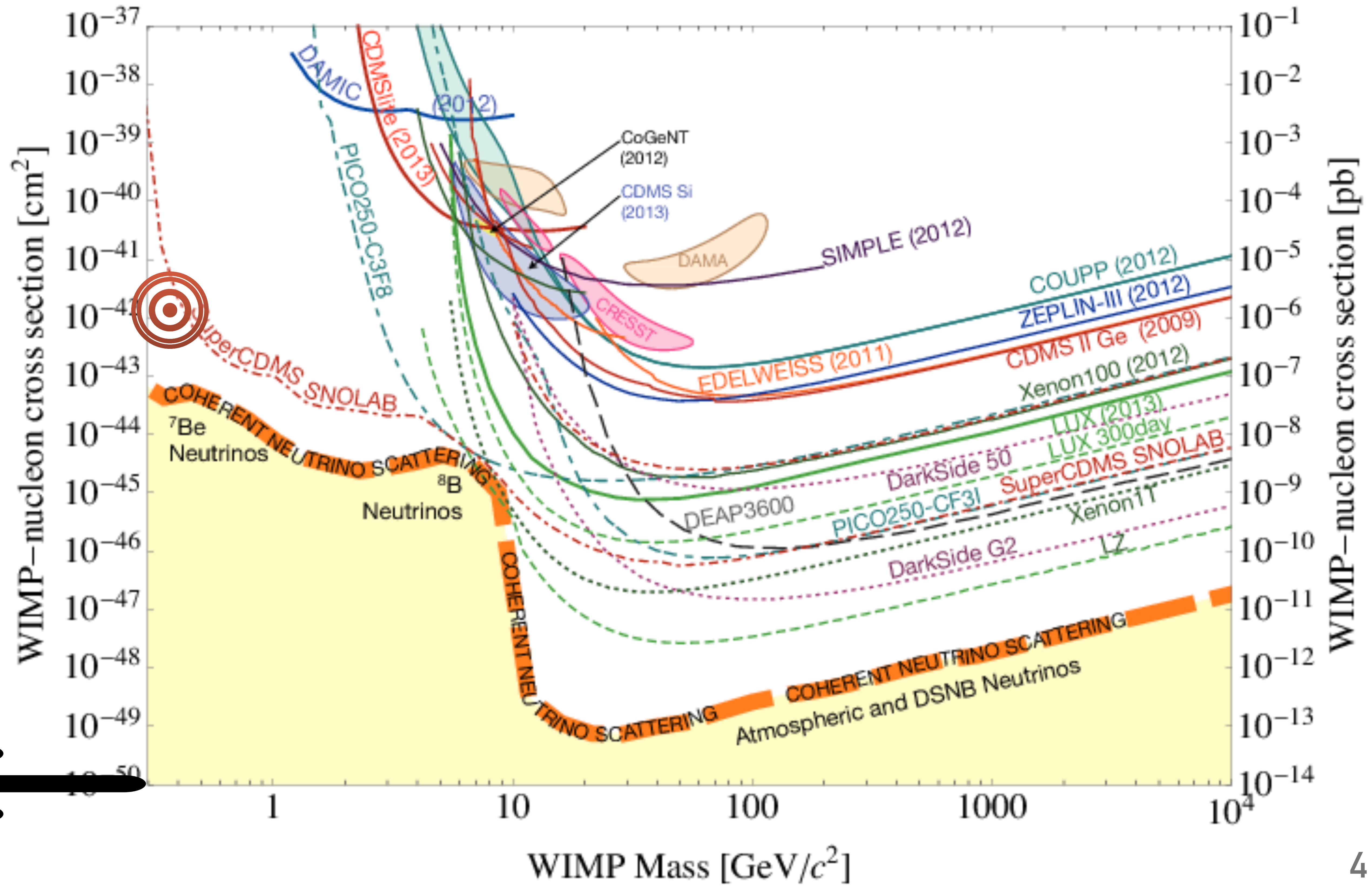
?



Toward lighter DM in direct detection

- Electron excitations.
  - Atoms (binding energy ~ 10eV).

Essig, Mardon, Volansky, 1108.5383.
Graham, Kaplan, Rajendran, Walters, 1203.2531.
Lee, Lisanti, Mishra-Sharma, Safdi, 1508.07361.
Essig, Fernandez-Serra, Mardon, Soto, Volansky, Yu, 1509.01598.
Essig, Volansky, Yu, 1703.00910.
Catena, Emken, Spaldin, Tarantino, 1912.08204.

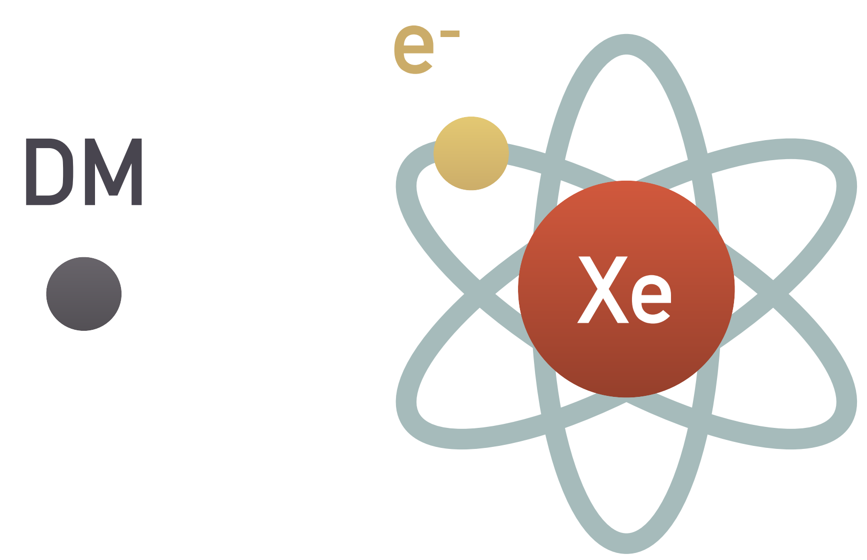


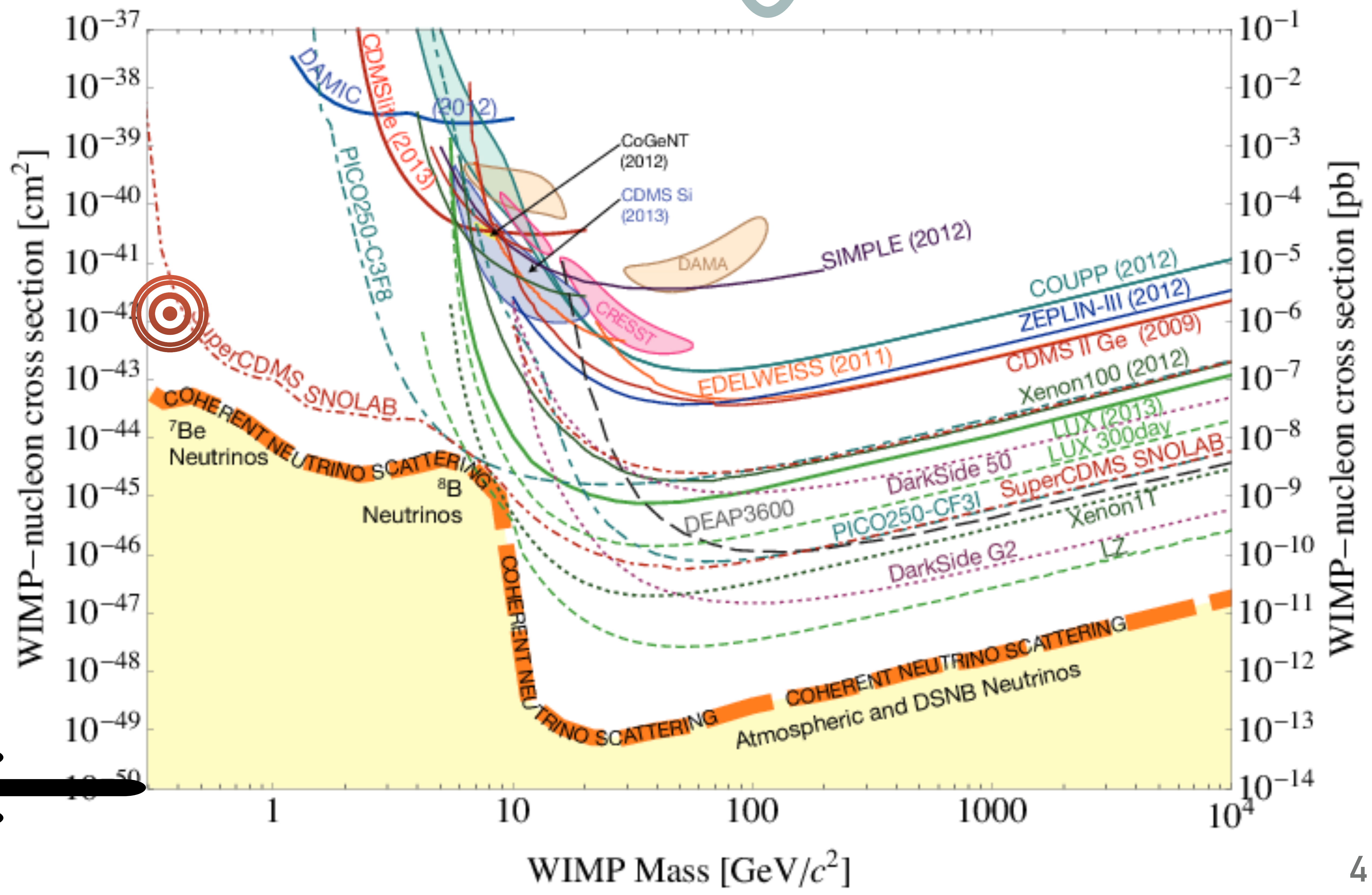
?

Toward lighter DM in direct detection

- Electron excitations.
  - Atoms (binding energy ~ 10eV).

Essig, Mardon, Volansky, 1108.5383.
Graham, Kaplan, Rajendran, Walters, 1203.2531.
Lee, Lisanti, Mishra-Sharma, Safdi, 1508.07361.
Essig, Fernandez-Serra, Mardon, Soto, Volansky, Yu, 1509.01598.
Essig, Volansky, Yu, 1703.00910.
Catena, Emken, Spaldin, Tarantino, 1912.08204.



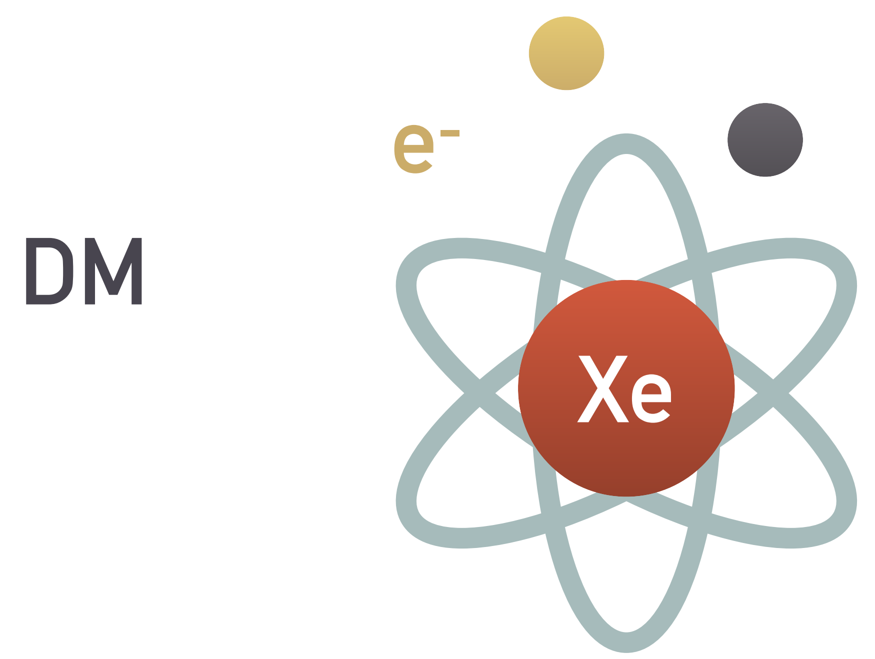


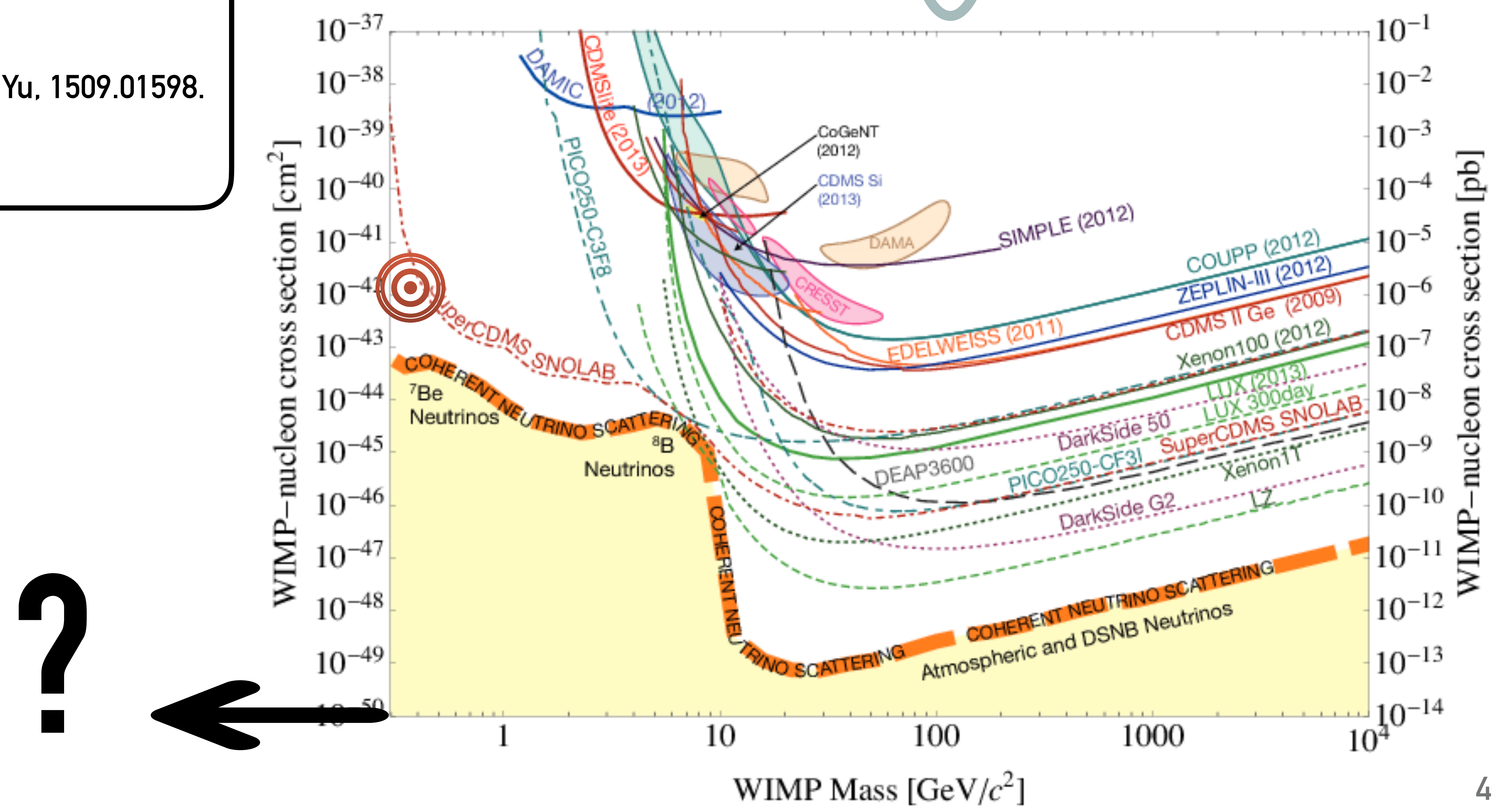
?

Toward lighter DM in direct detection

- Electron excitations.
  - Atoms (binding energy ~ 10eV).

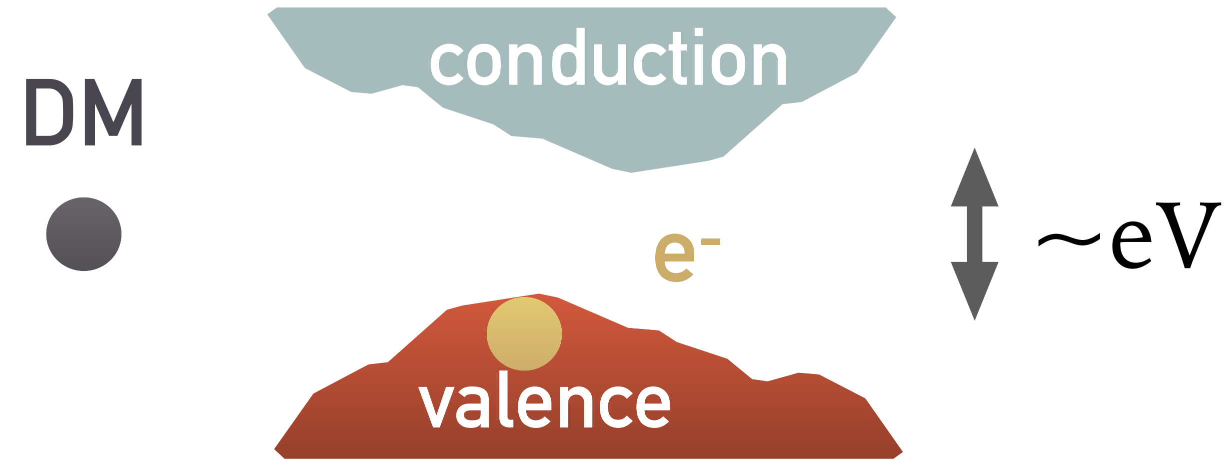
Essig, Mardon, Volansky, 1108.5383.
Graham, Kaplan, Rajendran, Walters, 1203.2531.
Lee, Lisanti, Mishra-Sharma, Safdi, 1508.07361.
Essig, Fernandez-Serra, Mardon, Soto, Volansky, Yu, 1509.01598.
Essig, Volansky, Yu, 1703.00910.
Catena, Emken, Spaldin, Tarantino, 1912.08204.





Toward lighter DM in direct detection

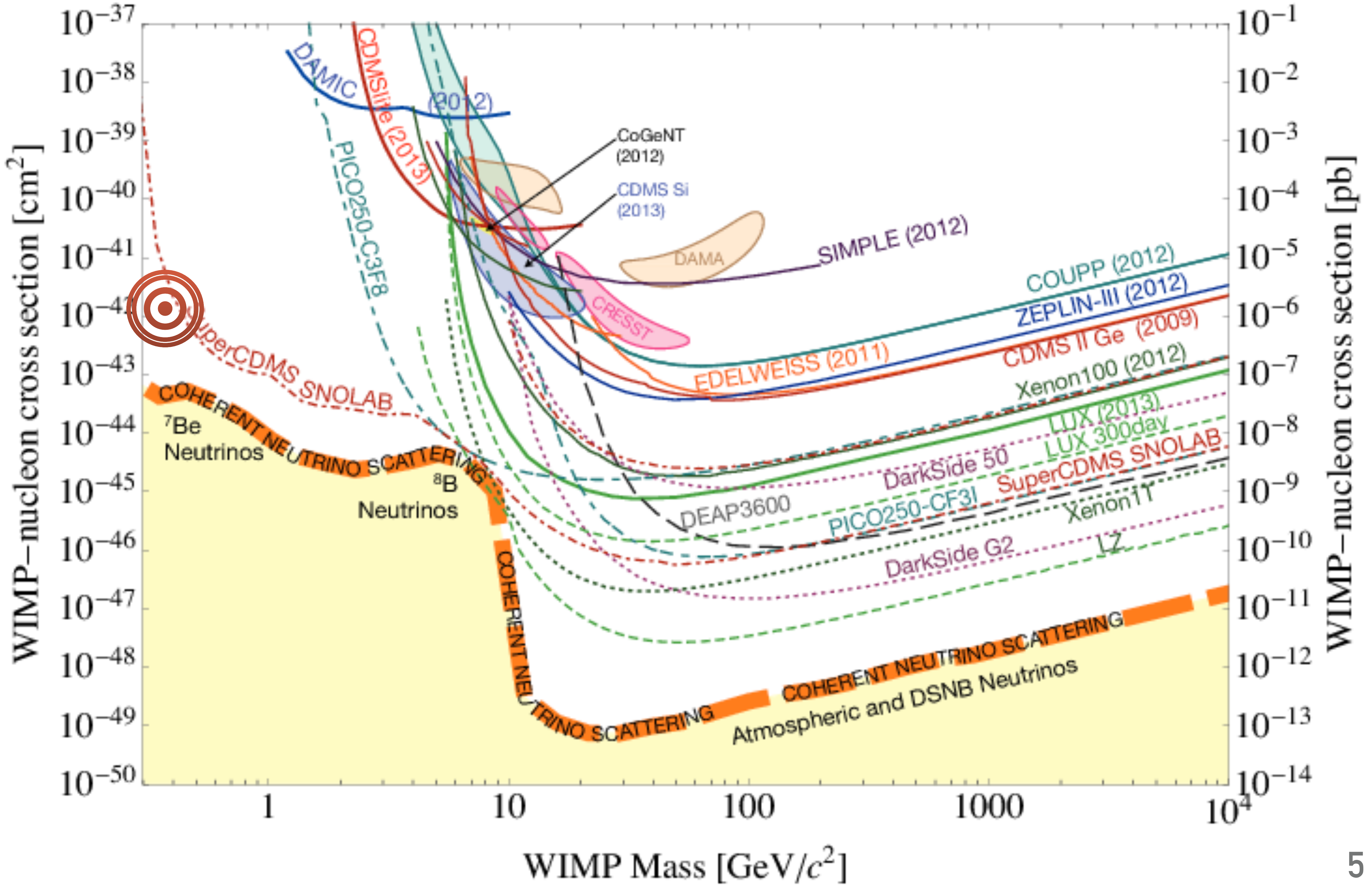
- Electron excitations.
  - Atoms (binding energy $\sim$ 10eV).
  - Semiconductors (gap $\sim$ eV).



Essig, Mardon, Volansky, 1108.5383.
Graham, Kaplan, Rajendran, Walters, 1203.2531.
Lee, Lisanti, Mishra-Sharma, Safdi, 1508.07361.
Essig, Fernandez-Serra, Mardon, Soto, Volansky, Yu, 1509.01598.
Derenzo, Essig, Massari, Soto, Yu, 1607.01009.
Kurinsky, Yu, Hochberg, Cabrera, 1901.07569.
Griffin, Inzani, Trickle, ZZ, Zurek, 1910.08092 + 1910.10716.
Andersson, Bökmark, Catena, Emken, Moberg, Åstrand, 2001.08910.

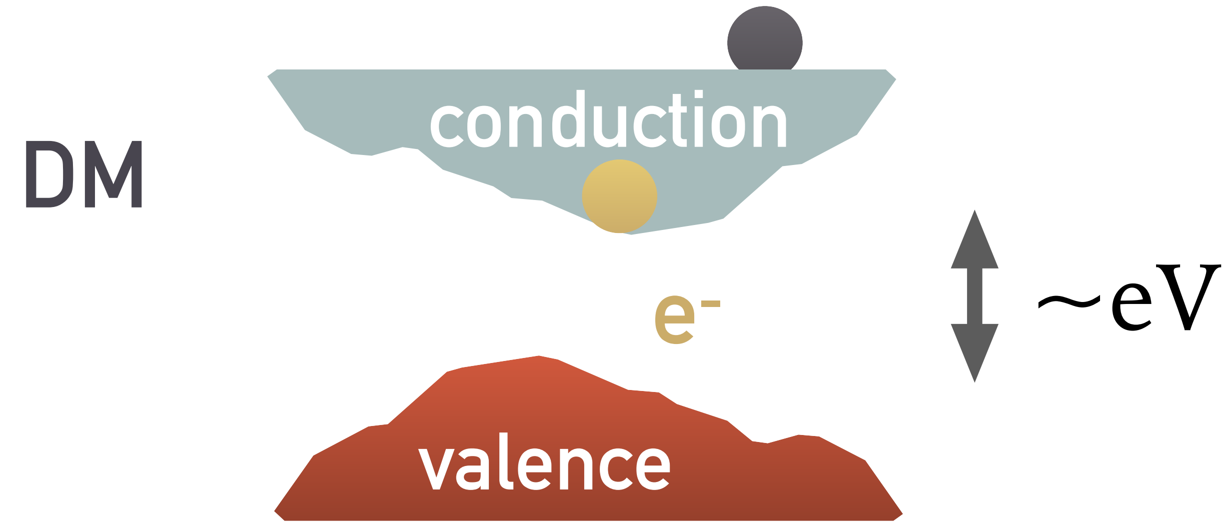
SuperCDMS, 1804.10697.
SENSEI, 1901.10478 + 2004.11378,
DAMIC, 1907.12628.

<u>Other similar proposals</u>
[Graphene] Hochberg, Kahn, Lisanti, Tully, Zurek, 1606.08849.
[Aromatic organic targets] Blanco, Collar, Kahn, Lillard, 1912.02822.
…



Toward lighter DM in direct detection

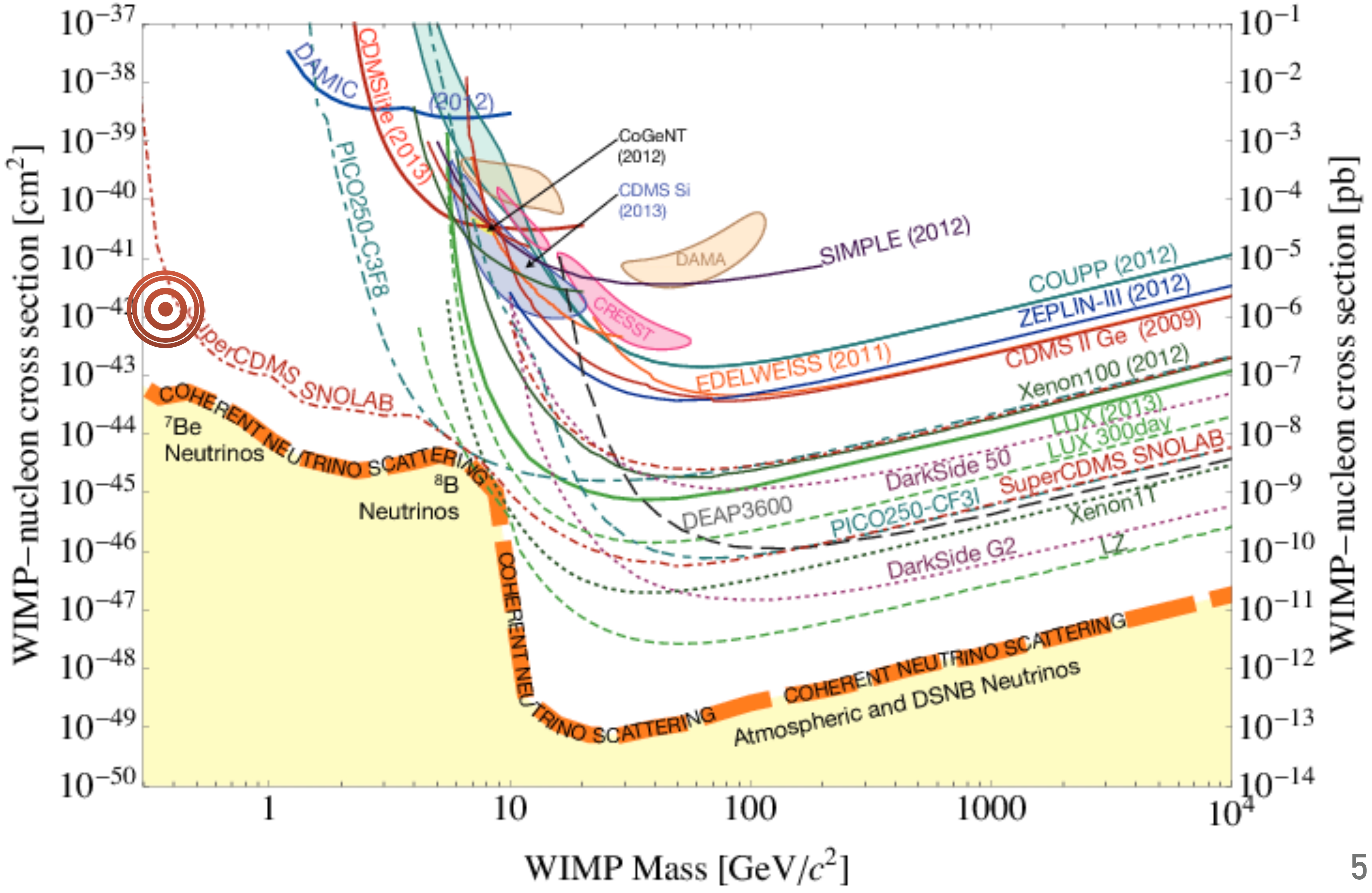
- Electron excitations.
  - Atoms (binding energy ~ 10eV).
  - Semiconductors (gap ~ eV).



Essig, Mardon, Volansky, 1108.5383.
Graham, Kaplan, Rajendran, Walters, 1203.2531.
Lee, Lisanti, Mishra-Sharma, Safdi, 1508.07361.
Essig, Fernandez-Serra, Mardon, Soto, Volansky, Yu, 1509.01598.
Derenzo, Essig, Massari, Soto, Yu, 1607.01009.
Kurinsky, Yu, Hochberg, Cabrera, 1901.07569.
Griffin, Inzani, Trickle, ZZ, Zurek, 1910.08092 + 1910.10716.
Andersson, Bökmark, Catena, Emken, Moberg, Åstrand, 2001.08910.

SuperCDMS, 1804.10697.
SENSEI, 1901.10478 + 2004.11378,
DAMIC, 1907.12628.

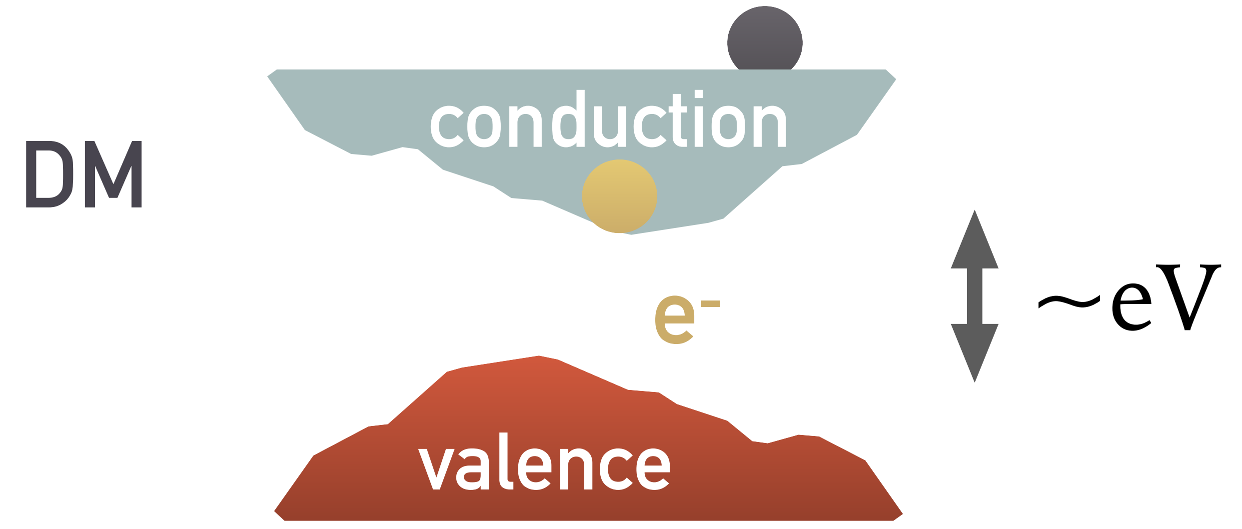
<u>Other similar proposals</u>
[Graphene] Hochberg, Kahn, Lisanti, Tully, Zurek, 1606.08849.
[Aromatic organic targets] Blanco, Collar, Kahn, Lillard, 1912.02822.
…

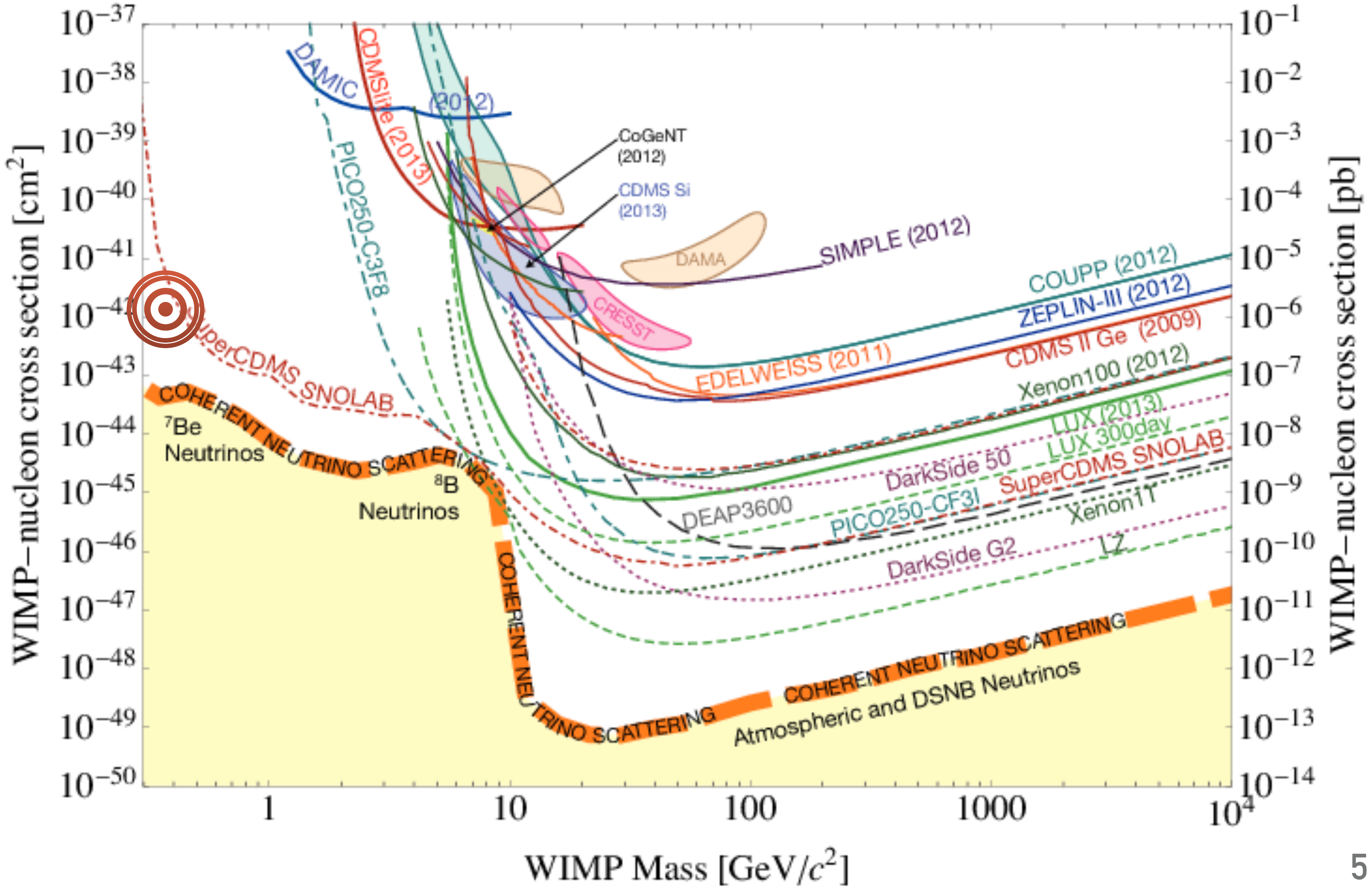


Toward lighter DM in direct detection

- Electron excitations.
  - Atoms (binding energy ~ 10eV).
  - Semiconductors (gap ~ eV).
  - Reach down to DM mass~MeV.
    - Recall $E \sim mv^2 \sim 10^{-6}\, m$.

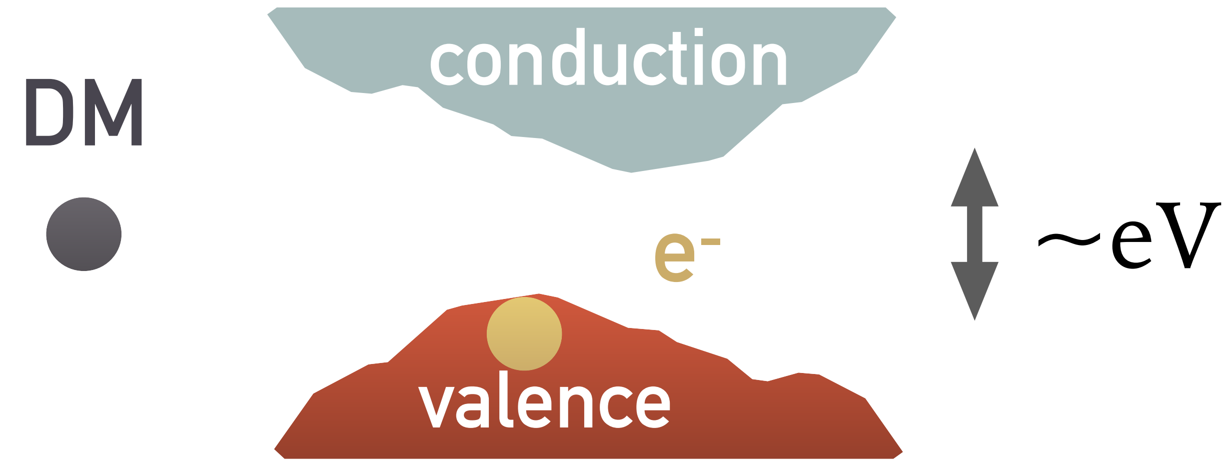
~MeV

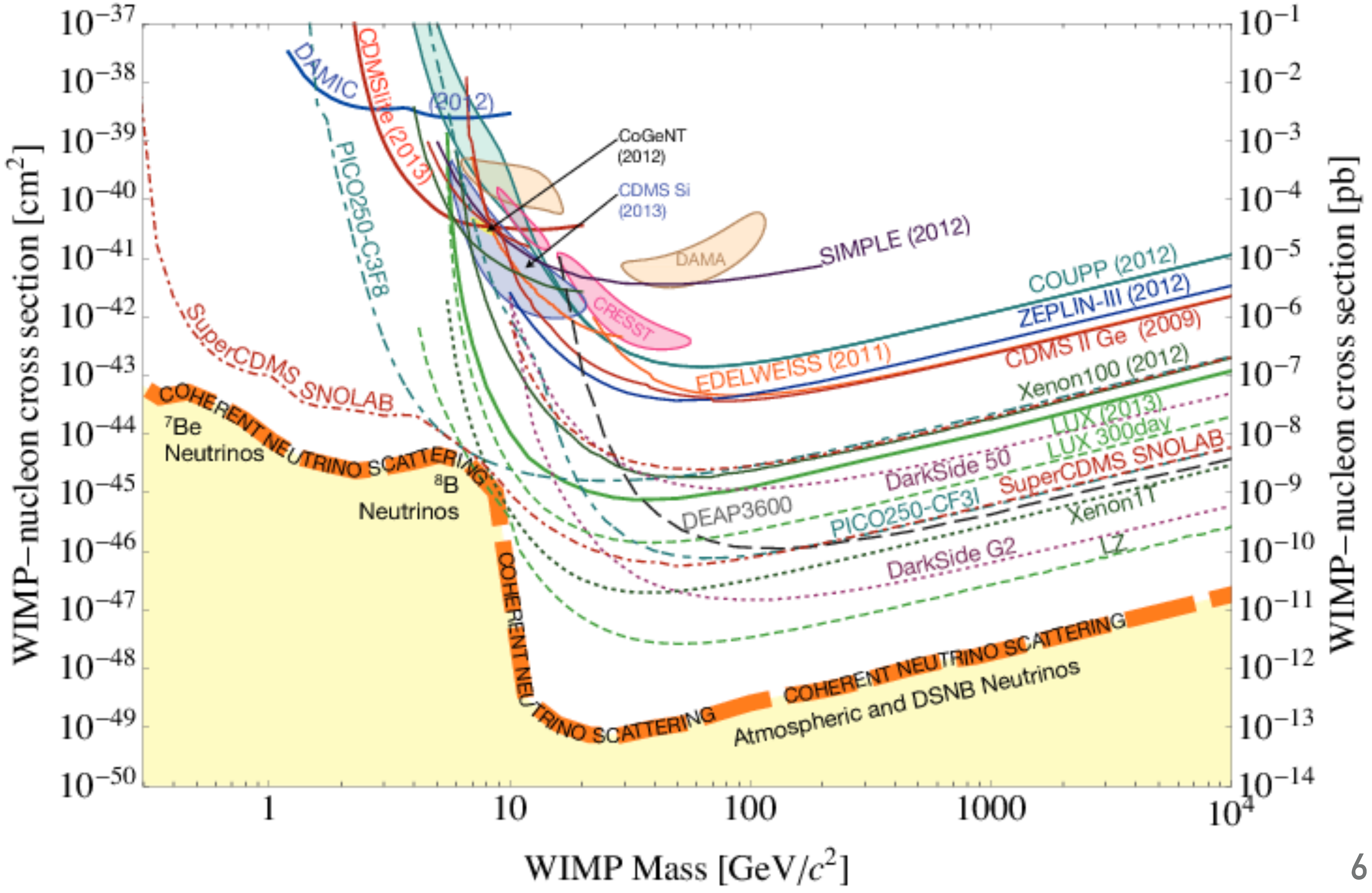




Toward lighter DM in direct detection

- Electron excitations.
  - Atoms (binding energy ~ 10eV).
  - Semiconductors (gap ~ eV).
  - Reach down to DM mass~MeV.
    - Recall $E \sim mv^2 \sim 10^{-6}\, m$.



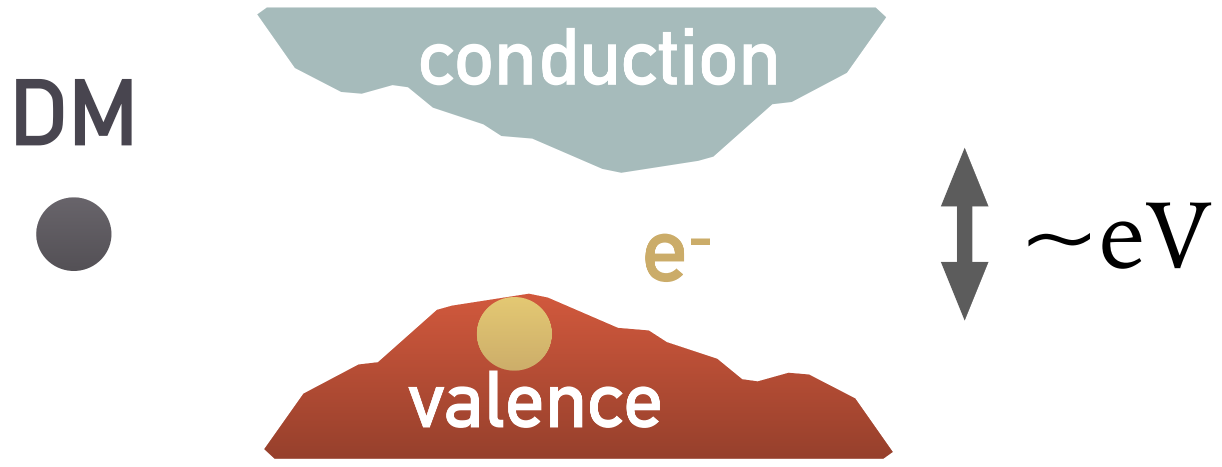


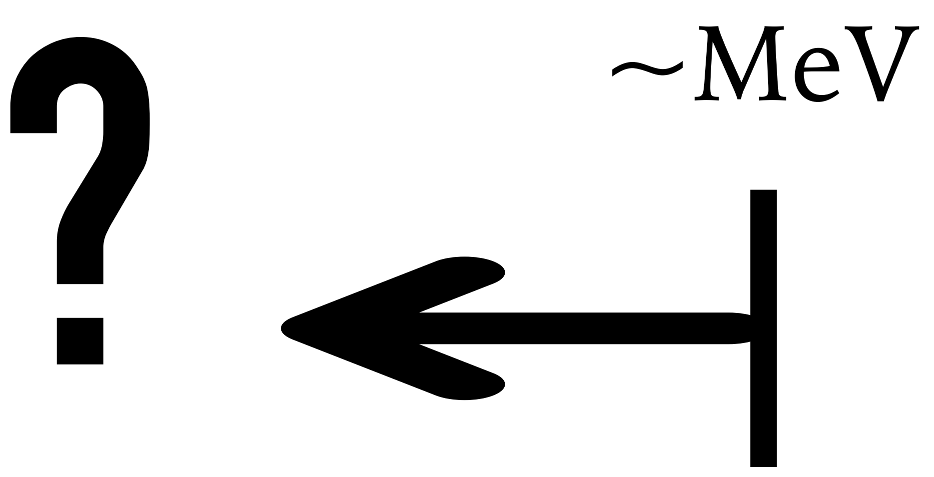
?

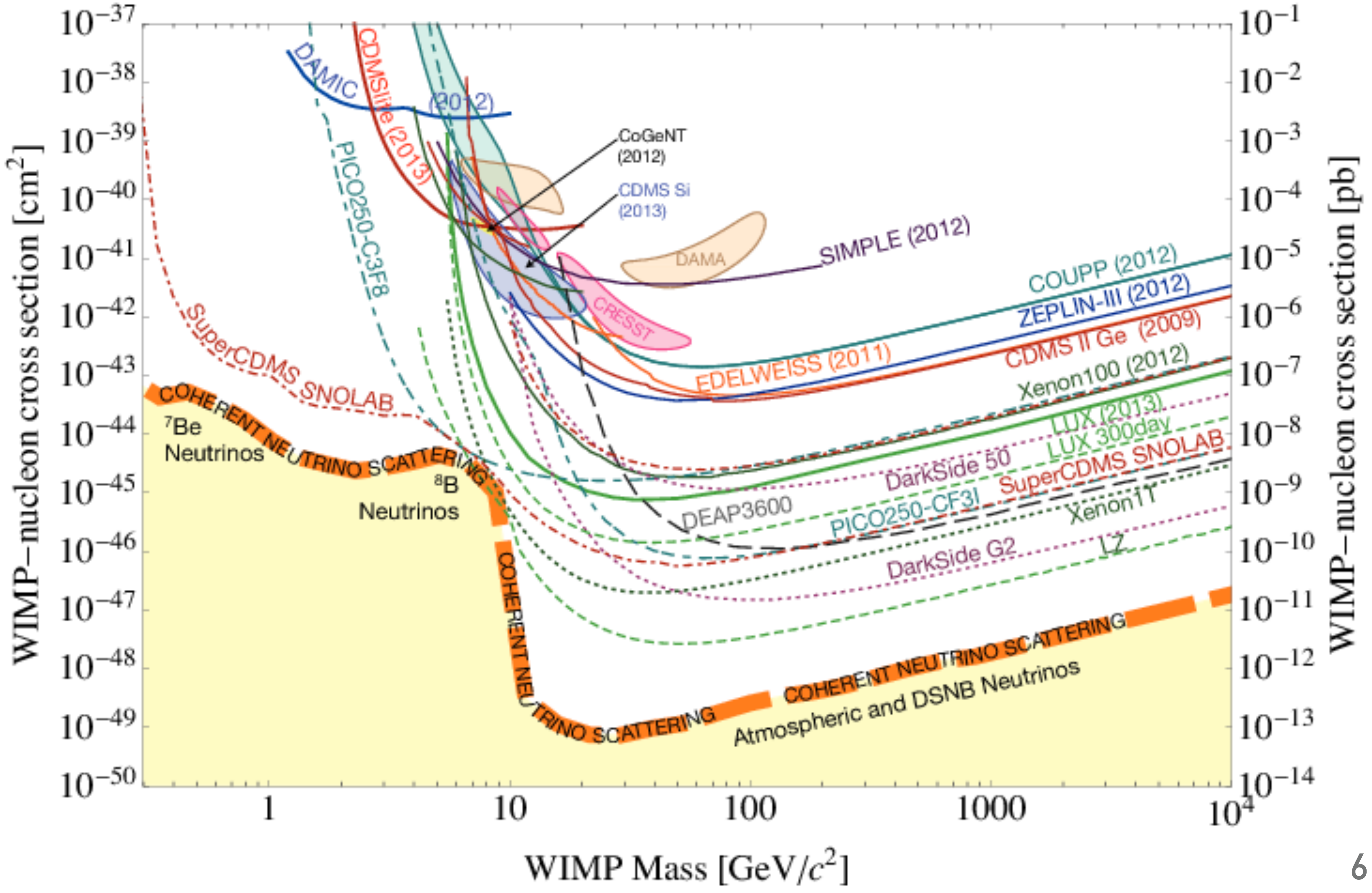
~MeV

Toward lighter DM in direct detection

- Electron excitations.
  - Atoms (binding energy ~ 10eV).
  - Semiconductors (gap ~ eV).
  - Reach down to DM mass~MeV.
    - Recall $E \sim mv^2 \sim 10^{-6}\, m$.

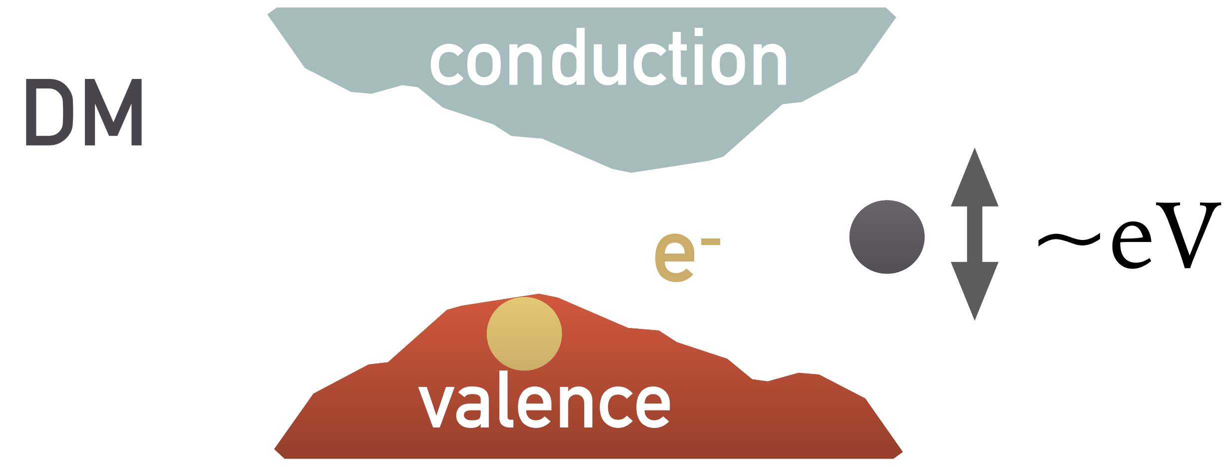


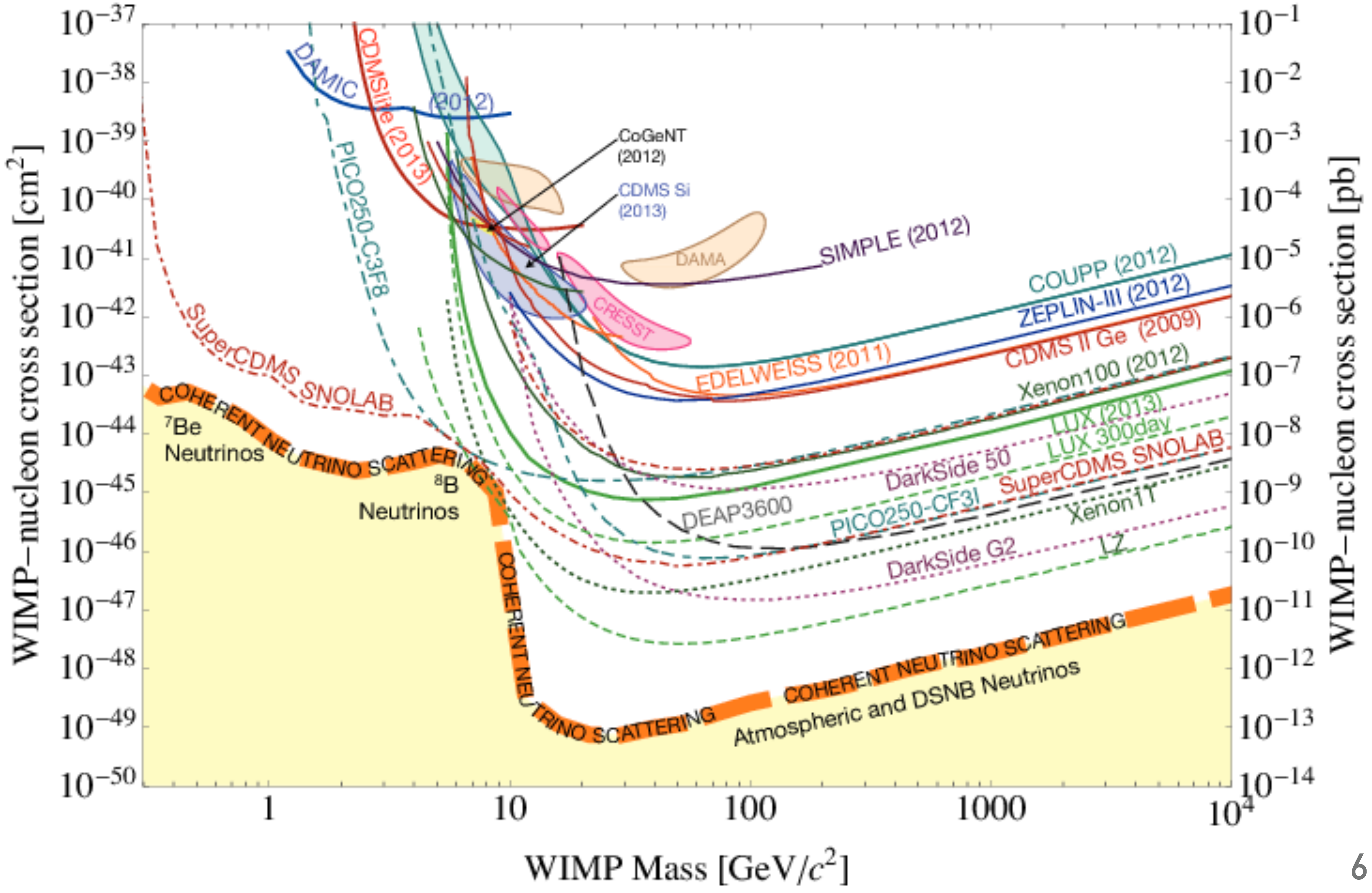




Toward lighter DM in direct detection

- Electron excitations.
  - Atoms (binding energy ~ 10eV).
  - Semiconductors (gap ~ eV).
  - Reach down to DM mass~MeV.
    - Recall $E \sim mv^2 \sim 10^{-6}\, m$.



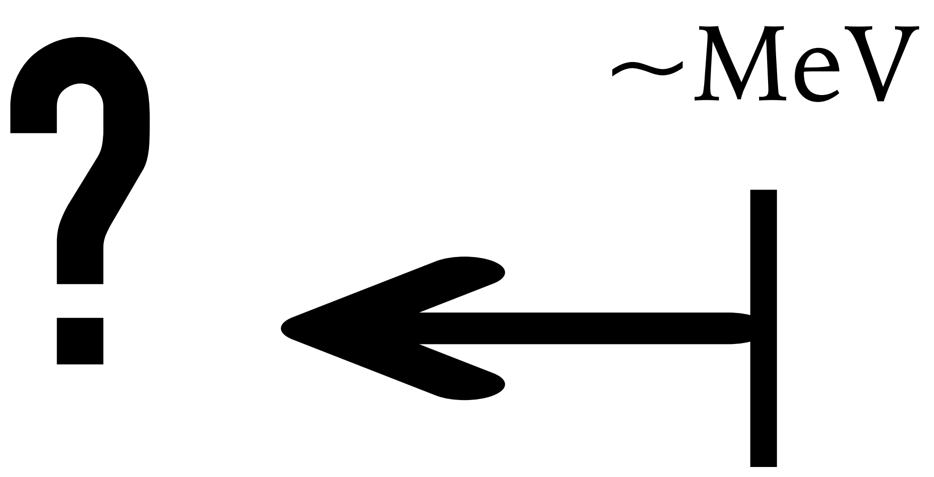


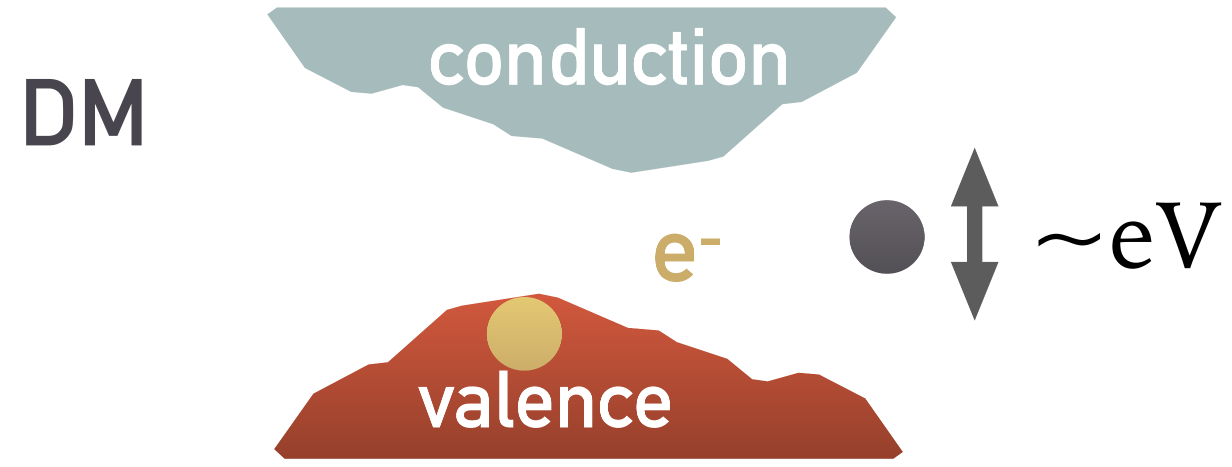
? ~MeV

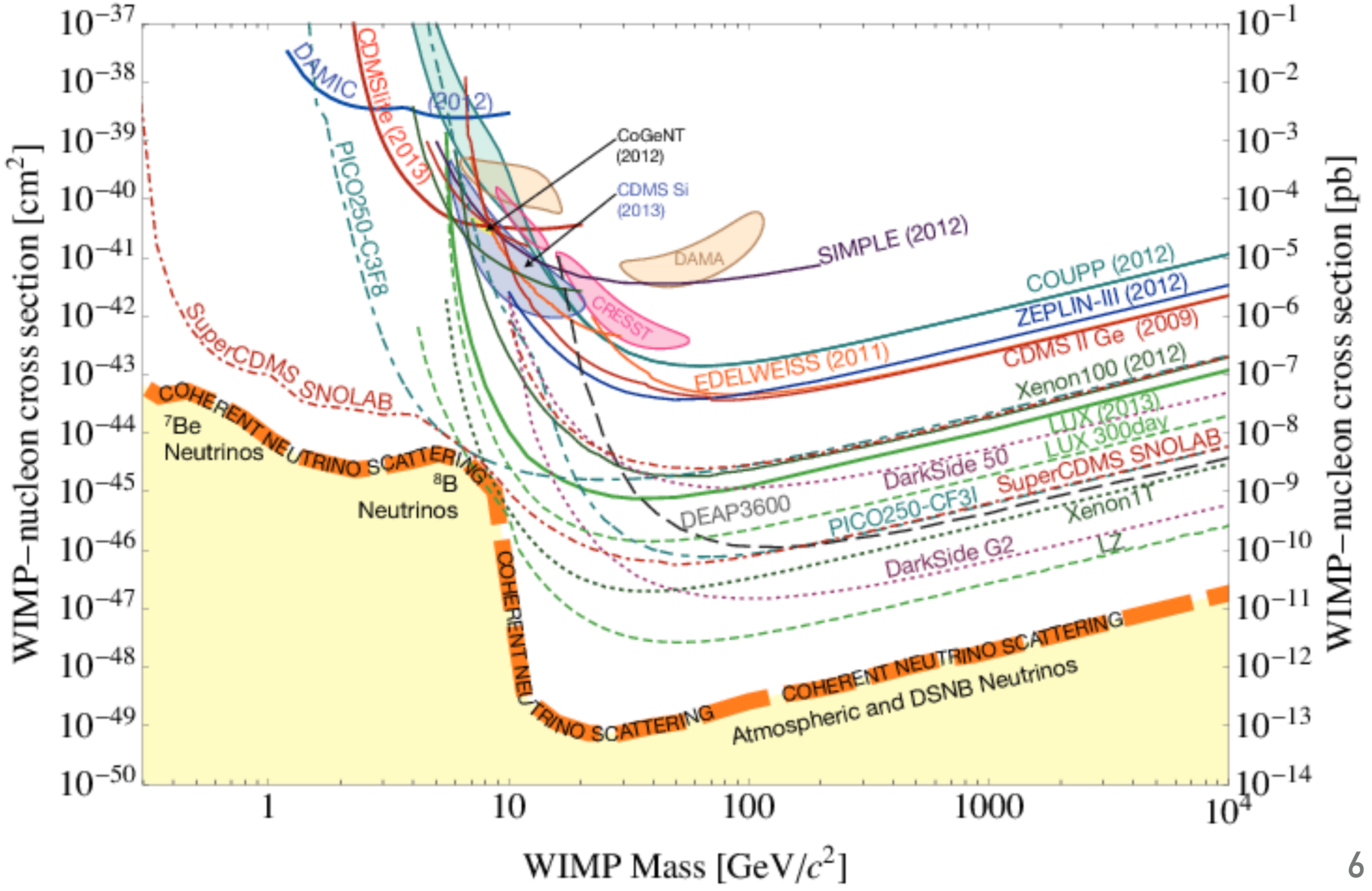
Toward lighter DM in direct detection

- Electron excitations.
  - Atoms (binding energy ~ 10eV).
  - Semiconductors (gap ~ eV).
  - Reach down to DM mass~MeV.
    - Recall $E \sim mv^2 \sim 10^{-6}\, m$.

<u>Proposed meV-gap targets (somewhat futuristic)</u>
[Superconductors]
Hochberg, Zhao, Zurek, 1504.07237.
Hochberg, Pyle, Zhao, Zurek, 1512.04533.
[Dirac materials]
Hochberg et al, 1708.08929.
Geilhufe, Kahlhoefer, Winkler, 1910.02091.
Coskuner, Mitridate, Olivares, Zurek, 1909.09170.

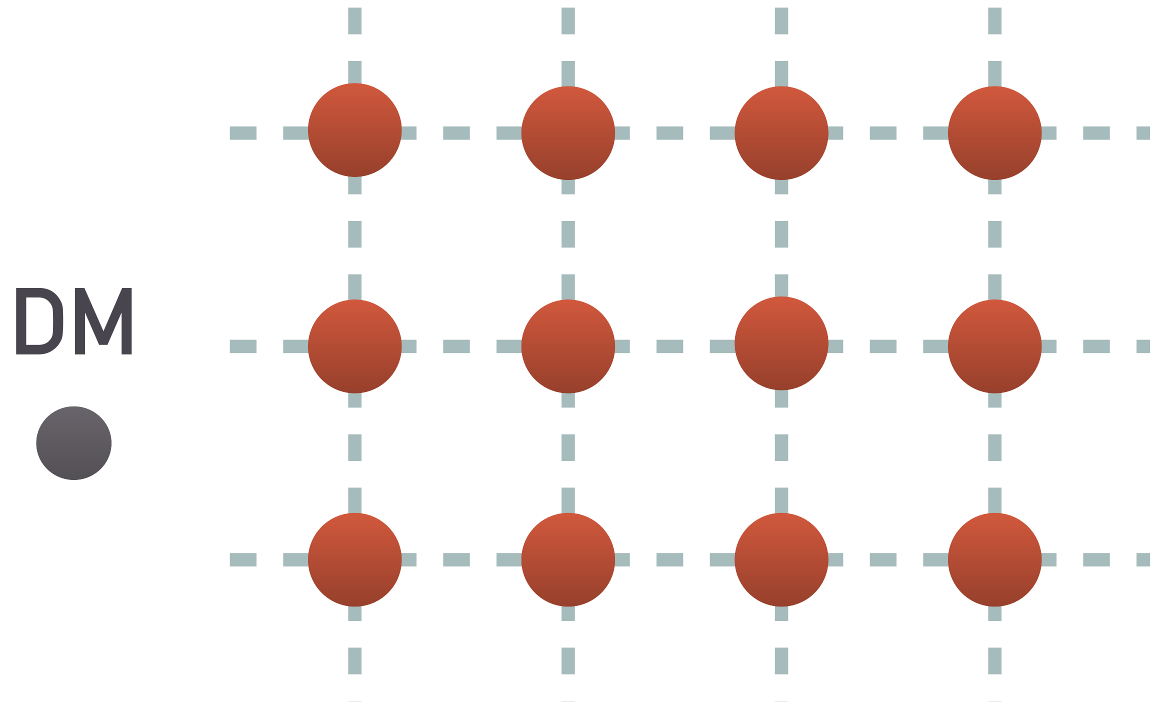




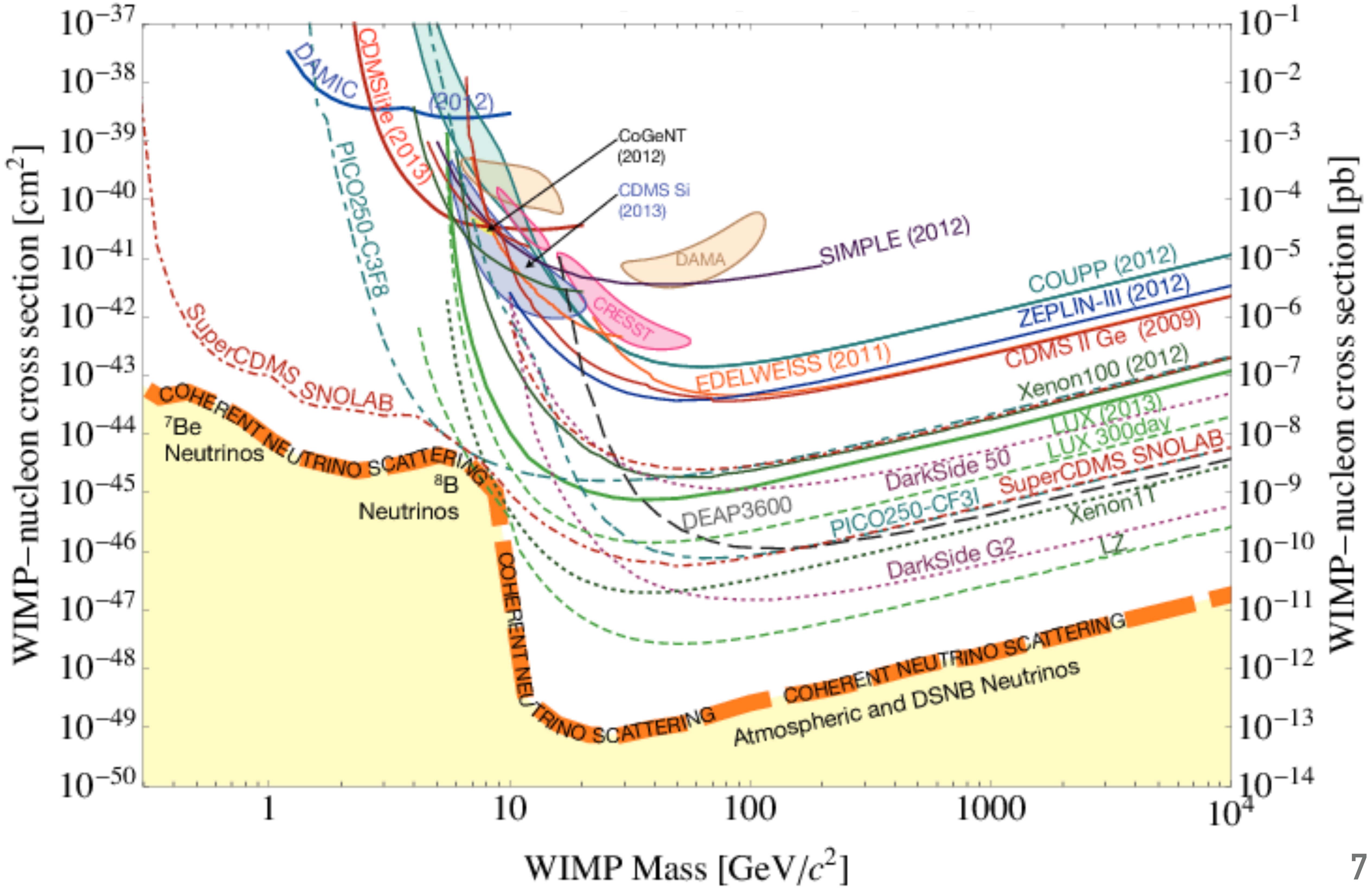


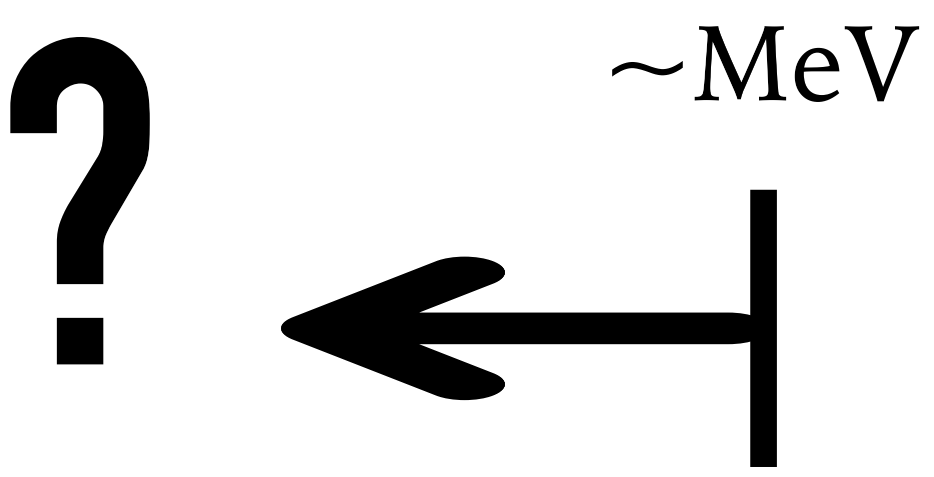
Toward lighter DM in direct detection

- Collective excitations (sub-eV energies).
  - Phonons in crystals (collective oscillations of ions).



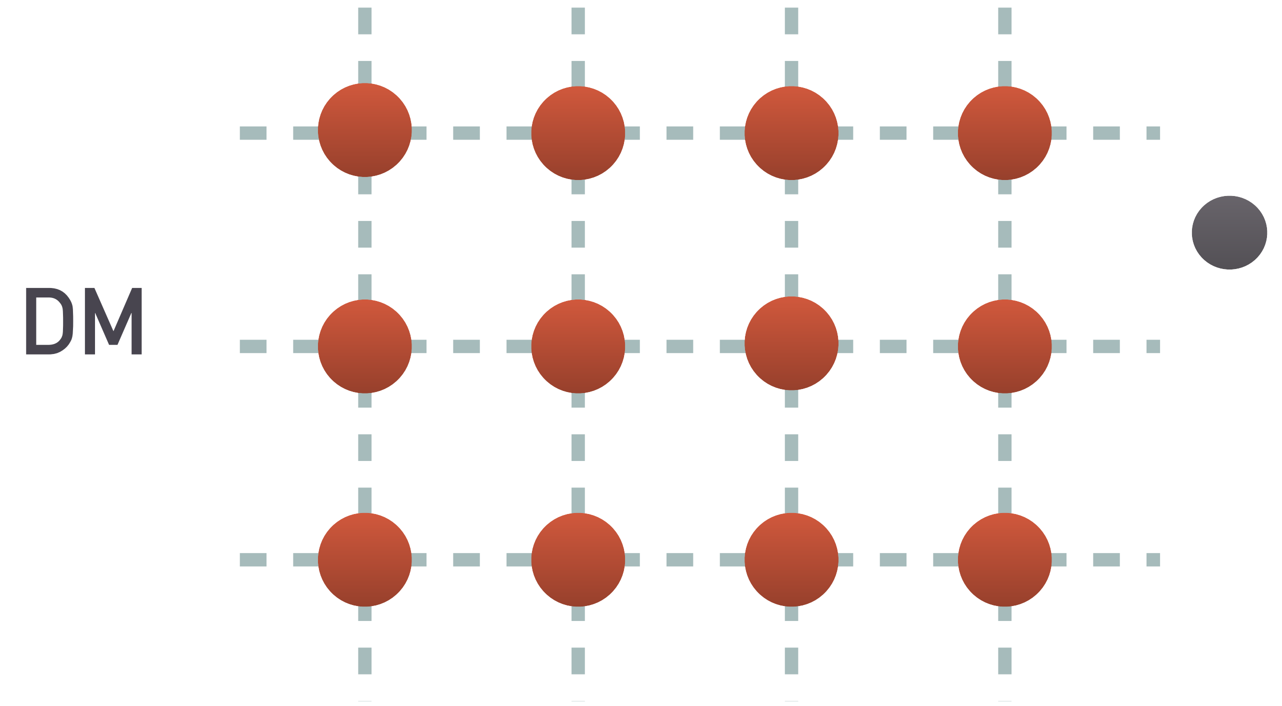
Knapen, Lin, Pyle, Zurek, 1712.06598.
Griffin, Knapen, Lin, Zurek, 1807.10291.
Griffin, Inzani, Trickle, ZZ, Zurek, 1910.08092 + 1910.10716.
Campbell-Deem, Cox, Knapen, Lin, Melia, 1911.03482.
Griffin, Hochberg, Inzani, Kurinsky, Lin, Yu, 2008.08560.





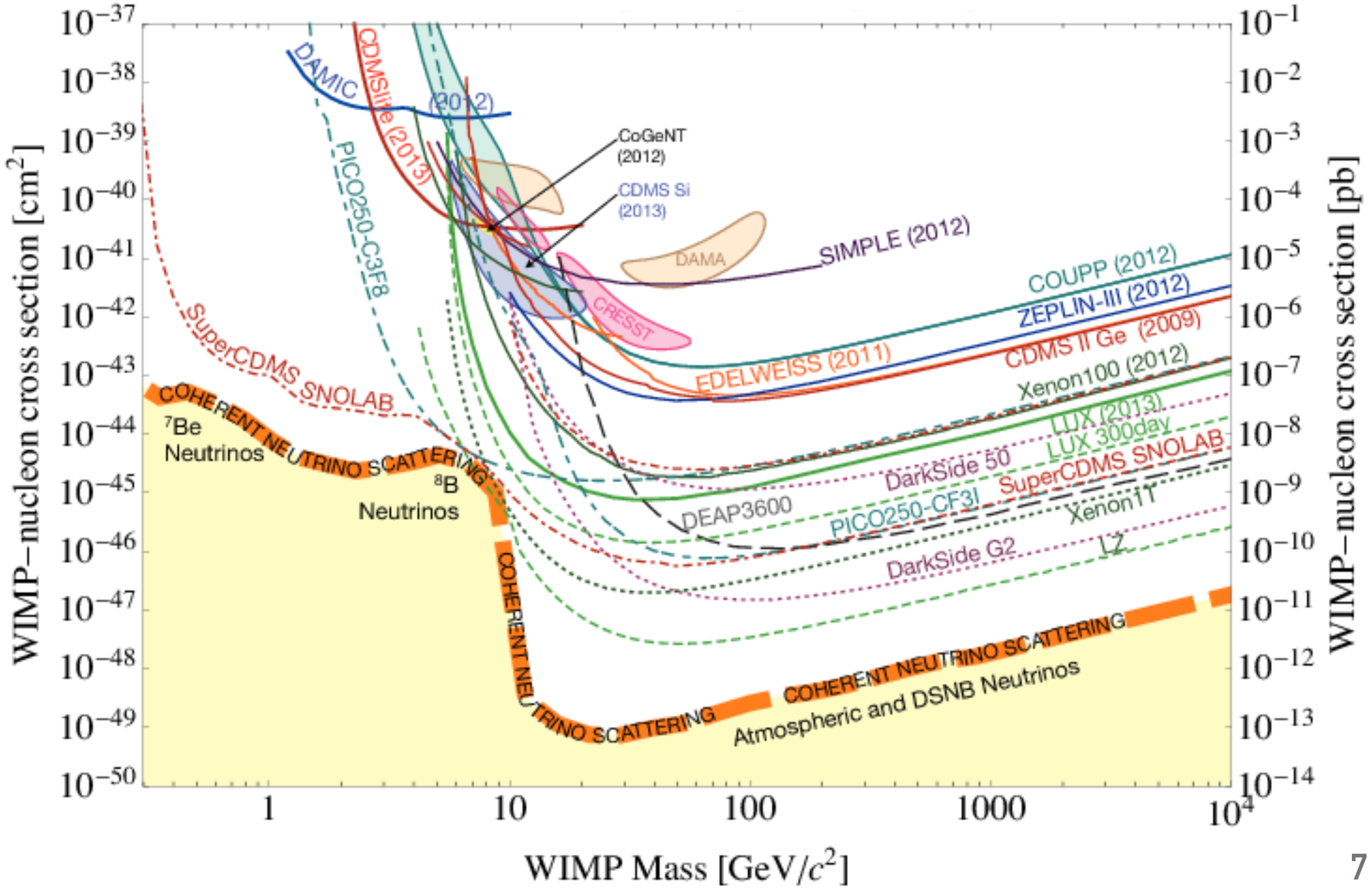
Toward lighter DM in direct detection

- Collective excitations (sub-eV energies).
  - Phonons in crystals (collective oscillations of ions).



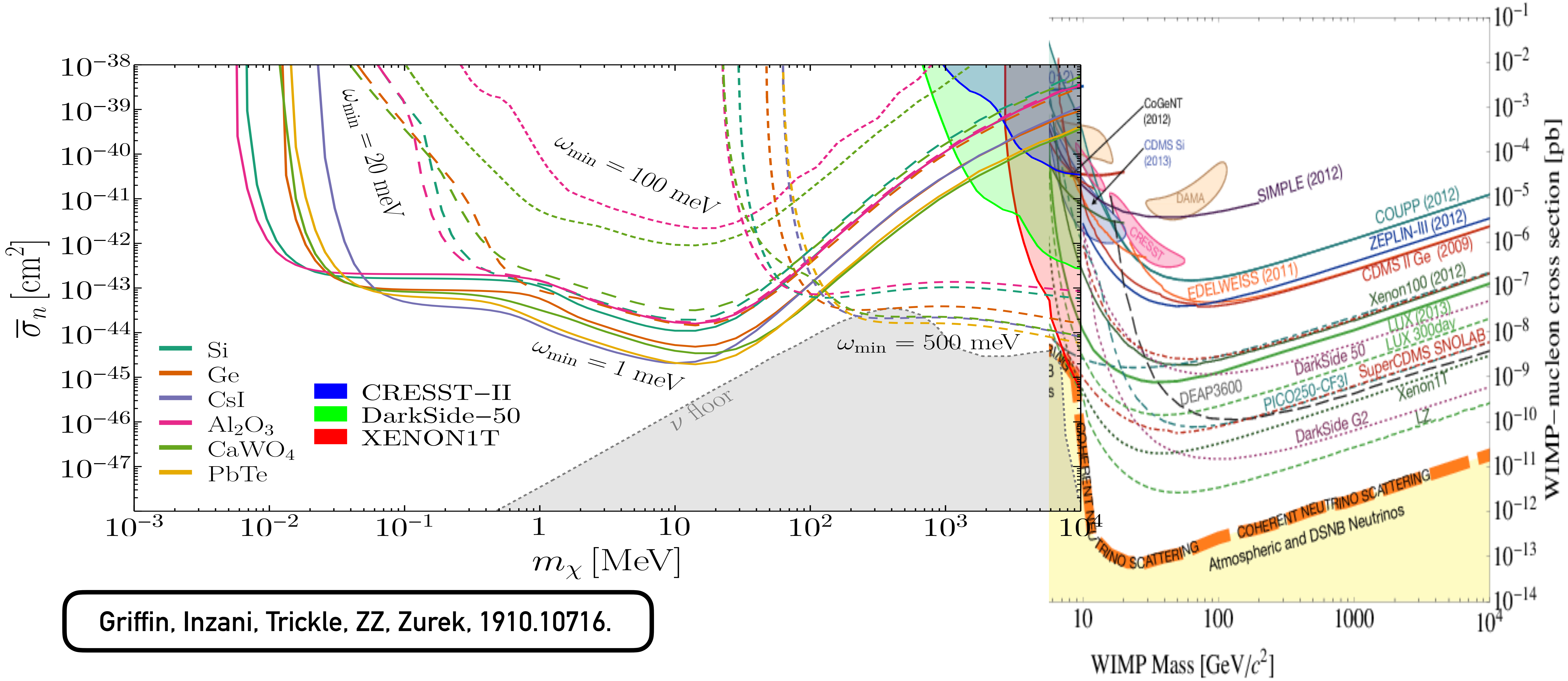
> Knapen, Lin, Pyle, Zurek, 1712.06598.
> Griffin, Knapen, Lin, Zurek, 1807.10291.
> Griffin, Inzani, Trickle, ZZ, Zurek, 1910.08092 + 1910.10716.
> Campbell-Deem, Cox, Knapen, Lin, Melia, 1911.03482.
> Griffin, Hochberg, Inzani, Kurinsky, Lin, Yu, 2008.08560.

? $\longleftarrow\!\!|$ $\sim$MeV



Toward lighter DM in direct detection

- Collective excitations (sub-eV energies).
  - Phonons in crystals (collective oscillations of ions).
    - Theoretical sensitivity demonstrated for a variety of targets.



Griffin, Inzani, Trickle, ZZ, Zurek, 1910.10716.

Toward lighter DM in direct detection

- Collective excitations (sub-eV energies).
  - Phonons in crystals (collective oscillations of ions).
    - Theoretical sensitivity demonstrated for a variety of targets.
    - Experiment in active R&D.

Snowmass2021 - Letter of Interest

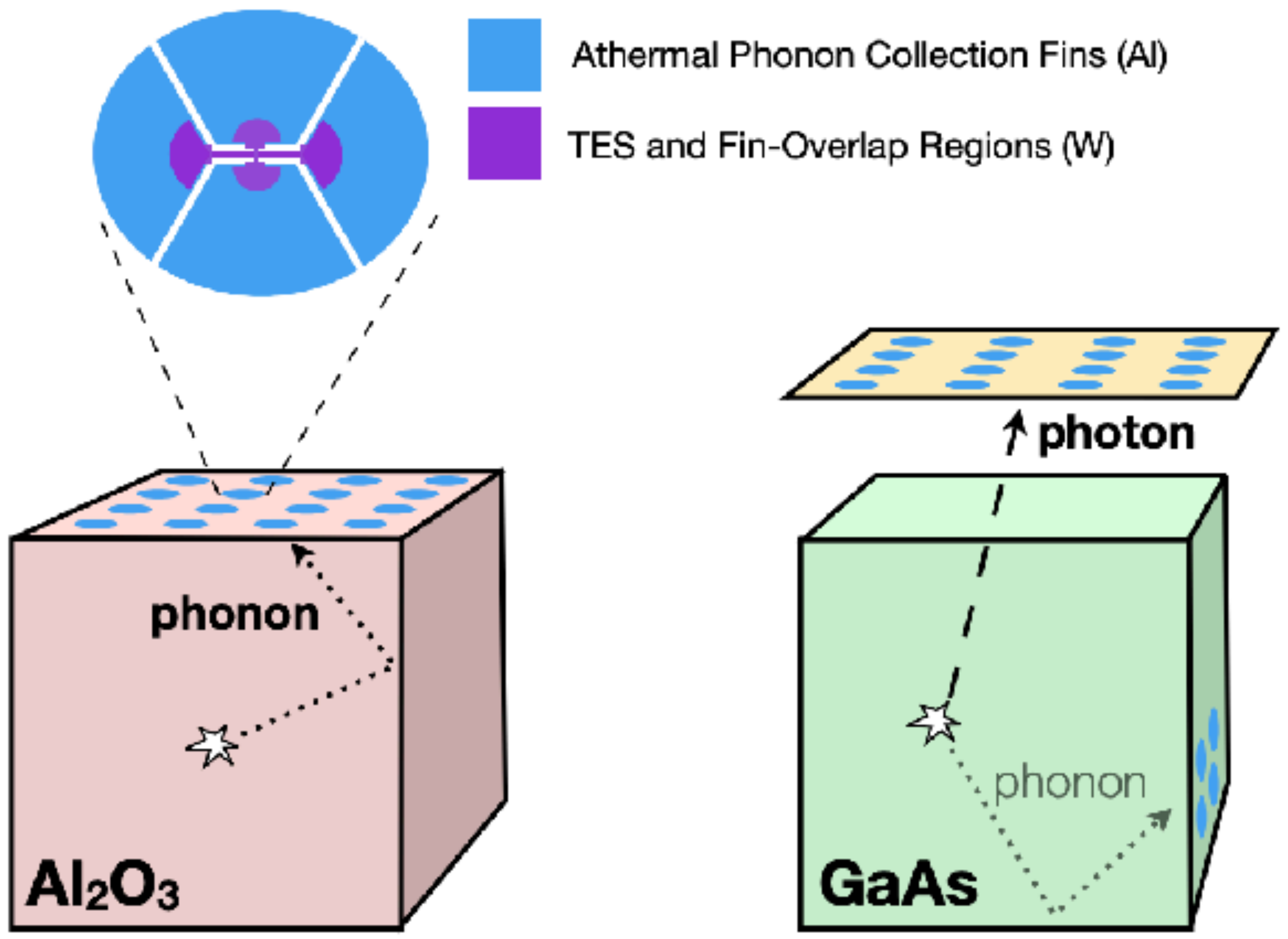
The TESSERACT Dark Matter Project

Thematic Areas:

- IF1 Quantum Sensors
- IF8 Noble Elements
- CF1 Dark Matter: Particle-like
- CF2 Dark Matter: Wavelike

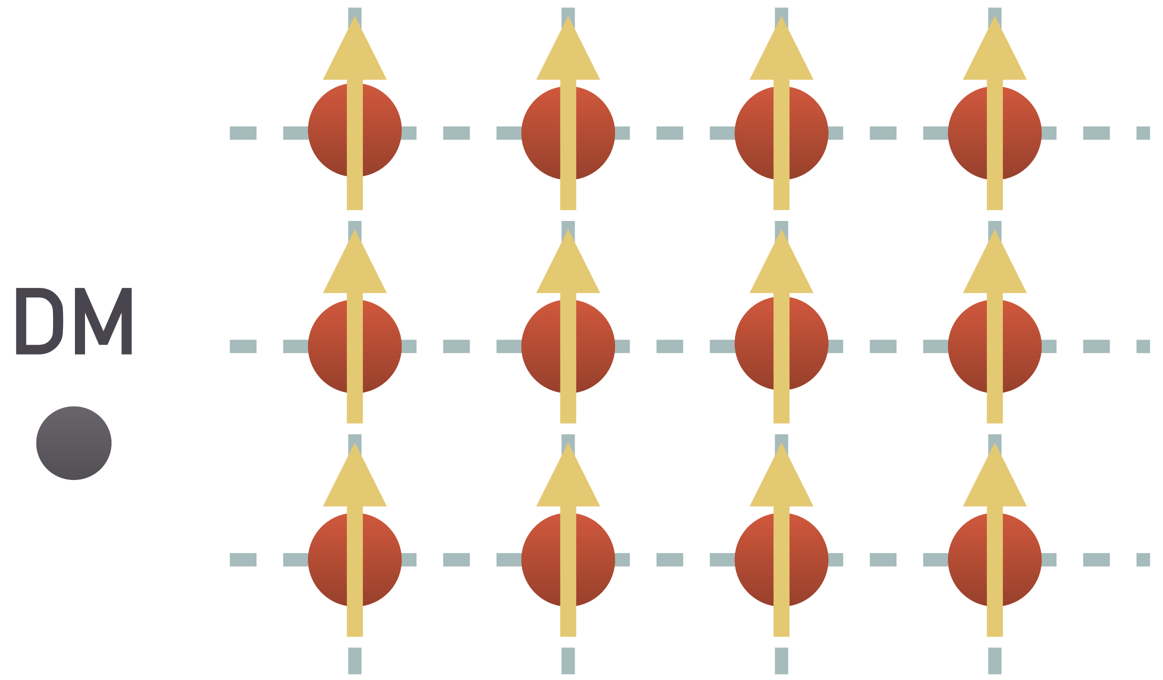
Contact Information:
Dan McKinsey (LBNL and UC Berkeley) [daniel.mckinsey@berkeley.edu]:
TESSERACT Collaboration

Authors:
C. Chang (ANL), S. Derenzo (LBNL), Y. Efremenko (ANL), W. Guo (Florida State University), S. Hertel (University of Massachusetts), M. Garcia-Sciveres, R. Mahapatra (Texas A&M University), D. N. McKinsey (LBNL and UC Berkeley), B. Penning (University of Michigan), M. Pyle (LBNL and UC Berkeley), P. Sorensen (LBNL), A. Suzuki (LBNL), G. Wang (ANL), K. Zurek (Caltech)



Toward lighter DM in direct detection

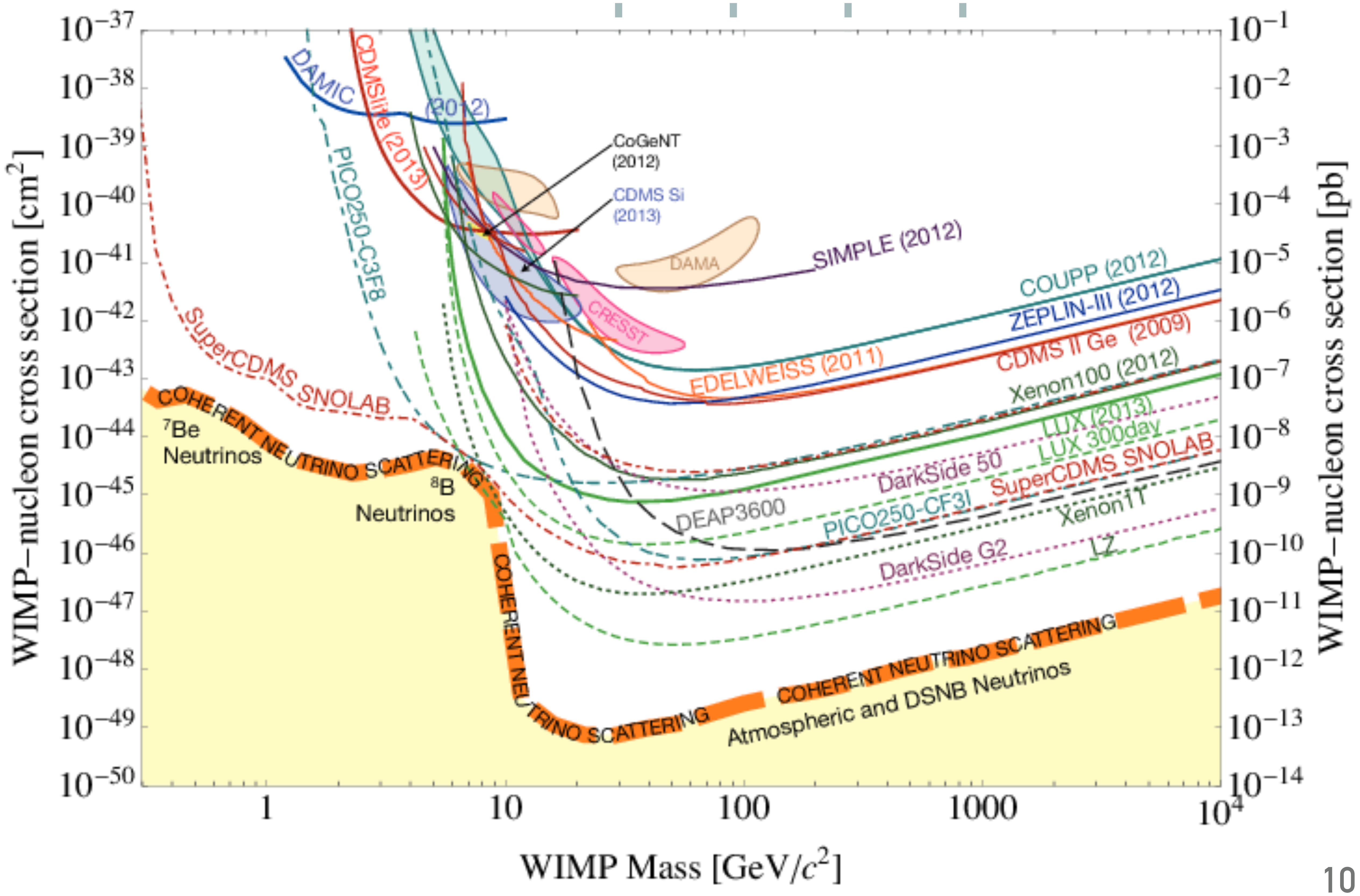
- Collective excitations (sub-eV energies).
  - Phonons in crystals (collective oscillations of ions).
  - Magnons (collective spin excitations in magnetically ordered materials).



Trickle, ZZ, Zurek, 1905.13744.

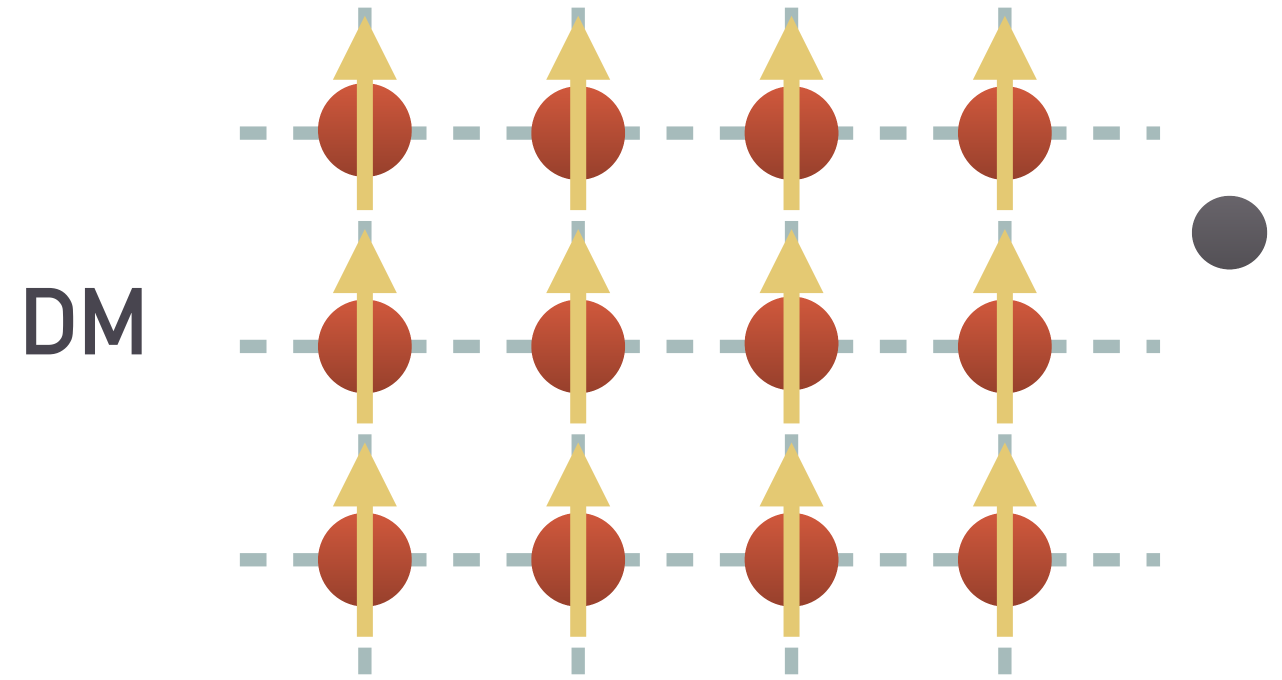
Also discussed for axion detection.
Chigusa, Moroi, Nakayama, 2001.10666.
Mitridate, Trickle, ZZ, Zurek, 2005.10256.

Ongoing experiment: QUAX (1511.09461, 1606.02201, 1806.00310, 1903.06547, 2001.08940) — cannot yet achieve single magnon sensitivity.



Toward lighter DM in direct detection

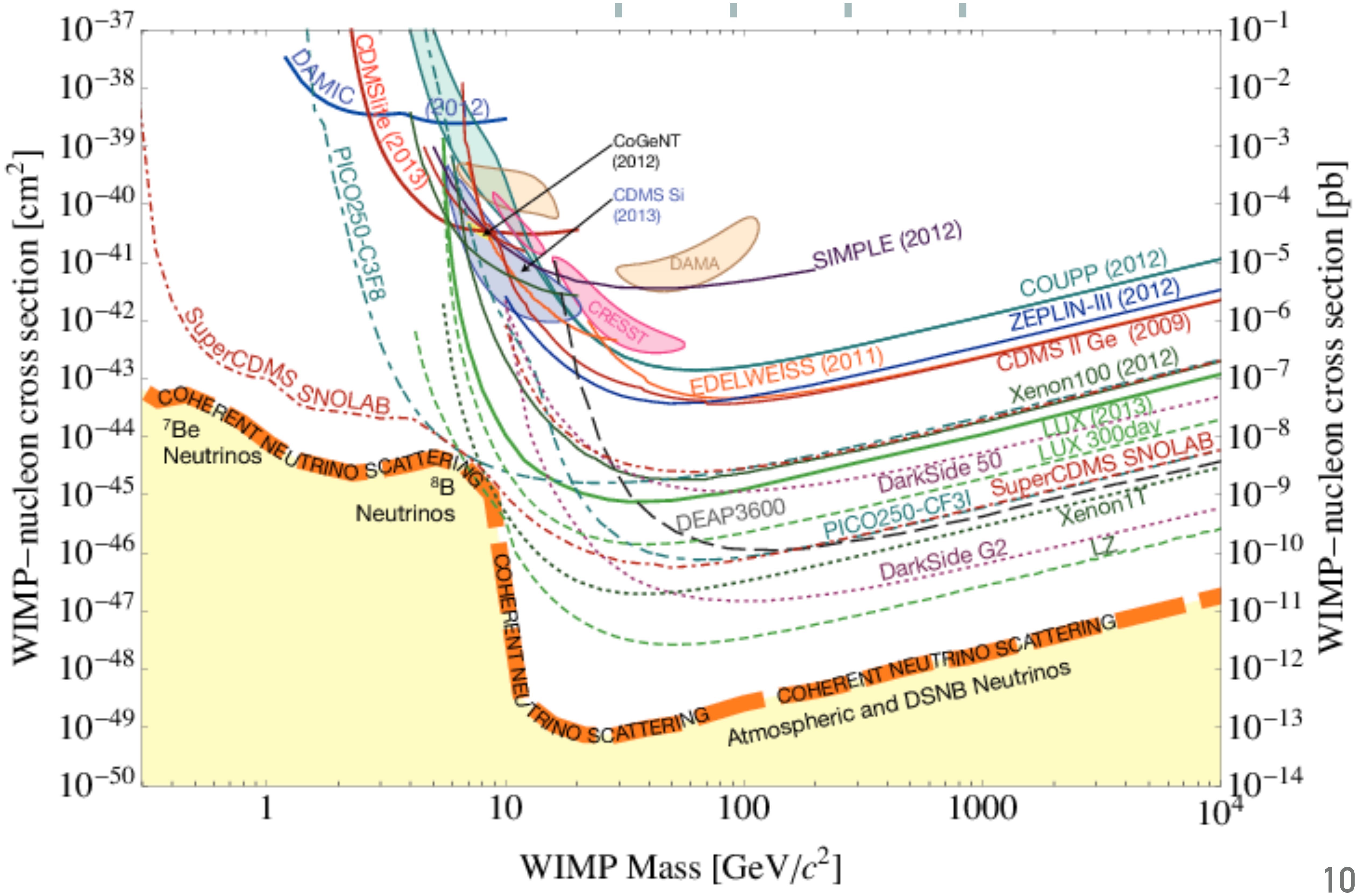
- Collective excitations (sub-eV energies).
  - Phonons in crystals (collective oscillations of ions).
  - Magnons (collective spin excitations in magnetically ordered materials).



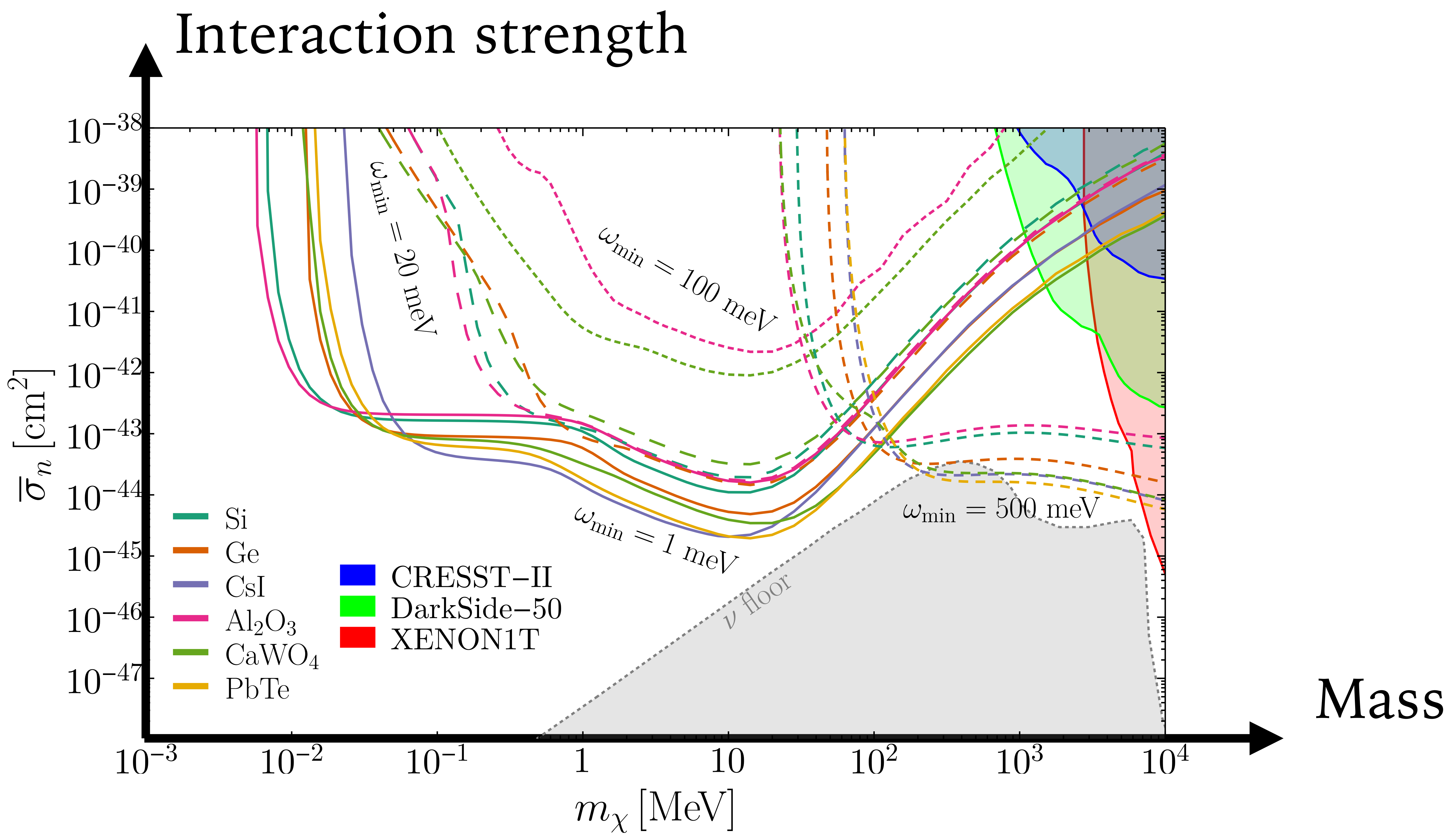
Trickle, ZZ, Zurek, 1905.13744.

<u>Also discussed for axion detection.</u>
Chigusa, Moroi, Nakayama, 2001.10666.
Mitridate, Trickle, ZZ, Zurek, 2005.10256.

<u>Ongoing experiment: QUAX</u> (1511.09461, 1606.02201, 1806.00310, 1903.06547, 2001.08940) — cannot yet achieve single magnon sensitivity.

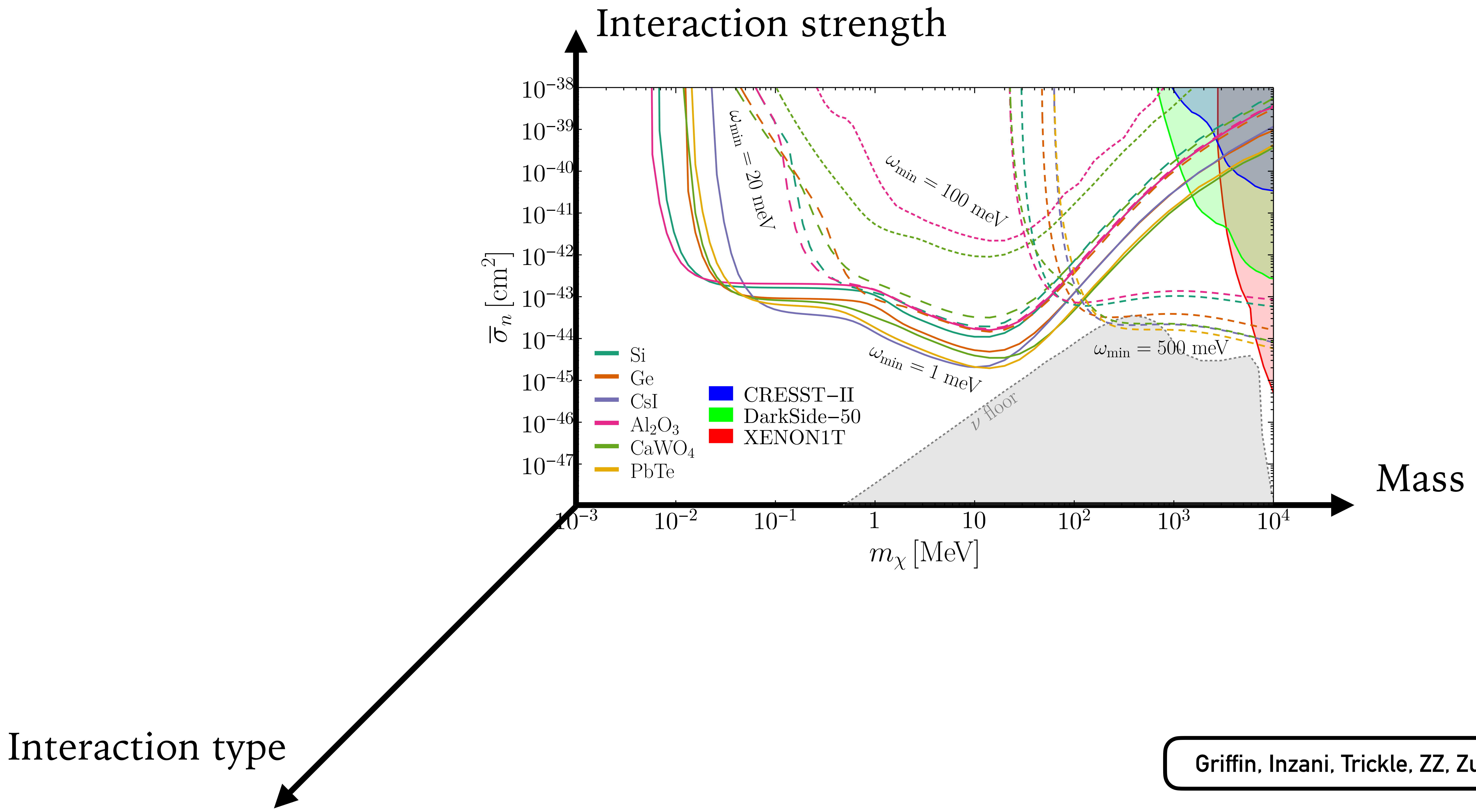


A 3rd axis of DM's parameter space



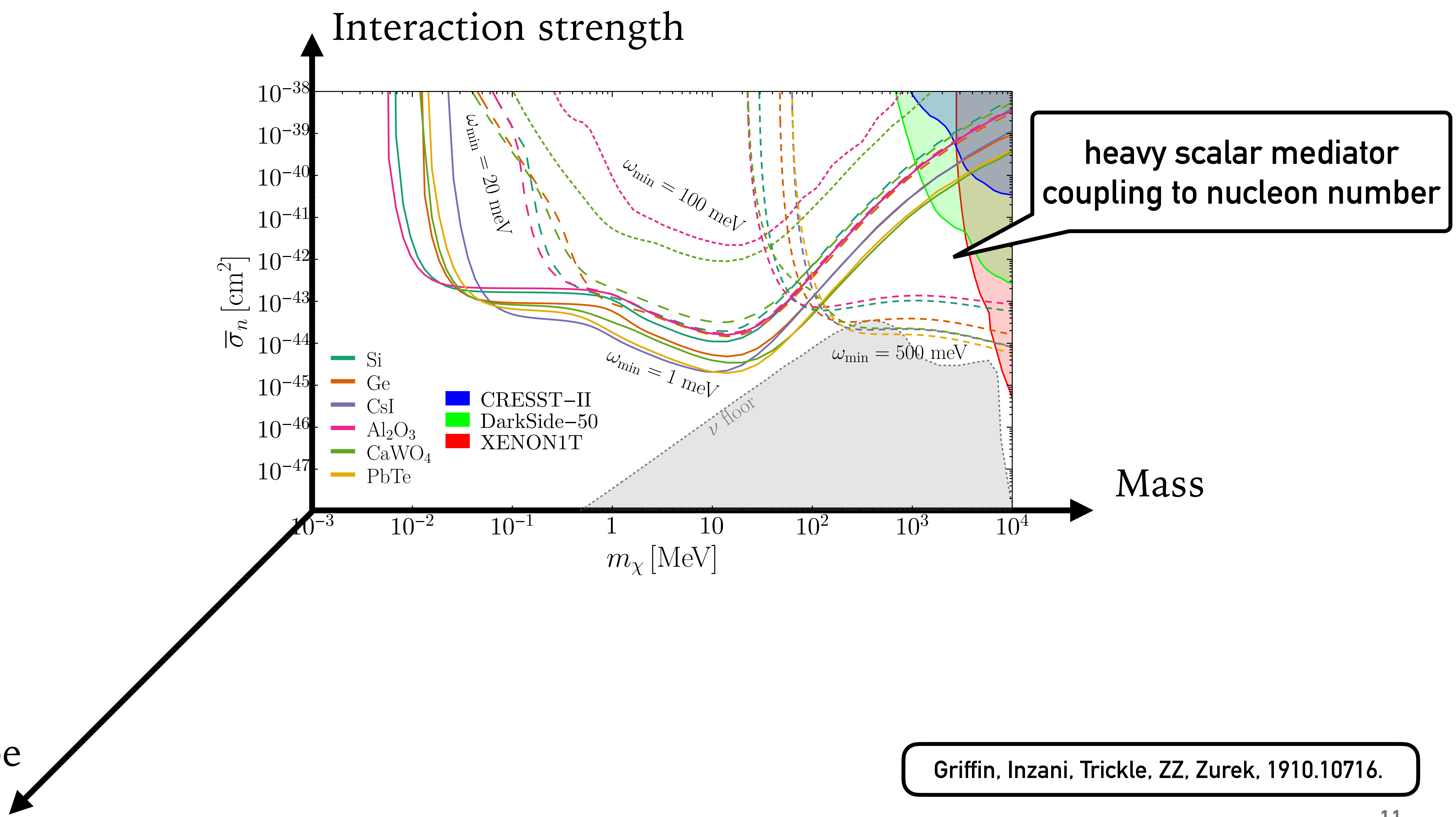
Griffin, Inzani, Trickle, ZZ, Zurek, 1910.10716.

A 3rd axis of DM's parameter space



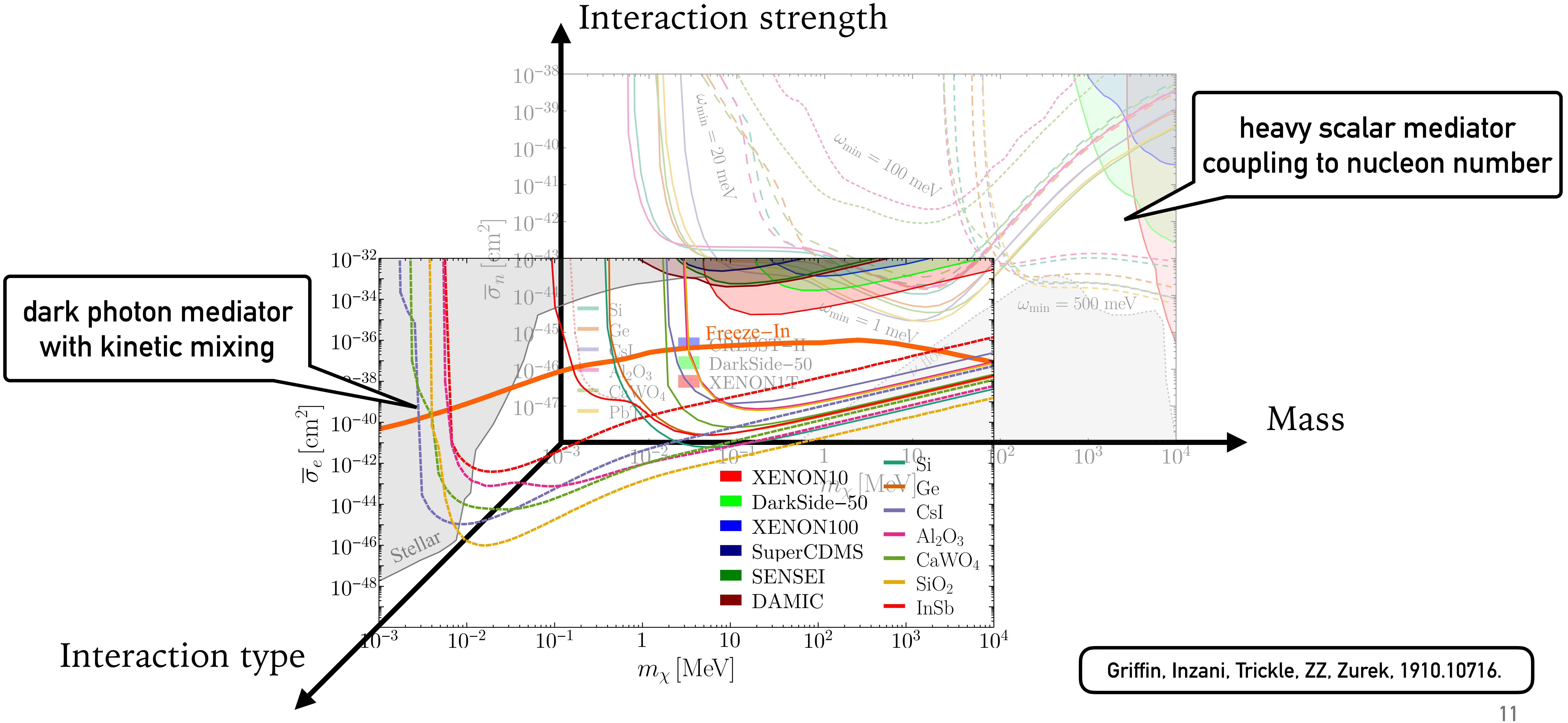
Griffin, Inzani, Trickle, ZZ, Zurek, 1910.10716.

A 3rd axis of DM's parameter space



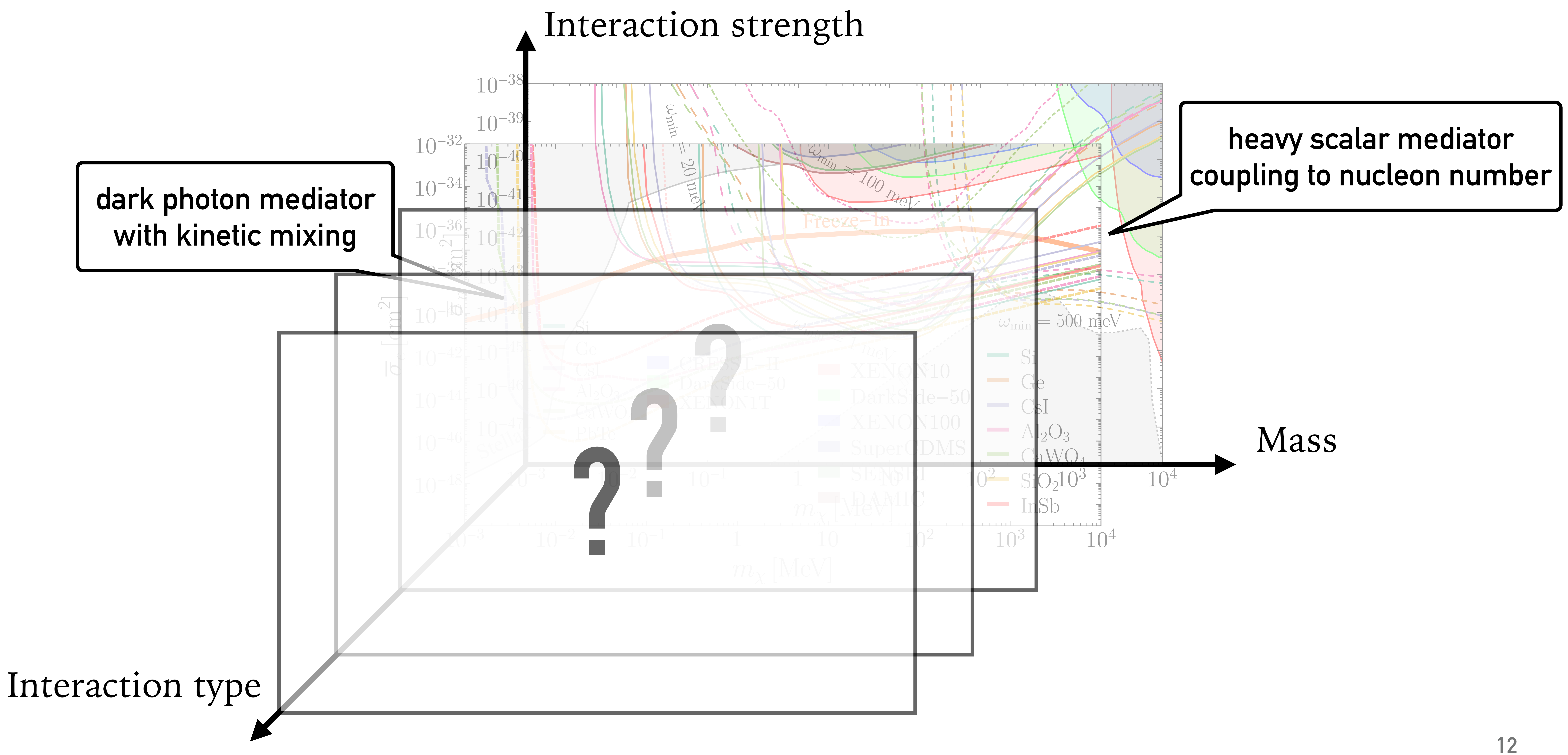
Griffin, Inzani, Trickle, ZZ, Zurek, 1910.10716.

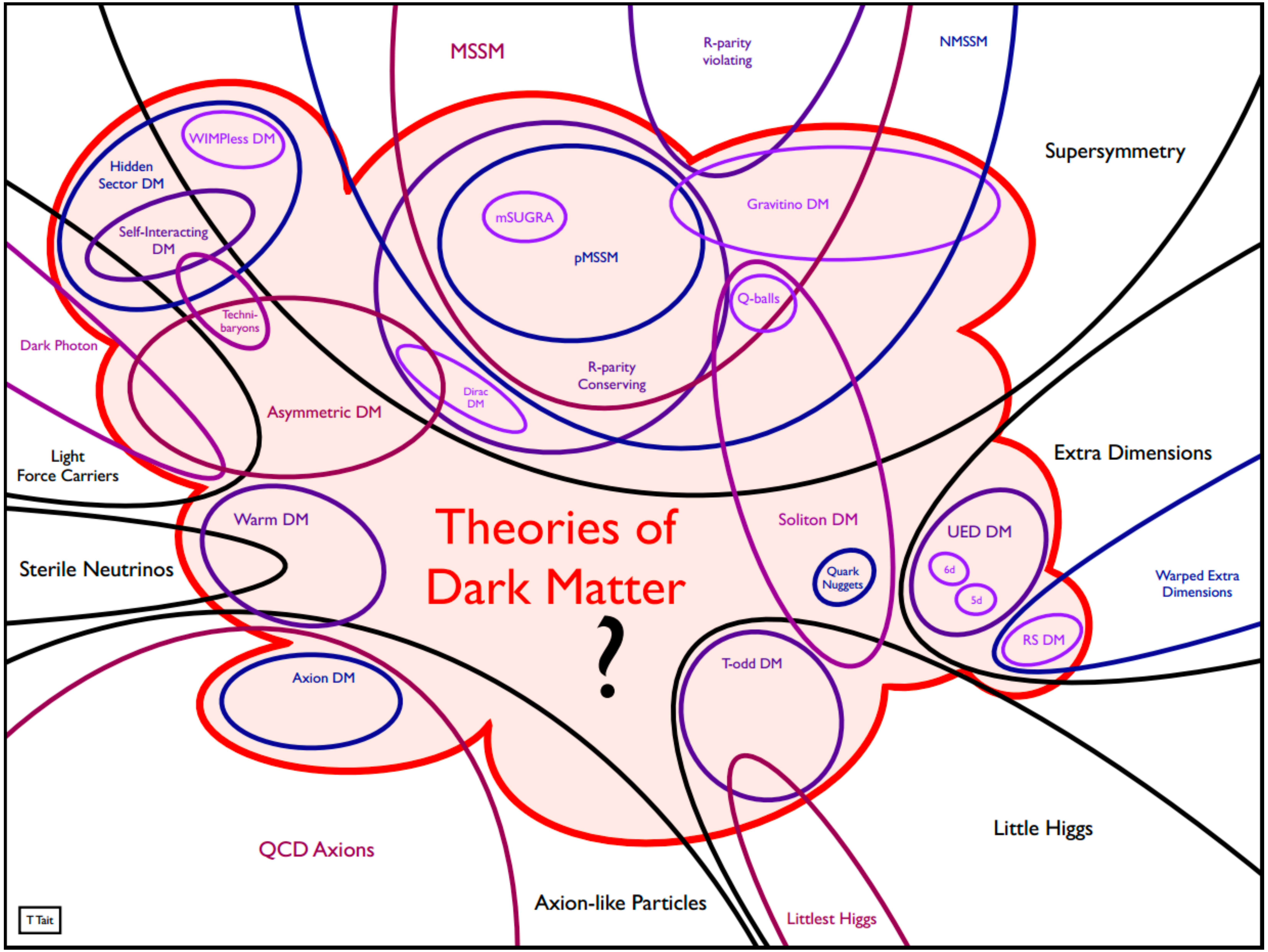
A 3rd axis of DM's parameter space



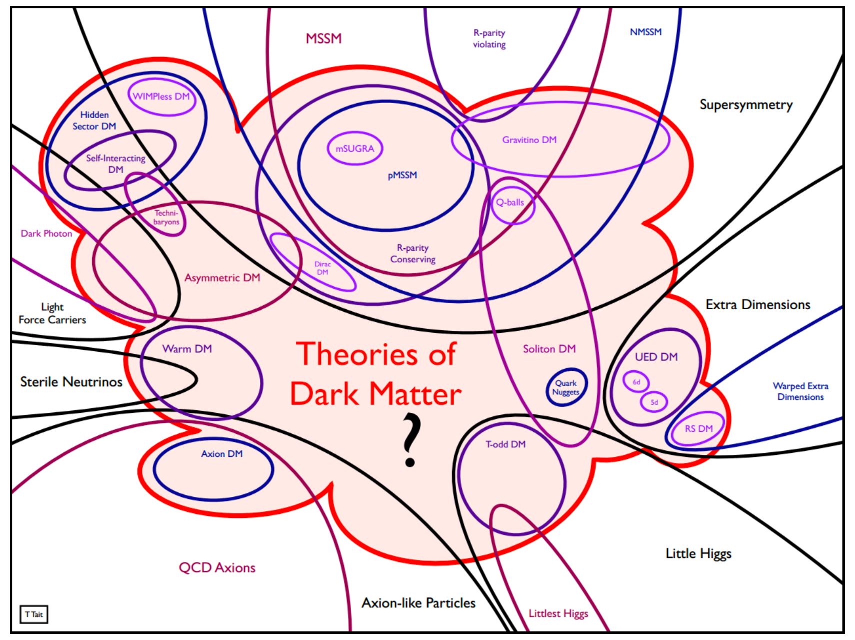
Griffin, Inzani, Trickle, ZZ, Zurek, 1910.10716.

A 3rd axis of DM's parameter space




MSSM
R-parity violating
NMSSM
Supersymmetry
WIMPless DM
Hidden Sector DM
Self-Interacting DM
Techni-baryons
mSUGRA
pMSSM
Gravitino DM
Q-balls
Dark Photon
Dirac DM
R-parity Conserving
Asymmetric DM
Light Force Carriers
Extra Dimensions
Warm DM
Theories of Dark Matter
?
Soliton DM
UED DM
Sterile Neutrinos
Quark Nuggets
6d
5d
Warped Extra Dimensions
RS DM
T-odd DM
Axion DM
Little Higgs
QCD Axions
Axion-like Particles
Littlest Higgs
T Tait

A common description at low energy



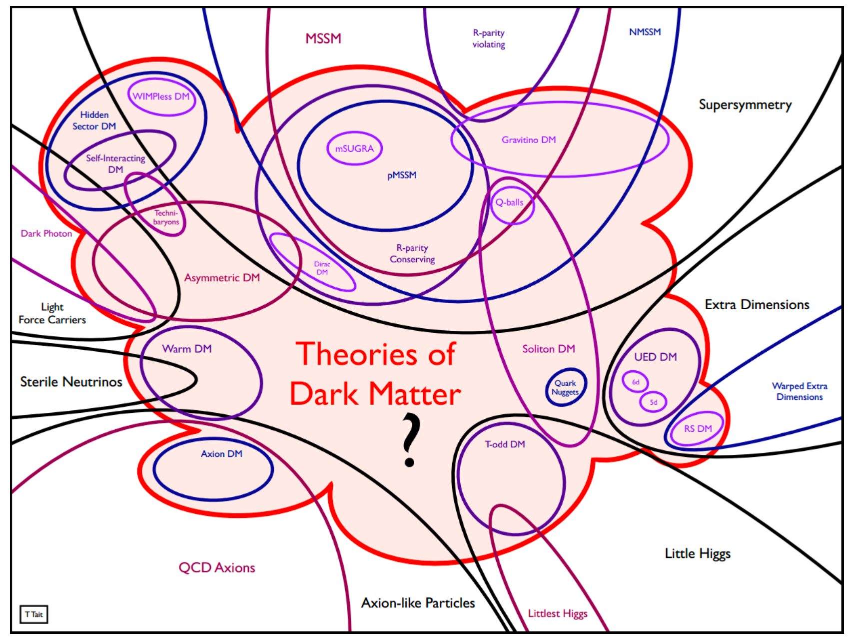
Nonrelativistic (NR) EFT of DM-SM interactions

$$\mathcal{O}_1^{(\psi)} = \mathbb{1}$$
$$\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}$$
$$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$$
$$\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp$$

$$\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$$
$$\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp$$
$$\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \boldsymbol{v}^\perp\right)$$
$$\mathcal{O}_{13}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$$
$$\mathcal{O}_{14}^{(\psi)} = \left(\boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$$
$$\mathcal{O}_{15}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$$

$$\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{S}_\psi$$
$$\mathcal{O}_6^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)\left(\boldsymbol{S}_\psi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)$$
$$\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \frac{i\boldsymbol{q}}{m_\psi}\right)$$
$$\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}$$

EFT of DM direct detection: preview



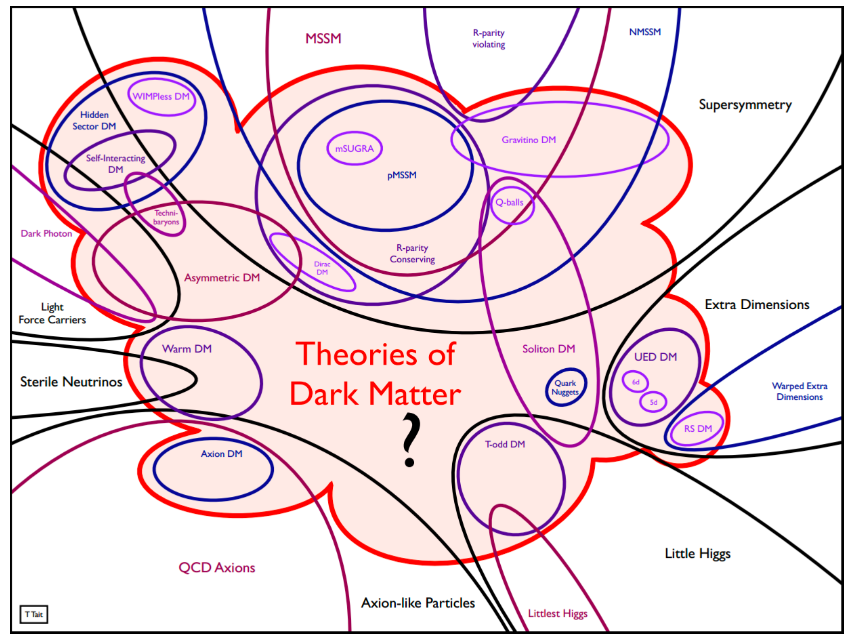
Nonrelativistic (NR) EFT of DM-SM interactions

Crystal responses

<u>DM couplings to lattice d.o.f.</u>

$\boldsymbol{N}$ (particle number)	$\boldsymbol{S}$ (spin)
$\boldsymbol{L}$ (orbital angular momentum)	$\boldsymbol{L}\otimes\boldsymbol{S}$ (spin-orbit coupling)

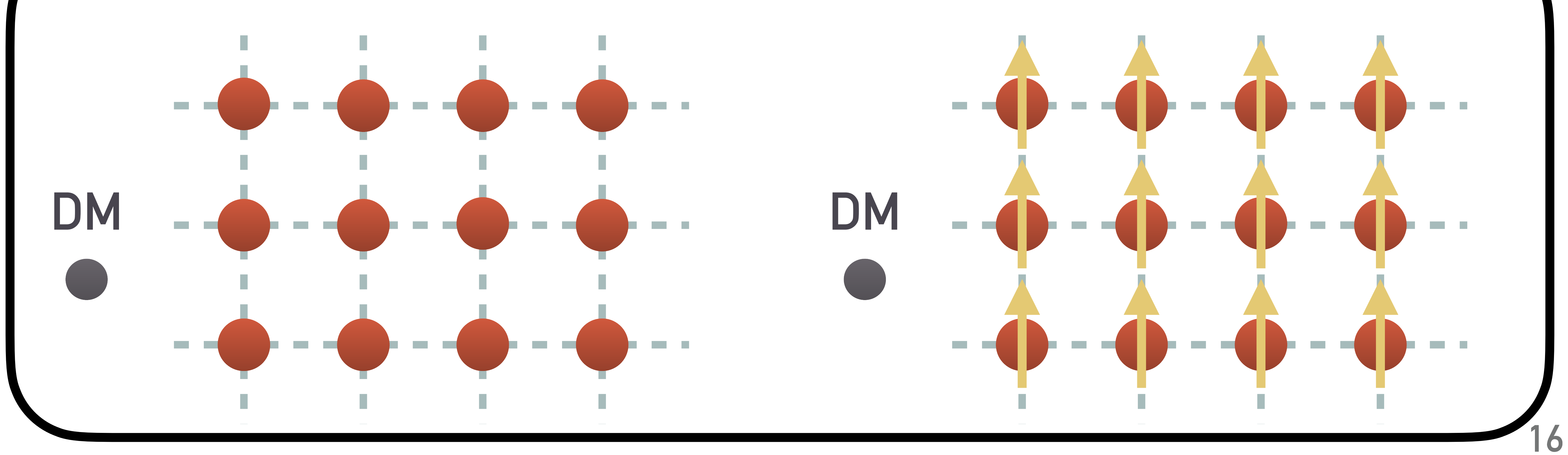
EFT of DM direct detection: preview



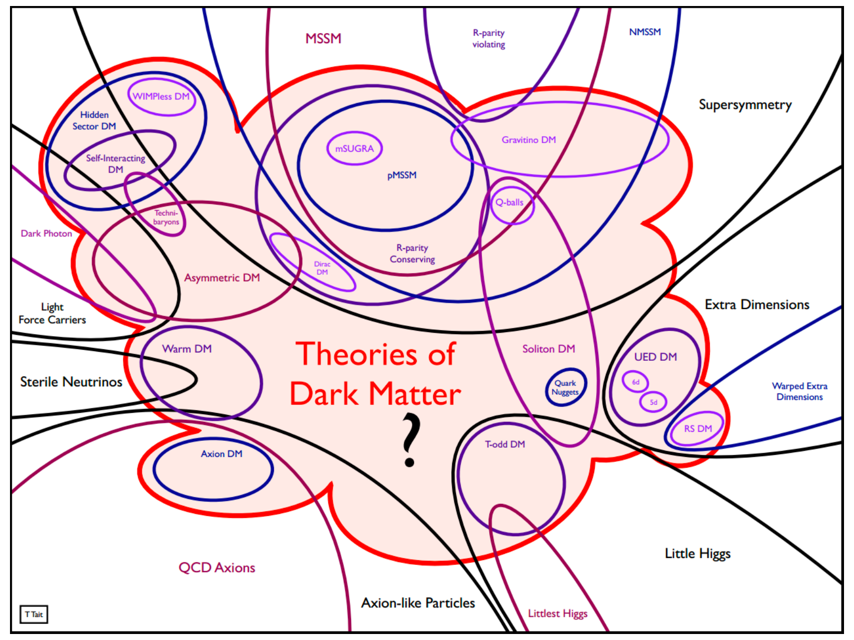
Crystal responses

Nonrelativistic (NR) EFT of DM-SM interactions

Phonon & magnon excitation rates



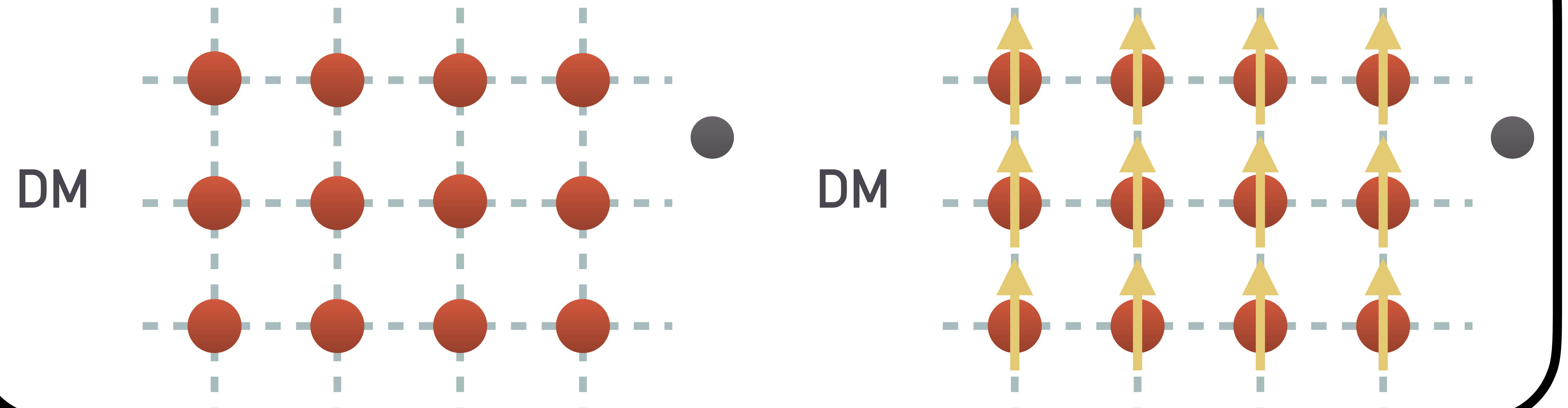
EFT of DM direct detection: preview



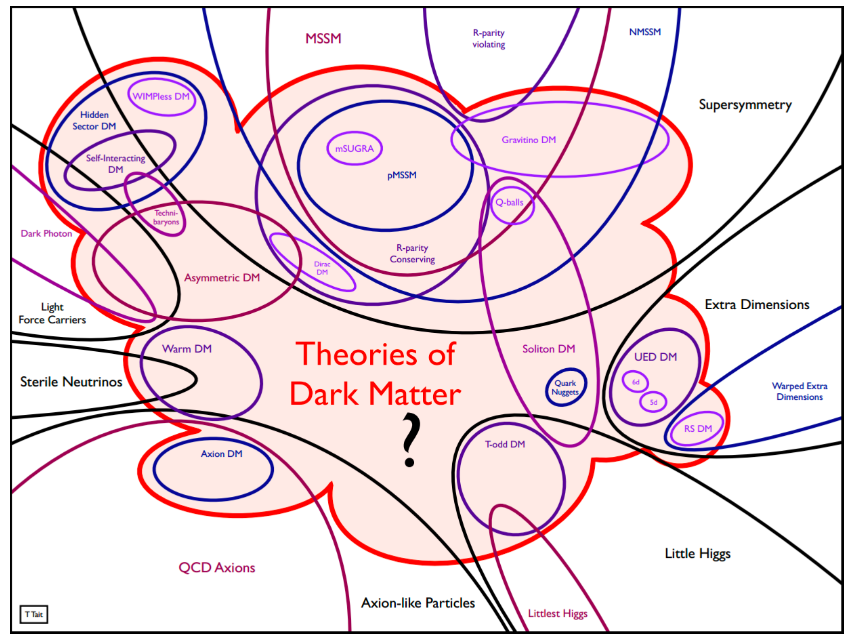
Crystal responses

Nonrelativistic (NR) EFT of DM-SM interactions

Phonon & magnon excitation rates



EFT of DM direct detection: preview



1

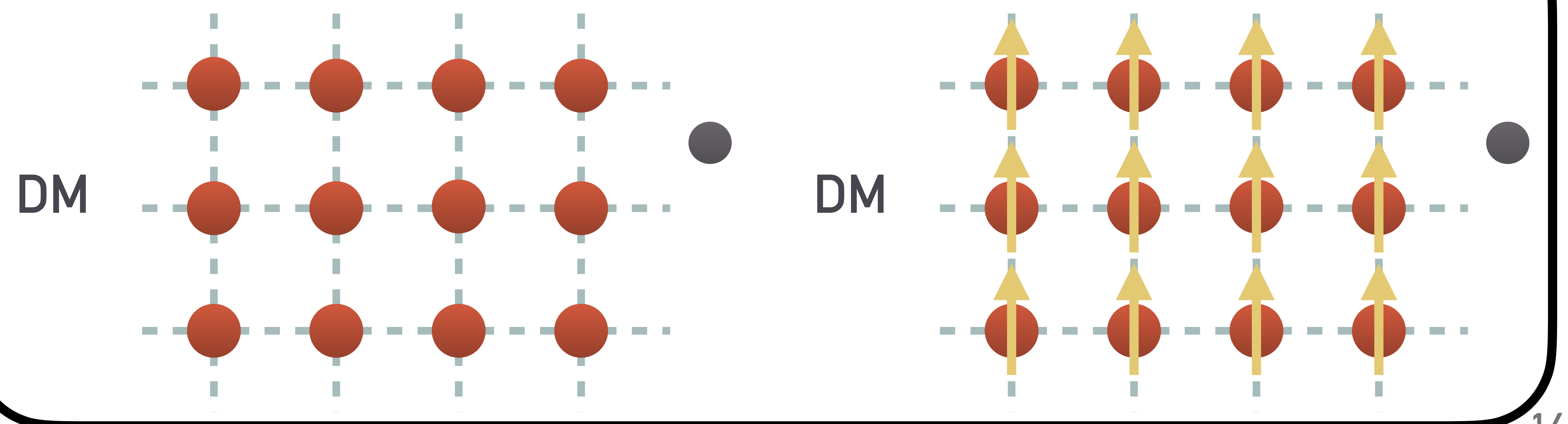
Nonrelativistic (NR) EFT of DM-SM interactions

2

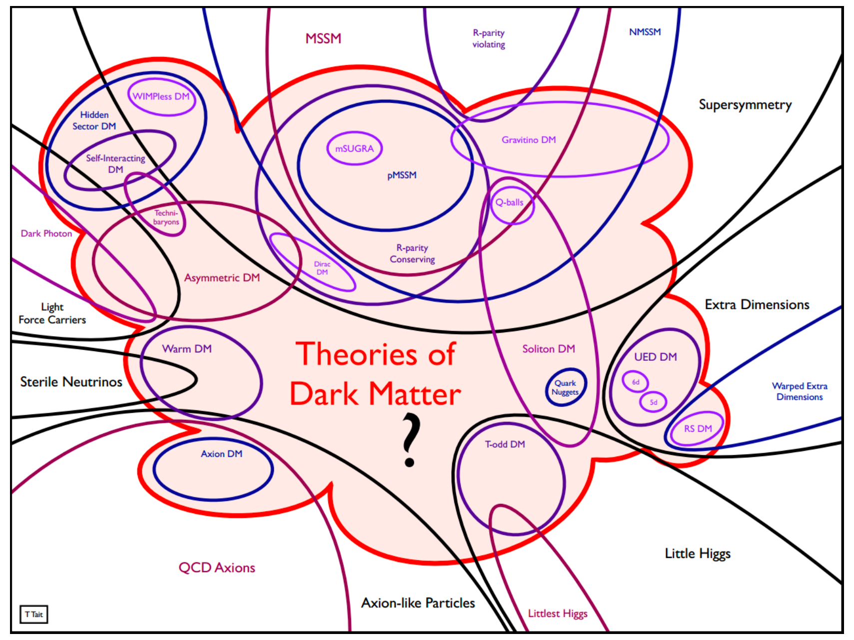
Crystal responses

3

Phonon & magnon excitation rates

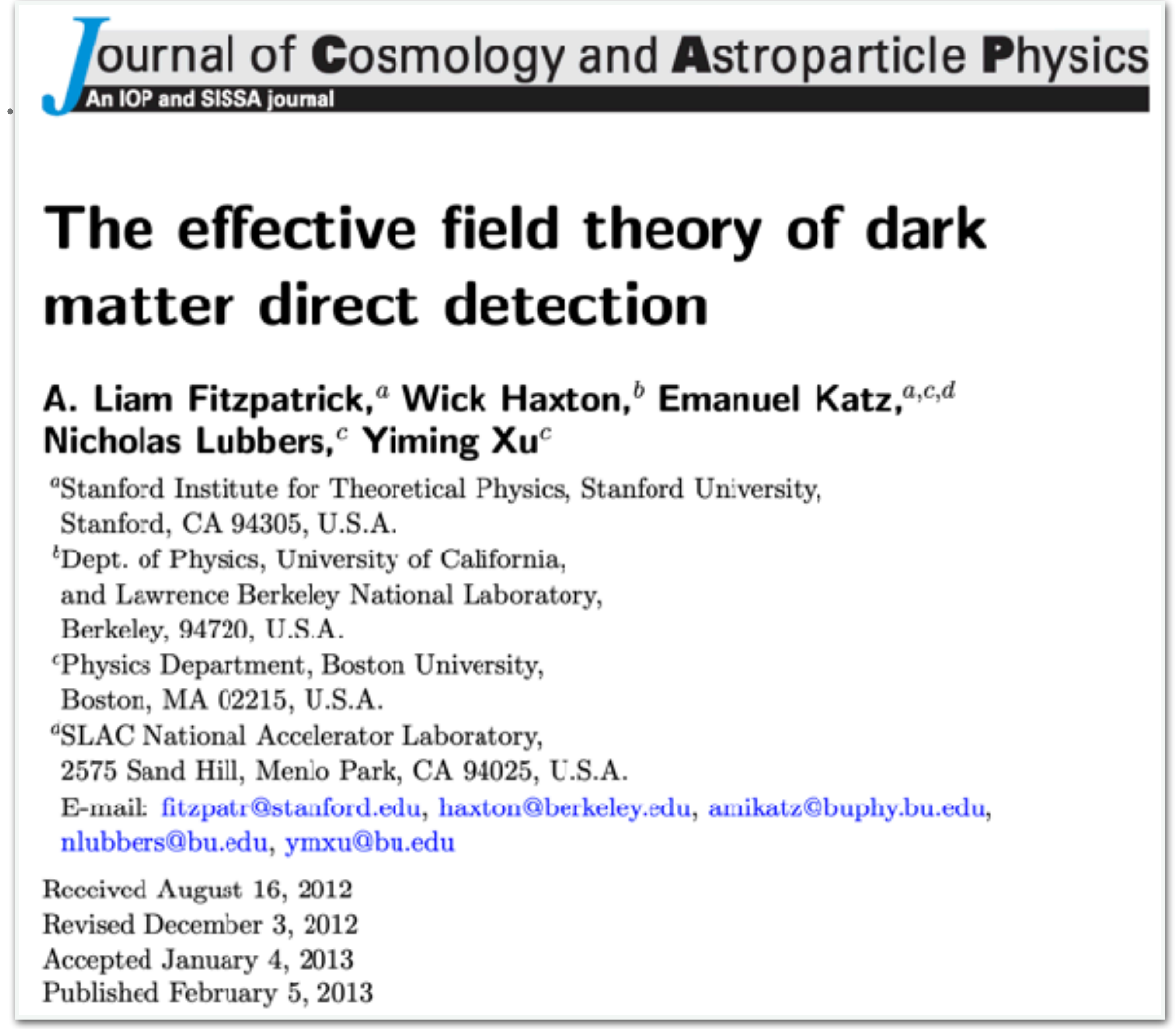


EFT of DM direct detection: preview



Similar situation in nuclear recoil calculations.

- At first, just spin-independent (SI) and spin-dependent (SD) benchmarks.
- Later on, extended to EFT.
- UV model ⇒ EFT ⇒ nuclear responses ⇒ rates.



Journal of Cosmology and Astroparticle Physics
An IOP and SISSA journal

The effective field theory of dark matter direct detection

A. Liam Fitzpatrick,[a] Wick Haxton,[b] Emanuel Katz,[a,c,d] Nicholas Lubbers,[c] Yiming Xu[c]

[a]Stanford Institute for Theoretical Physics, Stanford University, Stanford, CA 94305, U.S.A.
[b]Dept. of Physics, University of California, and Lawrence Berkeley National Laboratory, Berkeley, 94720, U.S.A.
[c]Physics Department, Boston University, Boston, MA 02215, U.S.A.
[d]SLAC National Accelerator Laboratory, 2575 Sand Hill, Menlo Park, CA 94025, U.S.A.

E-mail: fitzpatr@stanford.edu, haxton@berkeley.edu, amikatz@buphy.bu.edu, nlubbers@bu.edu, ymxu@bu.edu

Received August 16, 2012
Revised December 3, 2012
Accepted January 4, 2013
Published February 5, 2013

Crystal responses

1

2

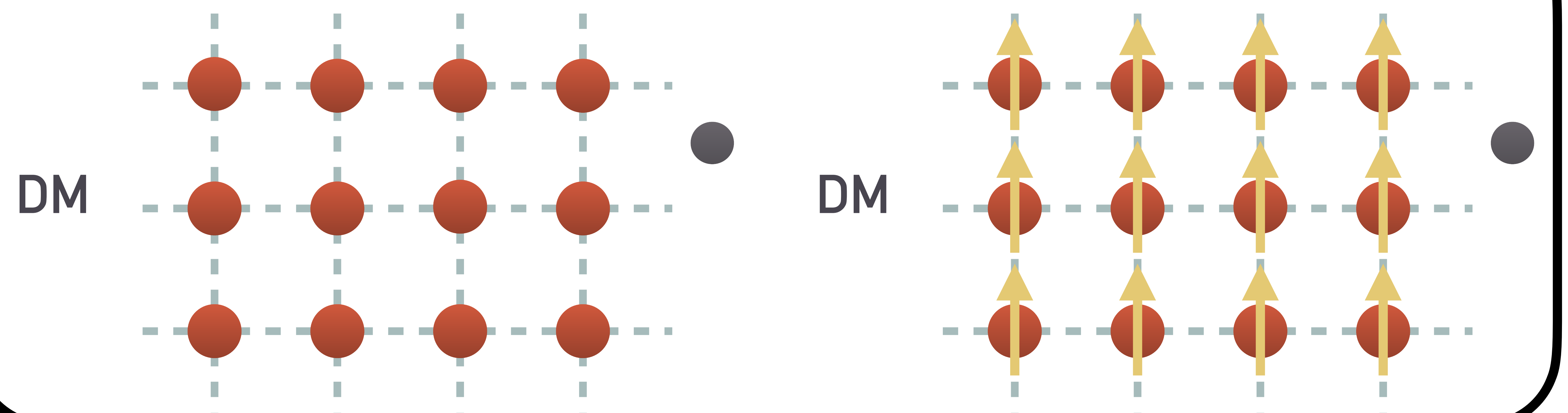
3

Nonrelativistic (NR) EFT of DM-SM interactions

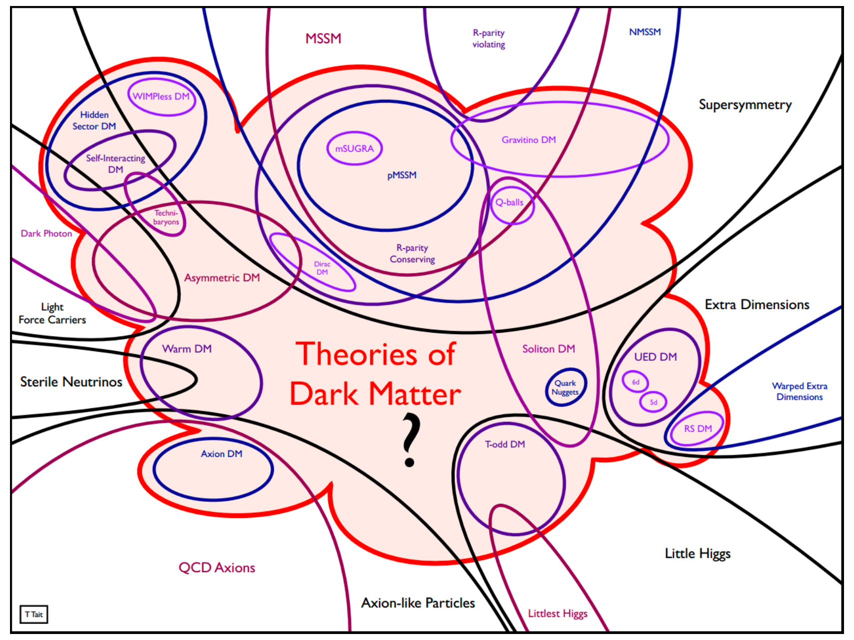
See also:
Cirelli, Del Nobile, Panci, 1307.5955.
Anand, Fitzpatrick, Haxton, 1308.6288 + 1405.6690.
Gresham, Zurek, 1401.3739.
Del Nobile, 1806.01291.

Similar calculation for electron excitations in atoms:
Catena, Emken, Spaldin, Tarantino, 1912.08204.

Phonon & magnon excitation rates



EFT of DM direct detection: preview



1

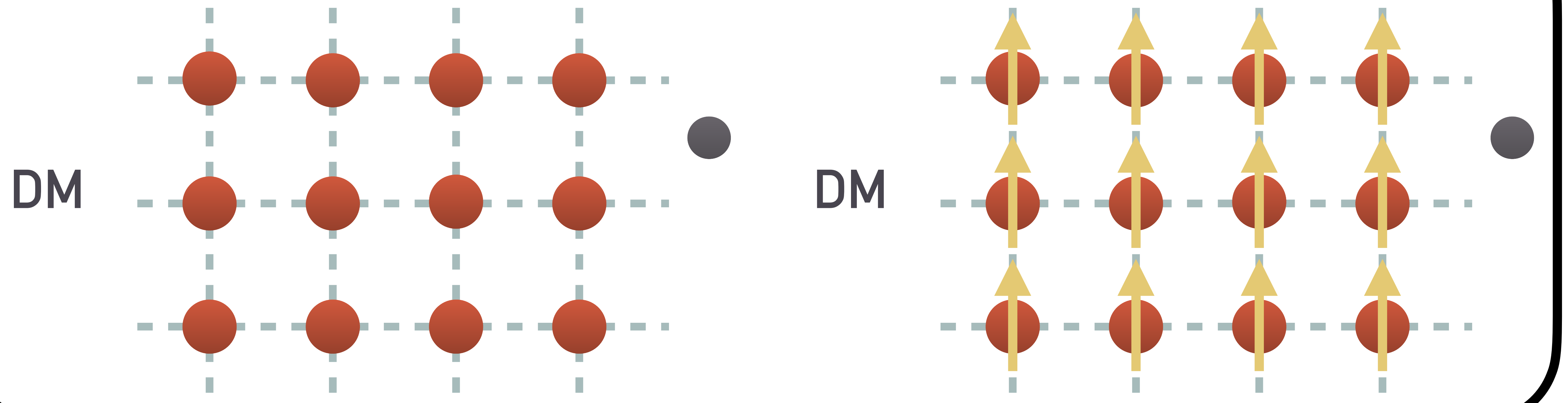
2

Crystal responses

3

Nonrelativistic (NR) EFT of DM-SM interactions

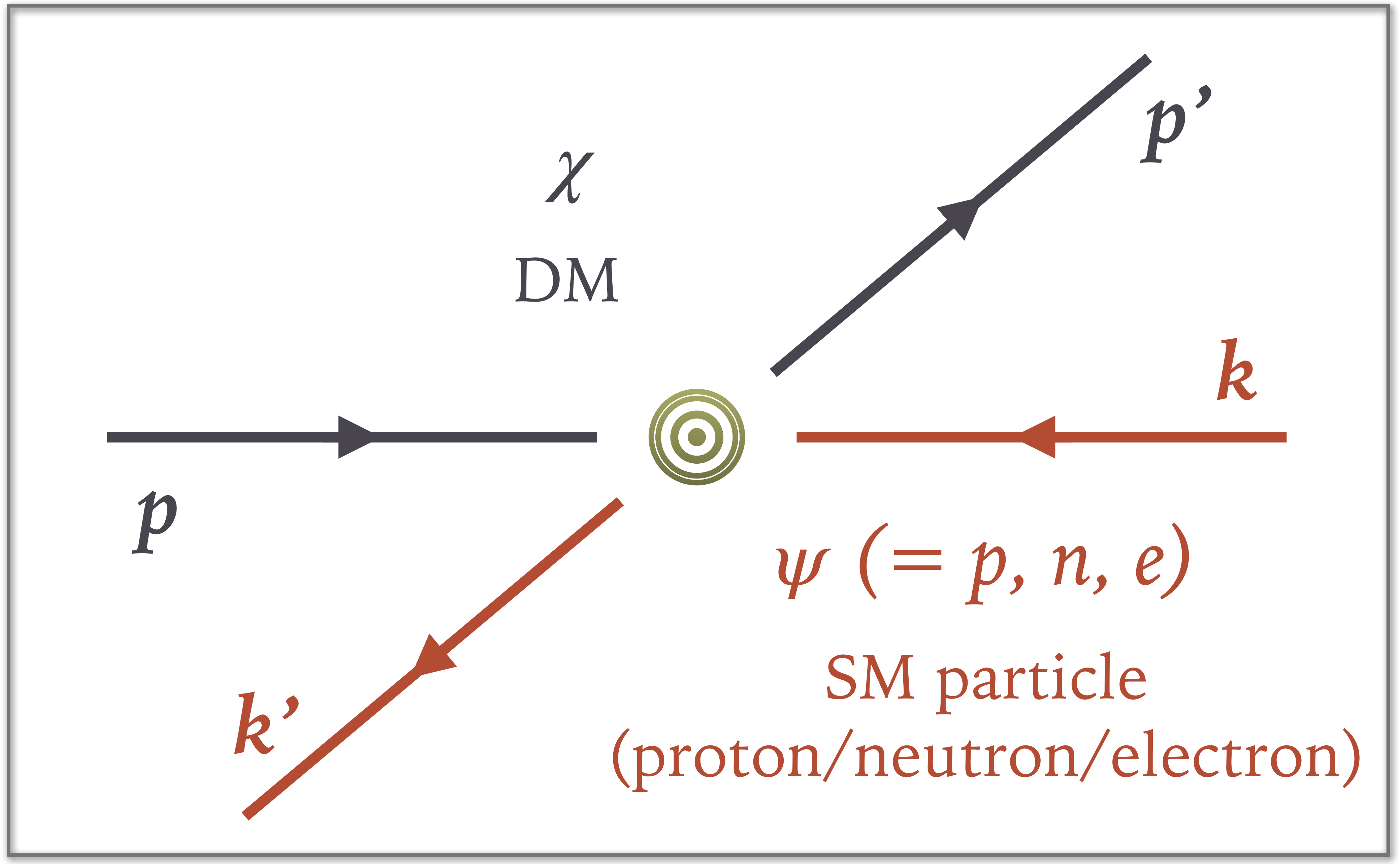
Phonon & magnon excitation rates



Step 1: building the NR EFT

- Bottom-up point of view.
  - Building blocks: spins + momenta.

S_χ S_ψ p p' k k'

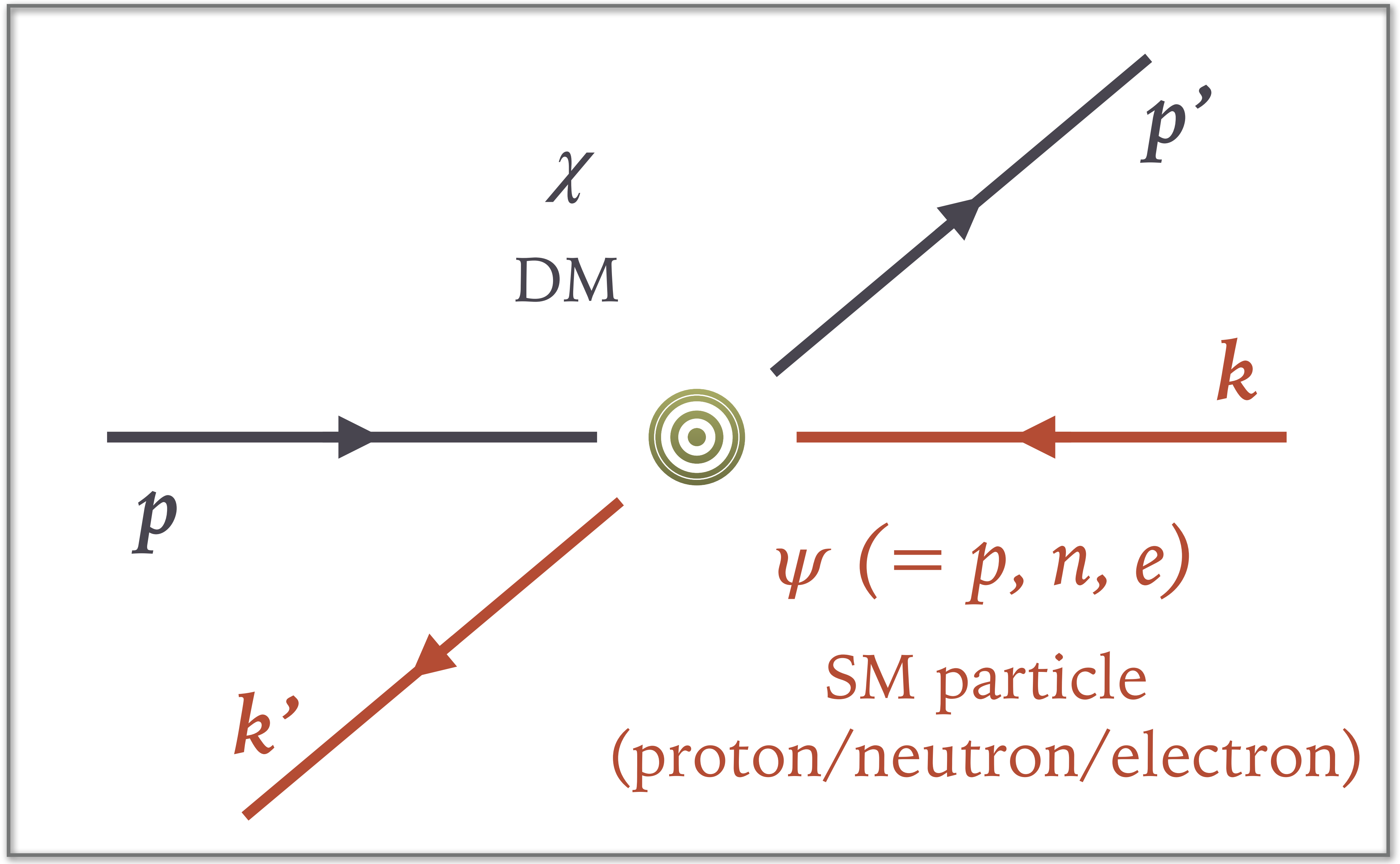


Step 1: building the NR EFT

- Bottom-up point of view.
  - Building blocks: spins + momenta.

    $\boldsymbol{S}_\chi \quad \boldsymbol{S}_\psi \quad \boldsymbol{p} \quad \boldsymbol{p}' \quad \boldsymbol{k} \quad \boldsymbol{k}'$

  - 2 constraints on 4 momenta.
    - Momentum conservation ⇒ $\boldsymbol{p}+\boldsymbol{p}'=\boldsymbol{k}+\boldsymbol{k}'$.
    - Galilean invariance ⇒ relative velocities only.
  - 2 independent kinematic variables chosen to be:



$$\boldsymbol{q} \equiv \boldsymbol{k}' - \boldsymbol{k} = \boldsymbol{p} - \boldsymbol{p}'$$

(momentum transfer to the SM target)

$\boldsymbol{P} = \boldsymbol{p}' + \boldsymbol{p}, \ \boldsymbol{K} = \boldsymbol{k}' + \boldsymbol{k}$

$$\boldsymbol{v}^\perp \equiv \frac{\boldsymbol{P}}{2m_\chi} - \frac{\boldsymbol{K}}{2m_\psi} = \boldsymbol{v} - \frac{\boldsymbol{k}}{m_\psi} - \frac{\boldsymbol{q}}{2\mu_{\chi\psi}}$$

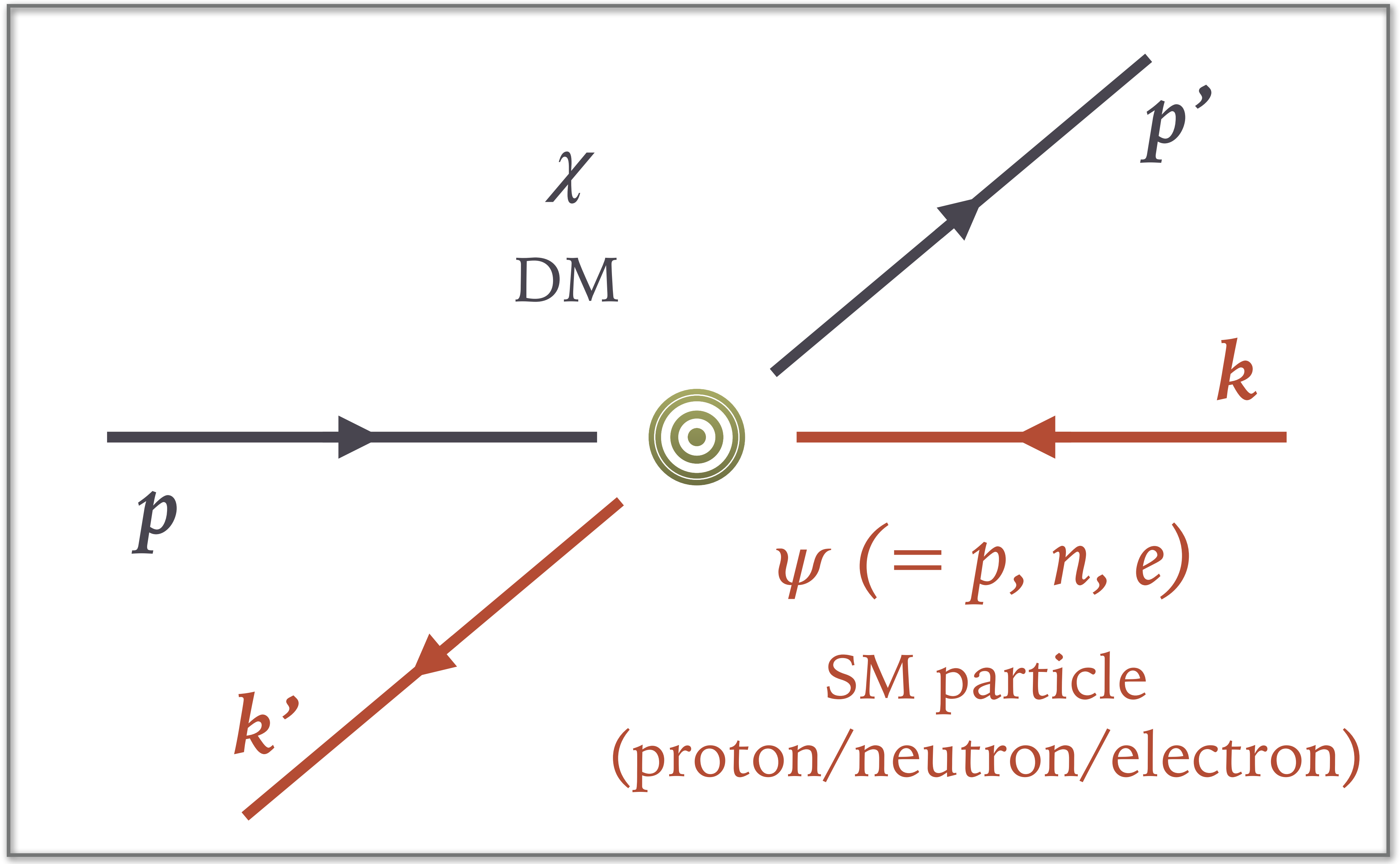
(component of relative velocity perpendicular to $\boldsymbol{q}$)

Step 1: building the NR EFT

- Bottom-up point of view.
  - Building blocks: spins + momenta.

$$\boxed{\boldsymbol{S}_\chi \quad \boldsymbol{S}_\psi \quad \boldsymbol{q} \quad \boldsymbol{v}^\perp}$$

  - 2 constraints on 4 momenta.
    - Momentum conservation ⇒ $\boldsymbol{p}+\boldsymbol{p}'=\boldsymbol{k}+\boldsymbol{k}'$.
    - Galilean invariance ⇒ relative velocities only.
  - 2 independent kinematic variables chosen to be:



$$\boldsymbol{q} \equiv \boldsymbol{k}' - \boldsymbol{k} = \boldsymbol{p} - \boldsymbol{p}'$$

(momentum transfer to the SM target)

$\boldsymbol{P} = \boldsymbol{p}' + \boldsymbol{p}, \ \boldsymbol{K} = \boldsymbol{k}' + \boldsymbol{k}$

$$\boldsymbol{v}^\perp \equiv \frac{\boldsymbol{P}}{2m_\chi} - \frac{\boldsymbol{K}}{2m_\psi} = \boldsymbol{v} - \frac{\boldsymbol{k}}{m_\psi} - \frac{\boldsymbol{q}}{2\mu_{\chi\psi}}$$

(component of relative velocity perpendicular to $\boldsymbol{q}$)

Step 1: building the NR EFT

- Bottom-up point of view.
  - Building blocks: spins + momenta.

$$\boldsymbol{S}_\chi \quad \boldsymbol{S}_\psi \quad \boldsymbol{q} \quad \boldsymbol{v}^\perp$$

  - Enumerate operators.

Fitzpatrick, Haxton, Katz, Lubbers, Xu, 1203.3542.
Del Nobile, 1806.01291.

  - Conveniently organize into 4 categories, according to whether the operator depends on $\boldsymbol{S}_\psi$ and $\boldsymbol{v}^\perp$.
    - More on this classification later.

Interaction Type	**NR Operators**
Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_1^{(\psi)} = \mathbb{1}$ $\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{S}_\psi$ $\mathcal{O}_6^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)\left(\boldsymbol{S}_\psi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp$ $\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_{13}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{14}^{(\psi)} = \left(\boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{15}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$

Step 1: building the NR EFT

- Top-down point of view.
  - Start from a UV theory and take the NR limit.
  - For a spin-1/2 fermion field:

$$\psi(\boldsymbol{x},t) = e^{-im_\psi t}\,\frac{1}{\sqrt{2}}\begin{pmatrix}\left(1-\frac{\boldsymbol{\sigma}\cdot\boxed{\boldsymbol{k}}}{2m_\psi+\varepsilon}\right)\psi^+(\boldsymbol{x},t)\\ \left(1+\frac{\boldsymbol{\sigma}\cdot\boxed{\boldsymbol{k}}}{2m_\psi+\varepsilon}\right)\psi^+(\boldsymbol{x},t)\end{pmatrix}$$

position space operators NR fields

$\boldsymbol{k} = -i\,\boldsymbol{\nabla} - \boldsymbol{A}\,,\ \ \varepsilon = i\,\partial_t - \Phi$

Step 1: building the NR EFT

- Top-down point of view.
  - Start from a UV theory and take the NR limit.
  - For a spin-1/2 fermion field:

$$\psi(\boldsymbol{x},t) = e^{-im_\psi t}\,\frac{1}{\sqrt{2}}\begin{pmatrix}\left(1-\frac{\boldsymbol{\sigma}\cdot\boxed{\boldsymbol{k}}}{2m_\psi+\boxed{\varepsilon}}\right)\boxed{\psi^+(\boldsymbol{x},t)}\\ \left(1+\frac{\boldsymbol{\sigma}\cdot\boxed{\boldsymbol{k}}}{2m_\psi+\boxed{\varepsilon}}\right)\boxed{\psi^+(\boldsymbol{x},t)}\end{pmatrix}$$

position space operators NR fields

$\boldsymbol{k} = -i\,\nabla - \boldsymbol{A}\,,\ \ \varepsilon = i\,\partial_t - \Phi$

Couplings to scalar & vector mediators

Lagrangian Term	Coupling Type	(Effective) Current → NR Limit
$g_S\phi\bar\psi\psi$	Scalar	$J_S = \bar\psi\psi \to \mathbb{1}$
$g_P\phi\bar\psi i\gamma^5\psi$	Pseudoscalar	$J_P = \bar\psi i\gamma^5\psi \to -\frac{i\boldsymbol{q}}{m_\psi}\cdot\boldsymbol{S}_\psi$
$g_V V_\mu\bar\psi\gamma^\mu\psi$	Vector	$J_V^\mu = \bar\psi\gamma^\mu\psi \to \left(\mathbb{1},\ \frac{\boldsymbol{K}}{2m_\psi}-\frac{i\boldsymbol{q}}{m_\psi}\times\boldsymbol{S}_\psi\right)$
$g_A V_\mu\bar\psi\gamma^\mu\gamma^5\psi$	Axial vector	$J_A^\mu = \bar\psi\gamma^\mu\gamma^5\psi \to \left(\frac{\boldsymbol{K}}{m_\psi}\cdot\boldsymbol{S}_\psi,\ 2\boldsymbol{S}_\psi\right)$
$\frac{g_{\rm edm}}{4m_\psi}V_{\mu\nu}\bar\psi\sigma^{\mu\nu}i\gamma^5\psi$	Electric dipole	$J^\mu_{\rm edm} = \frac{1}{2m_\psi}\partial_\nu\left(\bar\psi\sigma^{\mu\nu}i\gamma^5\psi\right) \to \left(-\frac{i\boldsymbol{q}}{m_\psi}\cdot\boldsymbol{S}_\psi,\ \frac{i\omega}{m_\psi}\boldsymbol{S}_\psi+\frac{i\boldsymbol{q}}{m_\psi}\times\left(\frac{\boldsymbol{K}}{2m_\psi}\times\boldsymbol{S}_\psi\right)\right)$
$\frac{g_{\rm mdm}}{4m_\psi}V_{\mu\nu}\bar\psi\sigma^{\mu\nu}\psi$	Magnetic dipole	$J^\mu_{\rm mdm} = \frac{1}{2m_\psi}\partial_\nu\left(\bar\psi\sigma^{\mu\nu}\psi\right) \to \left(\frac{i\boldsymbol{q}}{m_\psi}\cdot\left(\frac{\boldsymbol{K}}{2m_\psi}\times\boldsymbol{S}_\psi\right)-\frac{\boldsymbol{q}^2}{4m_\psi^2},\ -\frac{i\boldsymbol{q}}{m_\psi}\times\boldsymbol{S}_\psi\right)$
$\frac{g_{\rm ana}}{4m_\psi^2}(\partial^\nu V_{\mu\nu})(\bar\psi\gamma^\mu\gamma^5\psi)$	Anapole	$J^\mu_{\rm ana} = -\frac{1}{4m_\psi^2}(g^{\mu\nu}\partial^2-\partial^\mu\partial^\nu)(\bar\psi\gamma_\nu\gamma^5\psi) \to -\frac{\boldsymbol{q}^2}{4m_\psi^2}J_A^\mu+\left(\frac{\boldsymbol{q}}{m_\psi}\cdot\boldsymbol{S}_\psi\right)\frac{q^\mu}{2m_\psi}$
$\frac{g_{V2}}{4m_\psi^2}(\partial^\nu V_{\mu\nu})(\bar\psi\gamma^\mu\psi)$	Vector ($\mathcal{O}(q^2)$)	$J^\mu_{V2} = -\frac{1}{4m_\psi^2}\partial^2(\bar\psi\gamma^\mu\psi) \to -\frac{\boldsymbol{q}^2}{4m_\psi^2}J_V^\mu$

Step 1: building the NR EFT

- Top-down point of view.
  - Start from a UV theory and take the NR limit.
  - For a spin-1/2 fermion field:

$$\psi(\boldsymbol{x},t) = e^{-im_\psi t}\,\frac{1}{\sqrt{2}}\begin{pmatrix}\left(1-\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{2m_\psi+\varepsilon}\right)\psi^+(\boldsymbol{x},t)\\ \left(1+\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{2m_\psi+\varepsilon}\right)\psi^+(\boldsymbol{x},t)\end{pmatrix}$$

EFT building blocks emerge.

Couplings to scalar & vector mediators

Lagrangian Term	Coupling Type	(Effective) Current → NR Limit
$g_S\phi\bar\psi\psi$	Scalar	$J_S = \bar\psi\psi \to \mathbb{1}$
$g_P\phi\bar\psi i\gamma^5\psi$	Pseudoscalar	$J_P = \bar\psi i\gamma^5\psi \to -\frac{i\boldsymbol{q}}{m_\psi}\cdot\boldsymbol{S}_\psi$
$g_V V_\mu\bar\psi\gamma^\mu\psi$	Vector	$J_V^\mu = \bar\psi\gamma^\mu\psi \to \left(\mathbb{1},\, \frac{\boldsymbol{K}}{2m_\psi} - \frac{i\boldsymbol{q}}{m_\psi}\times\boldsymbol{S}_\psi\right)$
$g_A V_\mu\bar\psi\gamma^\mu\gamma^5\psi$	Axial vector	$J_A^\mu = \bar\psi\gamma^\mu\gamma^5\psi \to \left(\frac{\boldsymbol{K}}{m_\psi}\cdot\boldsymbol{S}_\psi,\, 2\boldsymbol{S}_\psi\right)$
$\frac{g_{\text{edm}}}{4m_\psi}V_{\mu\nu}\bar\psi\sigma^{\mu\nu}i\gamma^5\psi$	Electric dipole	$J^\mu_{\text{edm}} = \frac{1}{2m_\psi}\partial_\nu\left(\bar\psi\sigma^{\mu\nu}i\gamma^5\psi\right) \to \left(-\frac{i\boldsymbol{q}}{m_\psi}\cdot\boldsymbol{S}_\psi,\, \frac{i\omega}{m_\psi}\boldsymbol{S}_\psi + \frac{i\boldsymbol{q}}{m_\psi}\times\left(\frac{\boldsymbol{K}}{2m_\psi}\times\boldsymbol{S}_\psi\right)\right)$
$\frac{g_{\text{mdm}}}{4m_\psi}V_{\mu\nu}\bar\psi\sigma^{\mu\nu}\psi$	Magnetic dipole	$J^\mu_{\text{mdm}} = \frac{1}{2m_\psi}\partial_\nu\left(\bar\psi\sigma^{\mu\nu}\psi\right) \to \left(\frac{i\boldsymbol{q}}{m_\psi}\cdot\left(\frac{\boldsymbol{K}}{2m_\psi}\times\boldsymbol{S}_\psi\right) - \frac{\boldsymbol{q}^2}{4m_\psi^2},\, -\frac{i\boldsymbol{q}}{m_\psi}\times\boldsymbol{S}_\psi\right)$
$\frac{g_{\text{ana}}}{4m_\psi^2}(\partial^\nu V_{\mu\nu})(\bar\psi\gamma^\mu\gamma^5\psi)$	Anapole	$J^\mu_{\text{ana}} = -\frac{1}{4m_\psi^2}(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)(\bar\psi\gamma_\nu\gamma^5\psi) \to -\frac{\boldsymbol{q}^2}{4m_\psi^2}J_A^\mu + \left(\frac{\boldsymbol{q}}{m_\psi}\cdot\boldsymbol{S}_\psi\right)\frac{q^\mu}{2m_\psi}$
$\frac{g_{V2}}{4m_\psi^2}(\partial^\nu V_{\mu\nu})(\bar\psi\gamma^\mu\psi)$	Vector ($\mathcal{O}(q^2)$)	$J^\mu_{V2} = -\frac{1}{4m_\psi^2}\partial^2(\bar\psi\gamma^\mu\psi) \to -\frac{\boldsymbol{q}^2}{4m_\psi^2}J_V^\mu$

Step 1: building the NR EFT

- Top-down point of view.
  - Start from a UV theory and take the NR limit.
  - For a spin-1/2 fermion field:

$$\psi(\boldsymbol{x},t) = e^{-im_\psi t}\frac{1}{\sqrt{2}}\begin{pmatrix}\left(1-\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{2m_\psi+\varepsilon}\right)\psi^+(\boldsymbol{x},t)\\ \left(1+\frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{2m_\psi+\varepsilon}\right)\psi^+(\boldsymbol{x},t)\end{pmatrix}$$

EFT building blocks emerge.

Couplings to scalar & vector mediators

Lagrangian Term	Coupling Type	(Effective) Current → NR Limit
$g_S\phi\bar{\psi}\psi$	Scalar	$J_S = \bar{\psi}\psi \to \mathbb{1}$
$g_P\phi\bar{\psi}i\gamma^5\psi$	Pseudoscalar	$J_P = \bar{\psi}i\gamma^5\psi \to -\frac{i\boldsymbol{q}}{m_\psi}\cdot\boldsymbol{S}_\psi$
$g_V V_\mu\bar{\psi}\gamma^\mu\psi$	Vector	$J_V^\mu = \bar{\psi}\gamma^\mu\psi \to \left(\mathbb{1},\, \frac{\boldsymbol{K}}{2m_\psi} - \frac{i\boldsymbol{q}}{m_\psi}\times\boldsymbol{S}_\psi\right)$
$g_A V_\mu\bar{\psi}\gamma^\mu\gamma^5\psi$	Axial vector	$J_A^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi \to \left(\frac{\boldsymbol{K}}{m_\psi}\cdot\boldsymbol{S}_\psi,\, 2\boldsymbol{S}_\psi\right)$
$\frac{g_{\rm edm}}{4m_\psi}V_{\mu\nu}\bar{\psi}\sigma^{\mu\nu}i\gamma^5\psi$	Electric dipole	$J_{\rm edm}^\mu = \frac{1}{2m_\psi}\partial_\nu\left(\bar{\psi}\sigma^{\mu\nu}i\gamma^5\psi\right) \to \left(-\frac{i\boldsymbol{q}}{m_\psi}\cdot\boldsymbol{S}_\psi,\, \frac{i\omega}{m_\psi}\boldsymbol{S}_\psi + \frac{i\boldsymbol{q}}{m_\psi}\times\left(\frac{\boldsymbol{K}}{2m_\psi}\times\boldsymbol{S}_\psi\right)\right)$
$\frac{g_{\rm mdm}}{4m_\psi}V_{\mu\nu}\bar{\psi}\sigma^{\mu\nu}\psi$	Magnetic dipole	$J_{\rm mdm}^\mu = \frac{1}{2m_\psi}\partial_\nu\left(\bar{\psi}\sigma^{\mu\nu}\psi\right) \to \left(\frac{i\boldsymbol{q}}{m_\psi}\cdot\left(\frac{\boldsymbol{K}}{2m_\psi}\times\boldsymbol{S}_\psi\right) - \frac{\boldsymbol{q}^2}{4m_\psi^2},\, -\frac{i\boldsymbol{q}}{m_\psi}\times\boldsymbol{S}_\psi\right)$
[illegible]	[illegible]	[illegible]

$$\mathcal{L}_{\rm eff} = \chi^-\left[\varepsilon - \frac{\boldsymbol{p}^2}{2m_\chi} + \mathcal{O}(m_\chi^{-2})\right]\chi^+ + \psi^-\left[\varepsilon - \frac{\boldsymbol{k}^2}{2m_\psi} + \mathcal{O}(m_\psi^{-2})\right]\psi^+ + \sum_i \sum_{\psi=p,n,e} c_i^{(\psi)}\mathcal{O}_i^{(\psi)}\chi^-\chi^+\psi^-\psi^+$$

Step 1: building the NR EFT

- Top-down point of view.
  - Example: dark photon mediator.

$$\mathcal{L} \supset -g_e V_\mu J^\mu_{\mathrm{EM}} + \ldots$$

Several possibilities on how the DM couples.

Interaction Type	**NR Operators**
Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_1^{(\psi)} = \mathbb{1}$ $\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{S}_\psi$ $\mathcal{O}_6^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)\left(\boldsymbol{S}_\psi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp$ $\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_{13}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{14}^{(\psi)} = \left(\boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{15}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$

Step 1: building the NR EFT

- Top-down point of view.
  - Example: dark photon mediator.

$$\mathcal{L} \supset -g_e V_\mu J^\mu_{\rm EM} + \dots$$

Several possibilities on how the DM couples.

Millicharged DM

$$g_\chi V_\mu \bar{\chi}\gamma^\mu \chi$$

$$\Rightarrow \quad c_1^{(\psi)} = -\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2 + m_V^2} \qquad \text{(standard SI)}$$

where $\quad g_e^{\rm eff} = \frac{\boldsymbol{q}^2}{\boldsymbol{q}\cdot\boldsymbol{\varepsilon}\cdot\boldsymbol{q}}\, g_e = -g_p^{\rm eff} \quad$ (screened couplings)

Interaction Type	NR Operators
Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_1^{(\psi)} = \mathbb{1}$ $\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{S}_\psi$ $\mathcal{O}_6^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)\left(\boldsymbol{S}_\psi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp$ $\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_{13}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{14}^{(\psi)} = \left(\boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{15}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$

Step 1: building the NR EFT

- Top-down point of view.
  - Example: dark photon mediator.

$$\mathcal{L} \supset -g_e V_\mu J^\mu_{\rm EM} + \dots$$

Several possibilities on how the DM couples.

Millicharged DM

$$g_\chi V_\mu \bar{\chi}\gamma^\mu \chi$$

$$\Rightarrow \quad c_1^{(\psi)} = -\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2 + m_V^2} \qquad \text{(standard SI)}$$

where $\quad g_e^{\rm eff} = \frac{\boldsymbol{q}^2}{\boldsymbol{q}\cdot\varepsilon\cdot\boldsymbol{q}}\, g_e = -g_p^{\rm eff} \quad$ (screened couplings)

Interaction Type	NR Operators
Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\boxed{\mathcal{O}_1^{(\psi)} = \mathbb{1}}$ $\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{S}_\psi$ $\mathcal{O}_6^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)\left(\boldsymbol{S}_\psi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp$ $\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_{13}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{14}^{(\psi)} = \left(\boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{15}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$

Step 1: building the NR EFT

- Top-down point of view.
  - Example: dark photon mediator.

$$\mathcal{L} \supset -g_e V_\mu J^\mu_{\rm EM} + \ldots$$

Millicharged DM

$$c_1^{(\psi)} = -\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2 + m_V^2}$$

Interaction Type	NR Operators
Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_1^{(\psi)} = \mathbb{1}$ $\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{S}_\psi$ $\mathcal{O}_6^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)\left(\boldsymbol{S}_\psi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp$ $\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_{13}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{14}^{(\psi)} = \left(\boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{15}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$

Step 1: building the NR EFT

- Top-down point of view.
  - Example: dark photon mediator.

$$\mathcal{L} \supset -g_e V_\mu J^\mu_{\mathrm{EM}} + \dots$$

Millicharged DM

$$c_1^{(\psi)} = -\frac{g_\chi g_\psi^{\mathrm{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

Electric dipole DM

$$\frac{g_\chi}{4m_\chi} V_{\mu\nu} \bar{\chi} \sigma^{\mu\nu} i\gamma^5 \chi$$

$$\Rightarrow \quad c_{11}^{(\psi)} = -\frac{m_\psi}{m_\chi} \frac{g_\chi g_\psi^{\mathrm{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

Interaction Type	NR Operators
Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_1^{(\psi)} = \mathbb{1}$ $\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{S}_\psi$ $\mathcal{O}_6^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)\left(\boldsymbol{S}_\psi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp$ $\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_{13}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{14}^{(\psi)} = \left(\boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{15}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$

Step 1: building the NR EFT

- Top-down point of view.
  - Example: dark photon mediator.

$$\mathcal{L} \supset -g_e V_\mu J^\mu_{\rm EM} + \dots$$

Millicharged DM

$$c_1^{(\psi)} = -\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2 + m_V^2}$$

Electric dipole DM

$$\frac{g_\chi}{4m_\chi} V_{\mu\nu} \bar{\chi} \sigma^{\mu\nu} i\gamma^5 \chi$$

$$\Rightarrow \quad c_{11}^{(\psi)} = -\frac{m_\psi}{m_\chi} \frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2 + m_V^2}$$

Interaction Type	NR Operators
Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_1^{(\psi)} = \mathbb{1}$ $\boxed{\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}}$
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{S}_\psi$ $\mathcal{O}_6^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)\left(\boldsymbol{S}_\psi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp$ $\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_{13}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{14}^{(\psi)} = \left(\boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{15}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$

Step 1: building the NR EFT

- Top-down point of view.
  - Example: dark photon mediator.

$$\mathcal{L} \supset -g_e V_\mu J^\mu_{\mathrm{EM}} + \ldots$$

Millicharged DM

$$c_1^{(\psi)} = -\frac{g_\chi g_\psi^{\mathrm{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

Electric dipole DM

$$c_{11}^{(\psi)} = -\frac{m_\psi}{m_\chi} \frac{g_\chi g_\psi^{\mathrm{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

Interaction Type	**NR Operators**
Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_1^{(\psi)} = \mathbb{1}$ $\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{S}_\psi$ $\mathcal{O}_6^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)\left(\boldsymbol{S}_\psi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp$ $\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_{13}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{14}^{(\psi)} = \left(\boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{15}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$

Step 1: building the NR EFT

- Top-down point of view.
  - Example: dark photon mediator.

$$\mathcal{L} \supset -g_e V_\mu J^\mu_{\mathrm{EM}} + \dots$$

Millicharged DM

$$c_1^{(\psi)} = -\frac{g_\chi g_\psi^{\mathrm{eff}}}{\boldsymbol{q}^2+m_V^2}$$

Electric dipole DM

$$c_{11}^{(\psi)} = -\frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\mathrm{eff}}}{\boldsymbol{q}^2+m_V^2}$$

Magnetic dipole DM

$$\frac{g_\chi}{4m_\chi} V_{\mu\nu}\bar{\chi}\sigma^{\mu\nu}\chi$$

$$\Rightarrow \quad c_1^{(\psi)} = \frac{\boldsymbol{q}^2}{4m_\chi^2}\frac{g_\chi g_\psi^{\mathrm{eff}}}{\boldsymbol{q}^2+m_V^2} \qquad c_4^{(\psi)} = \tilde{\mu}_\psi^{\mathrm{eff}}\frac{\boldsymbol{q}^2}{m_\chi m_\psi}\frac{g_\chi g_\psi^{\mathrm{eff}}}{\boldsymbol{q}^2+m_V^2}$$

$$c_5^{(\psi)} = \frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\mathrm{eff}}}{\boldsymbol{q}^2+m_V^2} \qquad c_6^{(\psi)} = -\tilde{\mu}_\psi^{\mathrm{eff}}\frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\mathrm{eff}}}{\boldsymbol{q}^2+m_V^2}$$

where $\quad \tilde{\mu}_p^{\mathrm{eff}} \simeq 1 + 1.8\,(\hat{\boldsymbol{q}}\cdot\boldsymbol{\varepsilon}_\infty\cdot\hat{\boldsymbol{q}})\,, \quad \tilde{\mu}_e^{\mathrm{eff}} \simeq 1$

Interaction Type	NR Operators
Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_1^{(\psi)} = \mathbb{1}$ $\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi}\times\boldsymbol{v}^\perp\right)$ $\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{S}_\psi$ $\mathcal{O}_6^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)\left(\boldsymbol{S}_\psi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi}\times\boldsymbol{v}^\perp\right)$ $\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp$ $\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_{13}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{14}^{(\psi)} = \left(\boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{15}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi}\times\boldsymbol{v}^\perp\right)\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$

Step 1: building the NR EFT

- Top-down point of view.
  - Example: dark photon mediator.

$$\mathcal{L} \supset -g_e V_\mu J^\mu_{\rm EM} + \dots$$

Millicharged DM

$$c_1^{(\psi)} = -\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2 + m_V^2}$$

Electric dipole DM

$$c_{11}^{(\psi)} = -\frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2 + m_V^2}$$

Magnetic dipole DM

$$\frac{g_\chi}{4m_\chi} V_{\mu\nu}\bar{\chi}\sigma^{\mu\nu}\chi$$

$$\Rightarrow \quad c_1^{(\psi)} = \frac{\boldsymbol{q}^2}{4m_\chi^2}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2 + m_V^2} \qquad c_4^{(\psi)} = \tilde{\mu}_\psi^{\rm eff}\frac{\boldsymbol{q}^2}{m_\chi m_\psi}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_5^{(\psi)} = \frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2 + m_V^2} \qquad c_6^{(\psi)} = -\tilde{\mu}_\psi^{\rm eff}\frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2 + m_V^2}$$

where $\quad \tilde{\mu}_p^{\rm eff} \simeq 1 + 1.8\,(\hat{\boldsymbol{q}}\cdot\boldsymbol{\varepsilon}_\infty\cdot\hat{\boldsymbol{q}})\,, \quad \tilde{\mu}_e^{\rm eff} \simeq 1$

Interaction Type	**NR Operators**
Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\boxed{\mathcal{O}_1^{(\psi)} = \mathbb{1}}$ $\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\boxed{\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)}$ $\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\boxed{\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{S}_\psi}$ $\boxed{\mathcal{O}_6^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)\left(\boldsymbol{S}_\psi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)}$ $\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp$ $\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_{13}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{14}^{(\psi)} = \left(\boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{15}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$

Step 1: building the NR EFT

- Top-down point of view.
  - Example: dark photon mediator.

$$\mathcal{L} \supset -g_e V_\mu J^\mu_{\mathrm{EM}} + \dots$$

Millicharged DM

$$c_1^{(\psi)} = -\frac{g_\chi g_\psi^{\mathrm{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

Electric dipole DM

$$c_{11}^{(\psi)} = -\frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\mathrm{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

Magnetic dipole DM

$$c_1^{(\psi)} = \frac{\boldsymbol{q}^2}{4m_\chi^2}\frac{g_\chi g_\psi^{\mathrm{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_4^{(\psi)} = \widetilde{\mu}_\psi^{\mathrm{eff}}\frac{\boldsymbol{q}^2}{m_\chi m_\psi}\frac{g_\chi g_\psi^{\mathrm{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_5^{(\psi)} = \frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\mathrm{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_6^{(\psi)} = -\widetilde{\mu}_\psi^{\mathrm{eff}}\frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\mathrm{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

Interaction Type	NR Operators
Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_1^{(\psi)} = \mathbb{1}$ $\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{S}_\psi$ $\mathcal{O}_6^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)\left(\boldsymbol{S}_\psi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp$ $\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_{13}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{14}^{(\psi)} = \left(\boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{15}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$

Step 1: building the NR EFT

- Top-down point of view.
  - Example: dark photon mediator.

$$\mathcal{L} \supset -g_e V_\mu J^\mu_{\mathrm{EM}} + \dots$$

Millicharged DM

$$c_1^{(\psi)} = -\frac{g_\chi g_\psi^{\mathrm{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

Electric dipole DM

$$c_{11}^{(\psi)} = -\frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\mathrm{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

Magnetic dipole DM

$$c_1^{(\psi)} = \frac{\boldsymbol{q}^2}{4m_\chi^2}\frac{g_\chi g_\psi^{\mathrm{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_4^{(\psi)} = \widetilde{\mu}_\psi^{\mathrm{eff}}\frac{\boldsymbol{q}^2}{m_\chi m_\psi}\frac{g_\chi g_\psi^{\mathrm{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_5^{(\psi)} = \frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\mathrm{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_6^{(\psi)} = -\widetilde{\mu}_\psi^{\mathrm{eff}}\frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\mathrm{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

Anapole DM

$$\frac{g_\chi}{4m_\chi^2}(\partial^\nu V_{\mu\nu})(\bar{\chi}\sigma^{\mu\nu}\gamma^5\chi)$$

$$\Rightarrow \quad c_8^{(\psi)} = \frac{\boldsymbol{q}^2}{2m_\chi^2}\frac{g_\chi g_\psi^{\mathrm{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_9^{(\psi)} = -\widetilde{\mu}_\psi^{\mathrm{eff}}\frac{\boldsymbol{q}^2}{2m_\chi^2}\frac{g_\chi g_\psi^{\mathrm{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

Interaction Type	**NR Operators**
Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_1^{(\psi)} = \mathbb{1}$ $\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{S}_\psi$ $\mathcal{O}_6^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)\left(\boldsymbol{S}_\psi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp$ $\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_{13}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{14}^{(\psi)} = \left(\boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{15}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$

Step 1: building the NR EFT

- Top-down point of view.
  - Example: dark photon mediator.

$$\mathcal{L} \supset -g_e V_\mu J^\mu_{\rm EM} + \ldots$$

Millicharged DM

$$c_1^{(\psi)} = -\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2+m_V^2}$$

Electric dipole DM

$$c_{11}^{(\psi)} = -\frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2+m_V^2}$$

Magnetic dipole DM

$$c_1^{(\psi)} = \frac{\boldsymbol{q}^2}{4m_\chi^2}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2+m_V^2}$$

$$c_4^{(\psi)} = \tilde{\mu}_\psi^{\rm eff}\frac{\boldsymbol{q}^2}{m_\chi m_\psi}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2+m_V^2}$$

$$c_5^{(\psi)} = \frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2+m_V^2}$$

$$c_6^{(\psi)} = -\tilde{\mu}_\psi^{\rm eff}\frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2+m_V^2}$$

Anapole DM

$$\frac{g_\chi}{4m_\chi^2}(\partial^\nu V_{\mu\nu})(\bar\chi\sigma^{\mu\nu}\gamma^5\chi)$$

$$\Rightarrow \quad c_8^{(\psi)} = \frac{\boldsymbol{q}^2}{2m_\chi^2}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2+m_V^2}$$

$$c_9^{(\psi)} = -\tilde{\mu}_\psi^{\rm eff}\frac{\boldsymbol{q}^2}{2m_\chi^2}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2+m_V^2}$$

Interaction Type	NR Operators
Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_1^{(\psi)} = \mathbb{1}$ $\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi}\times\boldsymbol{v}^\perp\right)$ $\boxed{\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{S}_\psi$ $\mathcal{O}_6^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)\left(\boldsymbol{S}_\psi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)$ $\boxed{\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \frac{i\boldsymbol{q}}{m_\psi}\right)}$ $\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi}\times\boldsymbol{v}^\perp\right)$ $\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp$ $\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_{13}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{14}^{(\psi)} = \left(\boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{15}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi}\times\boldsymbol{v}^\perp\right)\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$

Step 1: building the NR EFT

- Top-down point of view.
  - Example: dark photon mediator.

$$\mathcal{L} \supset -g_e V_\mu J^\mu_{\rm EM} + \dots$$

Millicharged DM

$$c_1^{(\psi)} = -\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2 + m_V^2}$$

Electric dipole DM

$$c_{11}^{(\psi)} = -\frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2 + m_V^2}$$

Magnetic dipole DM

$$c_1^{(\psi)} = \frac{\boldsymbol{q}^2}{4m_\chi^2}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_4^{(\psi)} = \tilde{\mu}_\psi^{\rm eff}\frac{\boldsymbol{q}^2}{m_\chi m_\psi}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_5^{(\psi)} = \frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_6^{(\psi)} = -\tilde{\mu}_\psi^{\rm eff}\frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2 + m_V^2}$$

Anapole DM

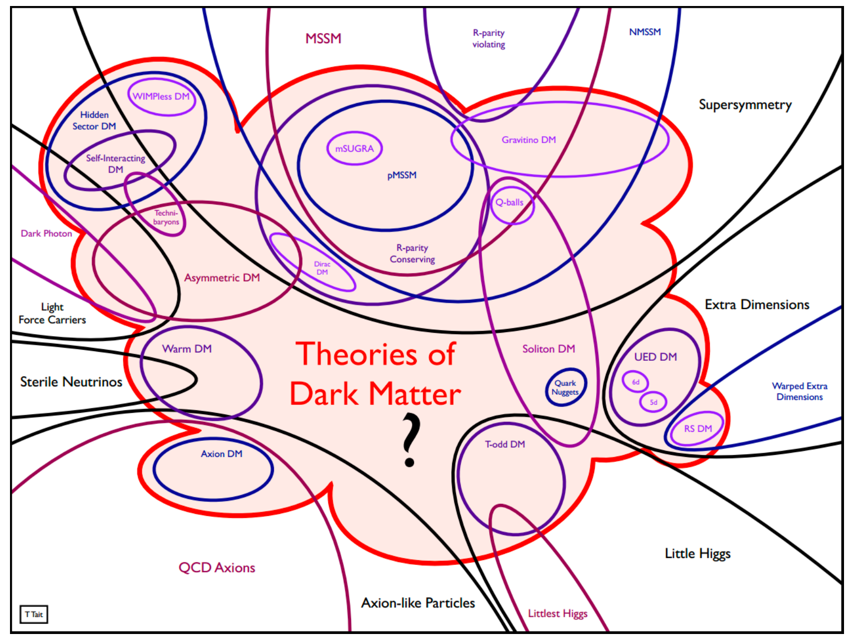
$$c_8^{(\psi)} = \frac{\boldsymbol{q}^2}{2m_\chi^2}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_9^{(\psi)} = -\tilde{\mu}_\psi^{\rm eff}\frac{\boldsymbol{q}^2}{2m_\chi^2}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2 + m_V^2}$$

More on these models later.

Interaction Type	NR Operators
Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_1^{(\psi)} = \mathbb{1}$ $\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{S}_\psi$ $\mathcal{O}_6^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)\left(\boldsymbol{S}_\psi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp$ $\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_{13}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{14}^{(\psi)} = \left(\boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{15}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$

EFT of DM direct detection: preview



1

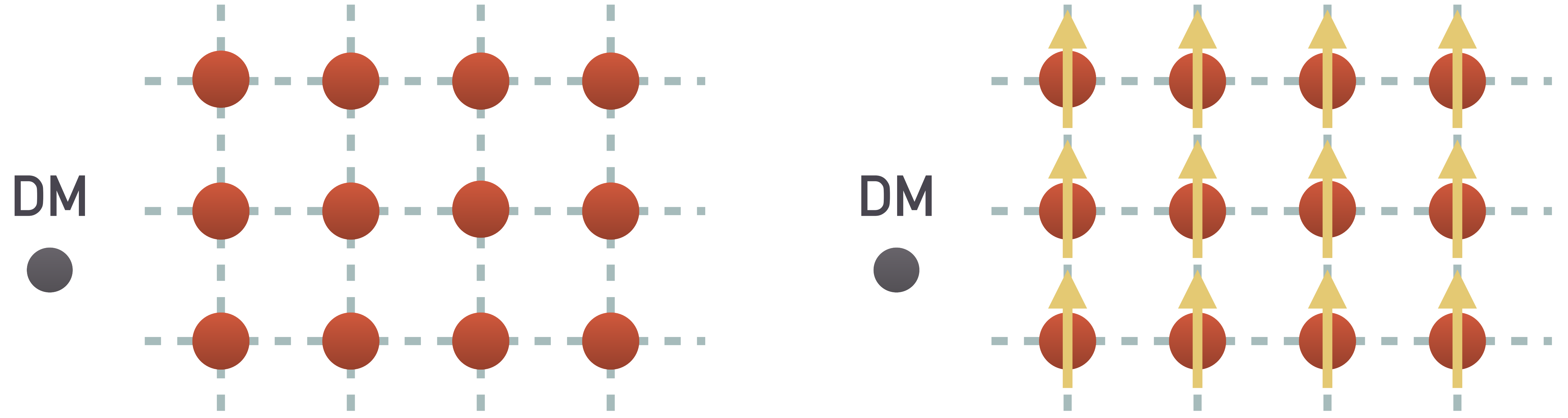
2

3

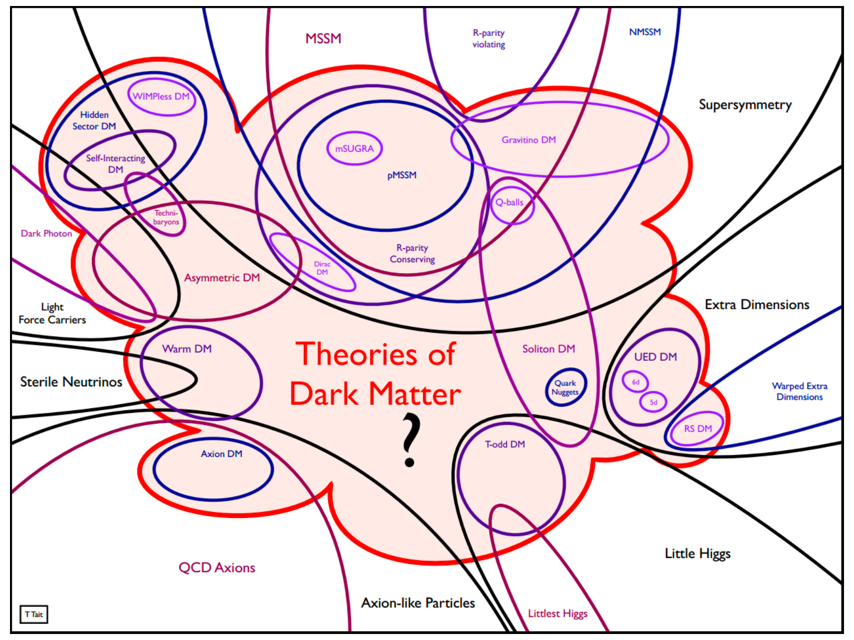
Crystal responses

Nonrelativistic (NR) EFT of DM-SM interactions

Phonon & magnon excitation rates



EFT of DM direct detection: preview



1

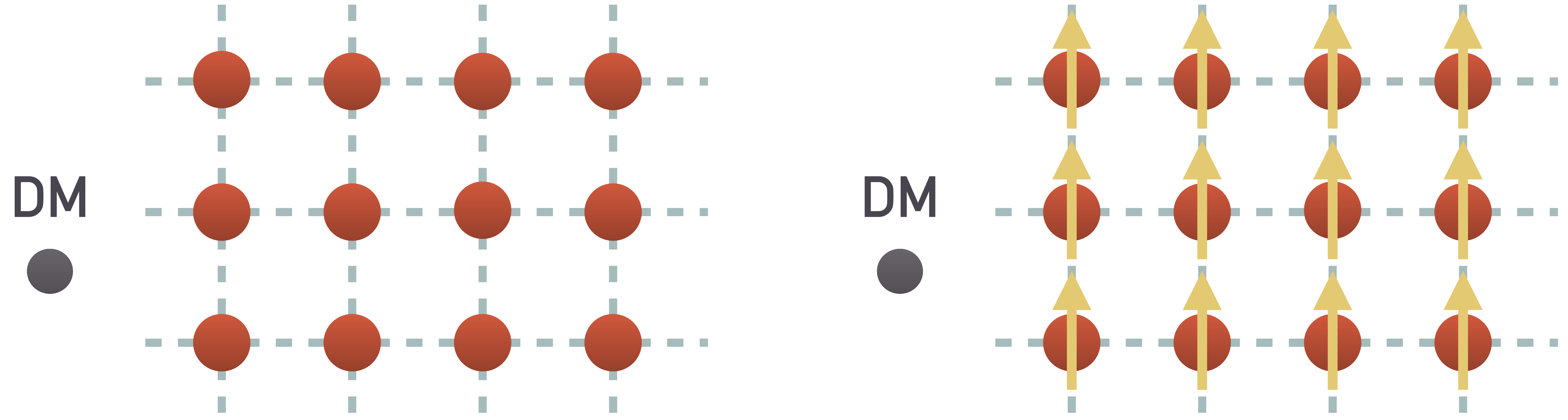
2

Crystal responses

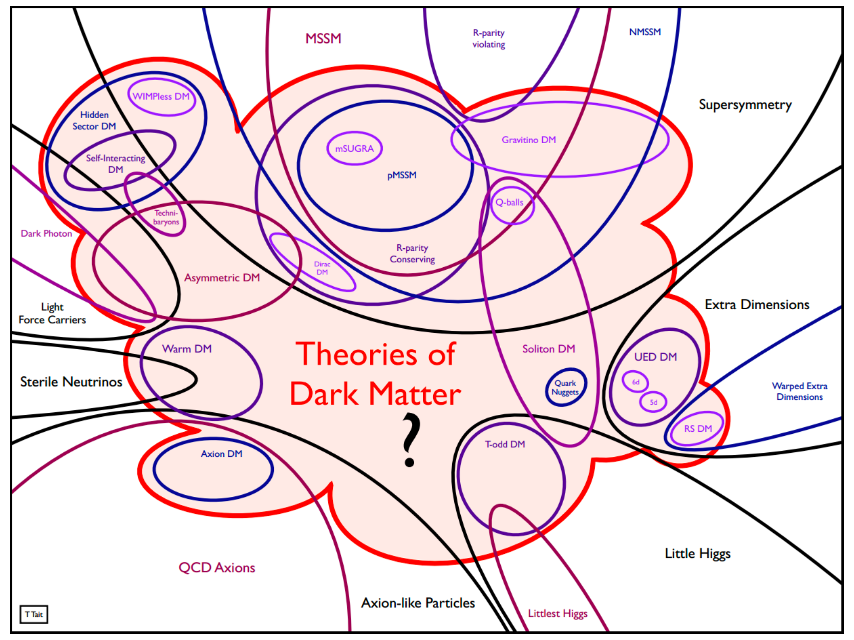
3

Nonrelativistic (NR) EFT of DM-SM interactions

Phonon & magnon excitation rates



EFT of DM direct detection: preview



1

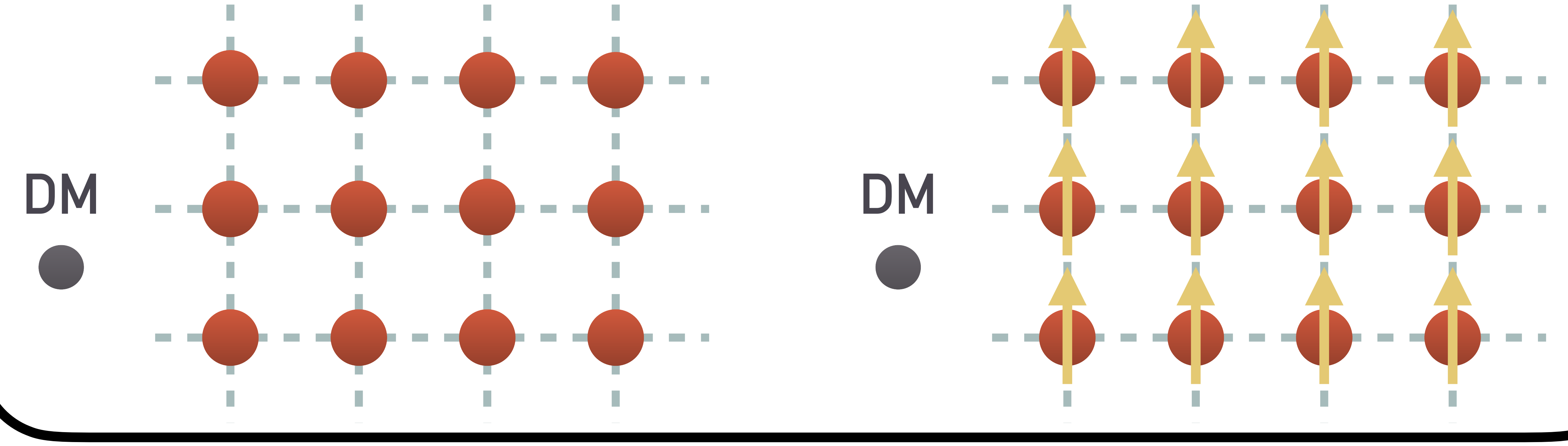
Nonrelativistic (NR) EFT of DM-SM interactions

2

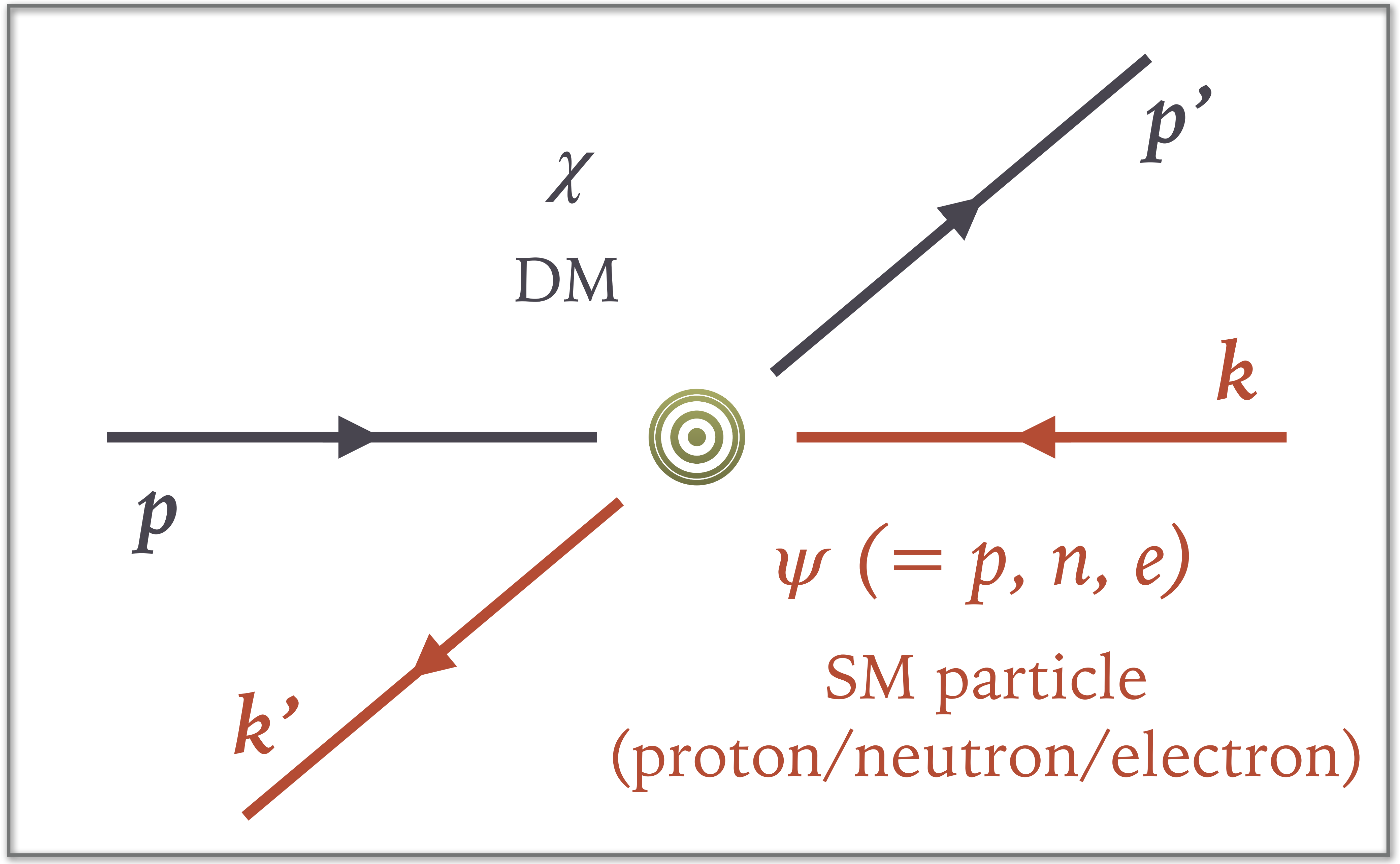
Crystal responses

3

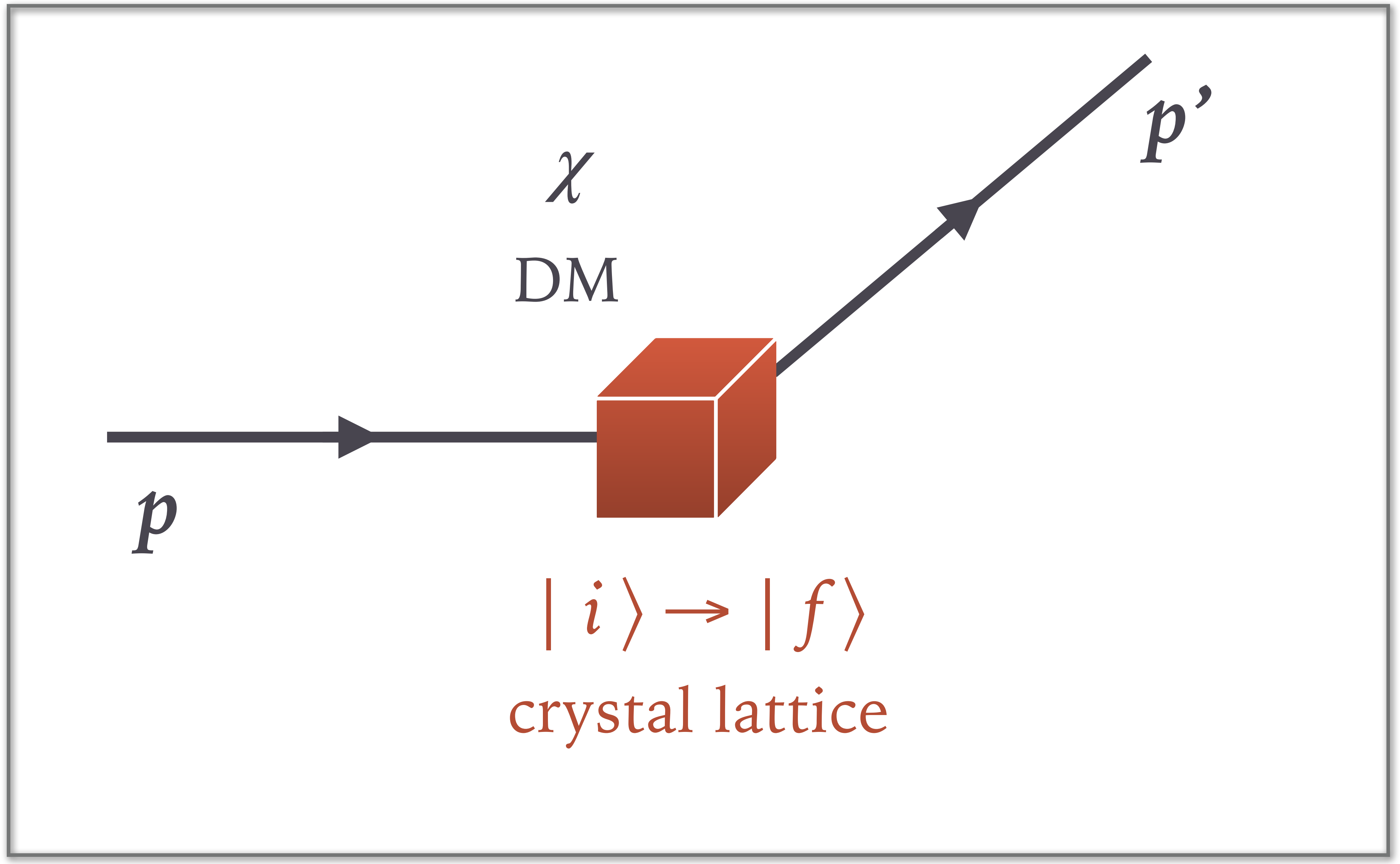
Phonon & magnon excitation rates



Step 2: matching onto lattice d.o.f.



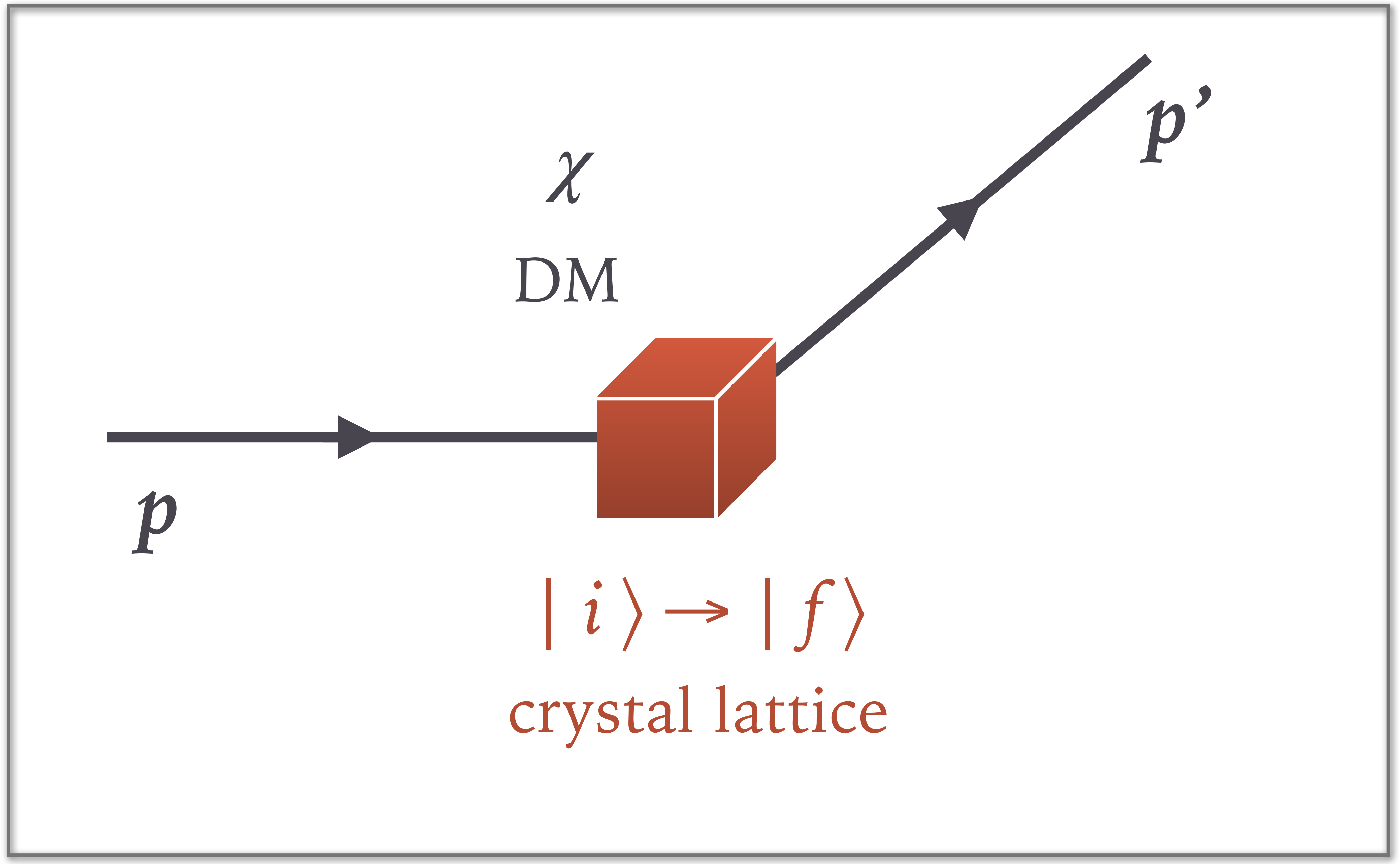
Step 2: matching onto lattice d.o.f.



Step 2: matching onto lattice d.o.f.

- By Fermi's golden rule,

$$\Gamma(\boldsymbol{v}) = \frac{1}{V}\int \frac{d^3q}{(2\pi)^3} \sum_f \left|\langle f|\, \widetilde{\mathcal{V}}(-\boldsymbol{q},\boldsymbol{v})\,|i\rangle\right|^2 2\pi\,\delta\big(E_f - E_i - \omega_{\boldsymbol{q}}\big)$$



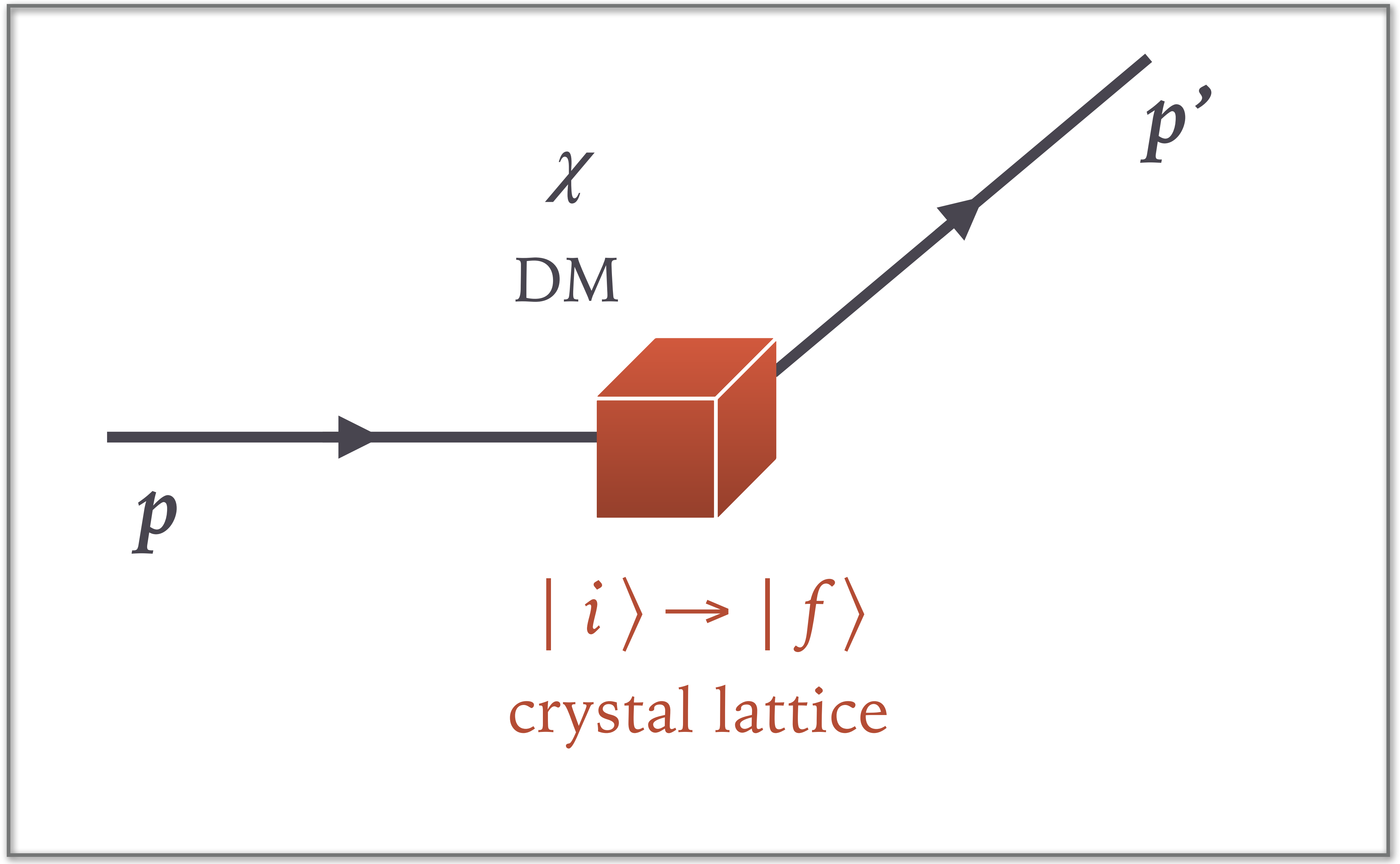
Step 2: matching onto lattice d.o.f.

- By Fermi's golden rule,

$$\Gamma(\boldsymbol{v}) = \frac{1}{V}\int \frac{d^3q}{(2\pi)^3} \sum_f \left|\langle f|\boxed{\widetilde{\mathcal{V}}(-\boldsymbol{q},\boldsymbol{v})}|i\rangle\right|^2 2\pi\,\delta\big(E_f - E_i - \omega_{\boldsymbol{q}}\big)$$

Fourier transform of lattice potential

[can be velocity dependent (non-static)]



Step 2: matching onto lattice d.o.f.

- By Fermi's golden rule,

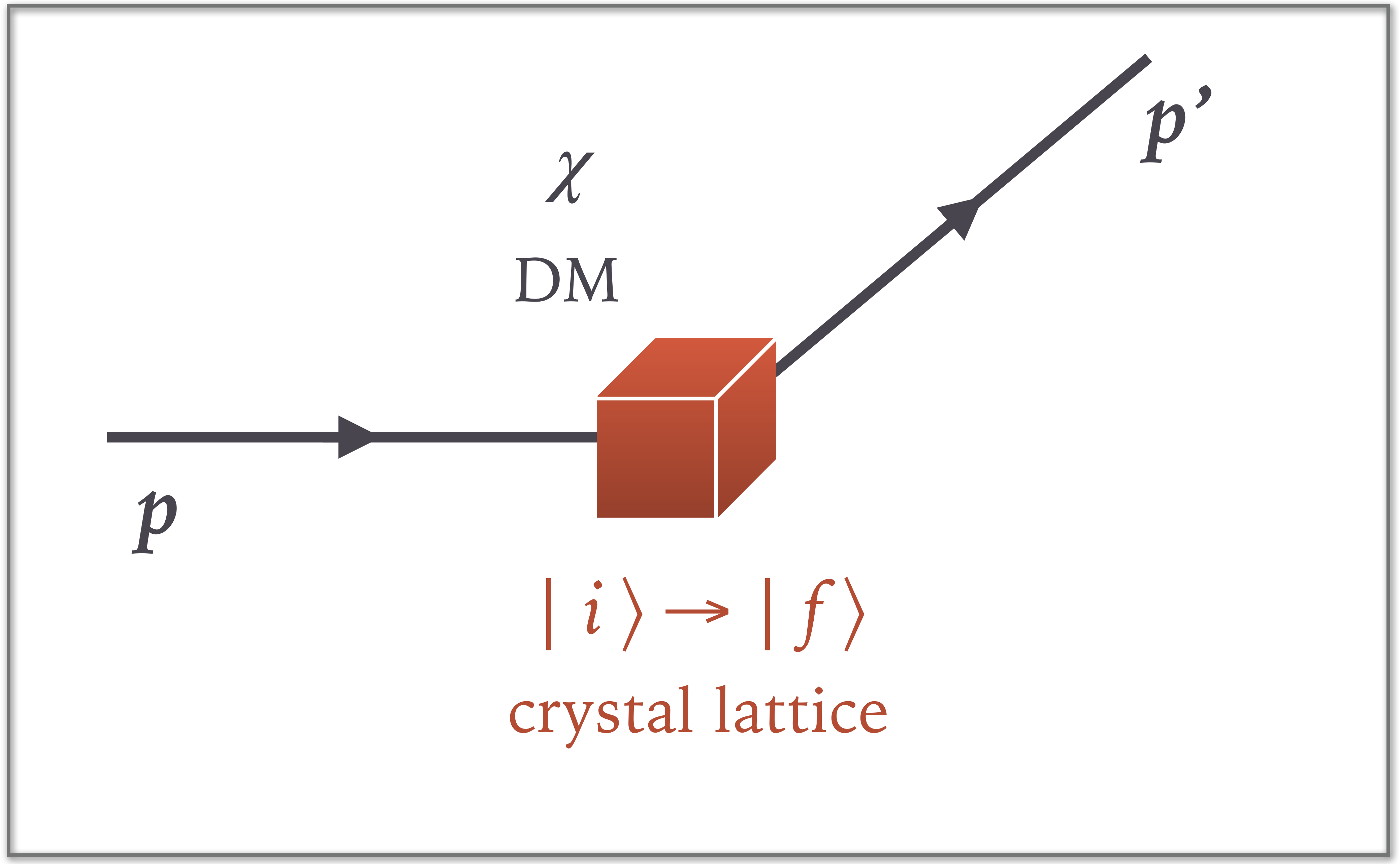
$$\Gamma(\boldsymbol{v}) = \frac{1}{V}\int \frac{d^3q}{(2\pi)^3}\sum_f \left|\langle f|\boxed{\widetilde{\mathcal{V}}(-\boldsymbol{q},\boldsymbol{v})}|i\rangle\right|^2 2\pi\,\delta\big(E_f - E_i - \boxed{\omega_{\boldsymbol{q}}}\big)$$

Fourier transform of lattice potential

[can be velocity dependent (non-static)]

energy deposition

$$\boxed{\omega_{\boldsymbol{q}} = \frac{1}{2}m_\chi v^2 - \frac{(m_\chi \boldsymbol{v} - \boldsymbol{q})^2}{2m_\chi} = \boldsymbol{q}\cdot\boldsymbol{v} - \frac{q^2}{2m_\chi}}$$



Step 2: matching onto lattice d.o.f.

- By Fermi's golden rule,

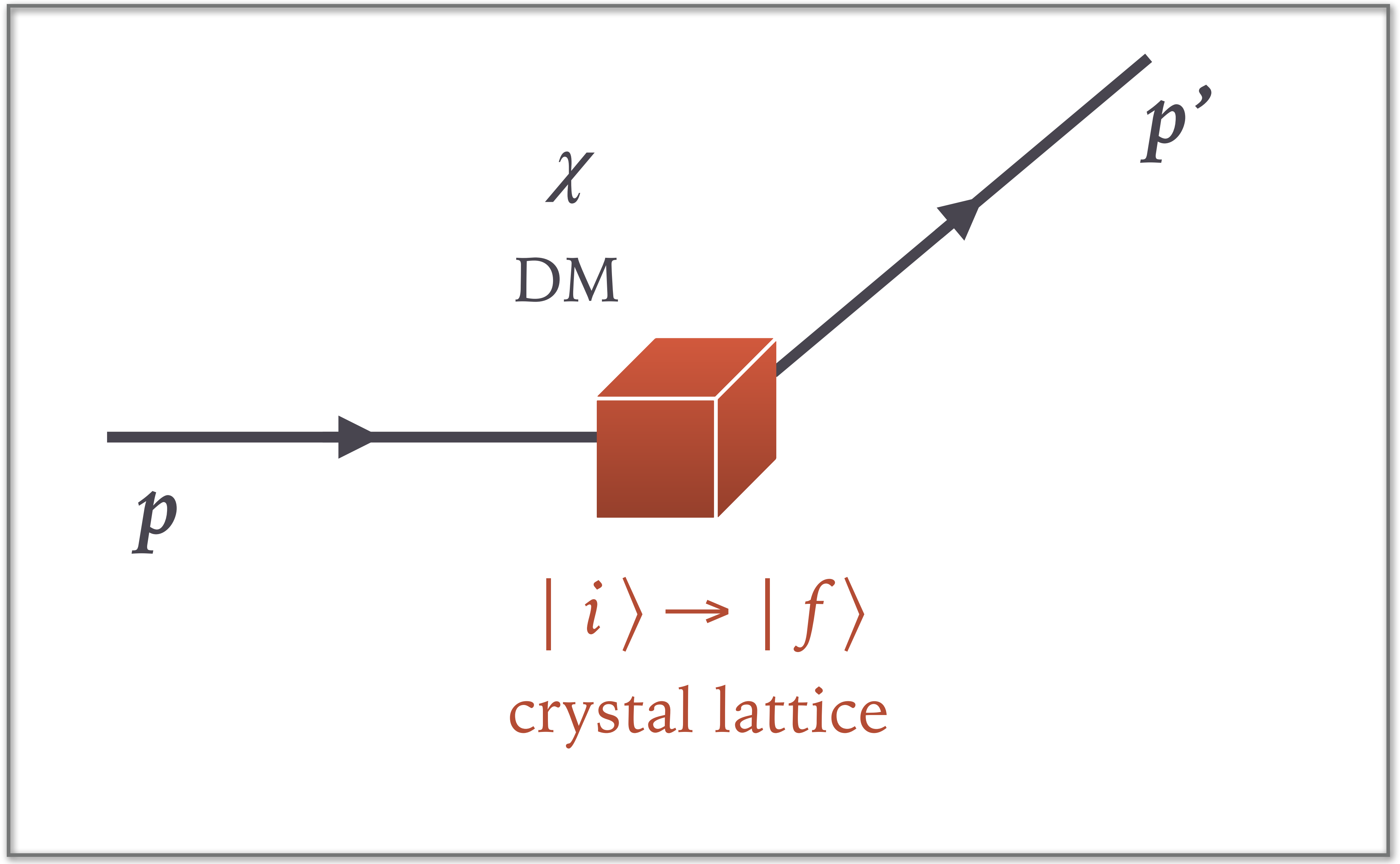
$$\Gamma(\boldsymbol{v}) = \frac{1}{V}\int \frac{d^3q}{(2\pi)^3} \sum_f \left|\langle f|\boxed{\widetilde{\mathcal{V}}(-\boldsymbol{q},\boldsymbol{v})}|i\rangle\right|^2 2\pi\,\delta\big(E_f - E_i - \boxed{\omega_{\boldsymbol{q}}}\big)$$

Fourier transform of lattice potential

[can be velocity dependent (non-static)]

energy deposition

$$\boxed{\omega_{\boldsymbol{q}} = \frac{1}{2}m_\chi v^2 - \frac{(m_\chi \boldsymbol{v} - \boldsymbol{q})^2}{2m_\chi} = \boldsymbol{q}\cdot\boldsymbol{v} - \frac{q^2}{2m_\chi}}$$



- For a periodic lattice,

$$\mathcal{V}(\boldsymbol{x},\boldsymbol{v}) = \sum_{lj} \mathcal{V}_{lj}(\boldsymbol{x} - \boldsymbol{x}_{lj}, \boldsymbol{v})$$

$$\widetilde{\mathcal{V}}(-\boldsymbol{q},\boldsymbol{v}) = \int d^3x\, e^{i\boldsymbol{q}\cdot\boldsymbol{x}}\, \mathcal{V}(\boldsymbol{x},\boldsymbol{v}) = \sum_{l,j} e^{i\boldsymbol{q}\cdot\boldsymbol{x}_{lj}}\, \widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v})$$

Step 2: matching onto lattice d.o.f.

- By Fermi's golden rule,

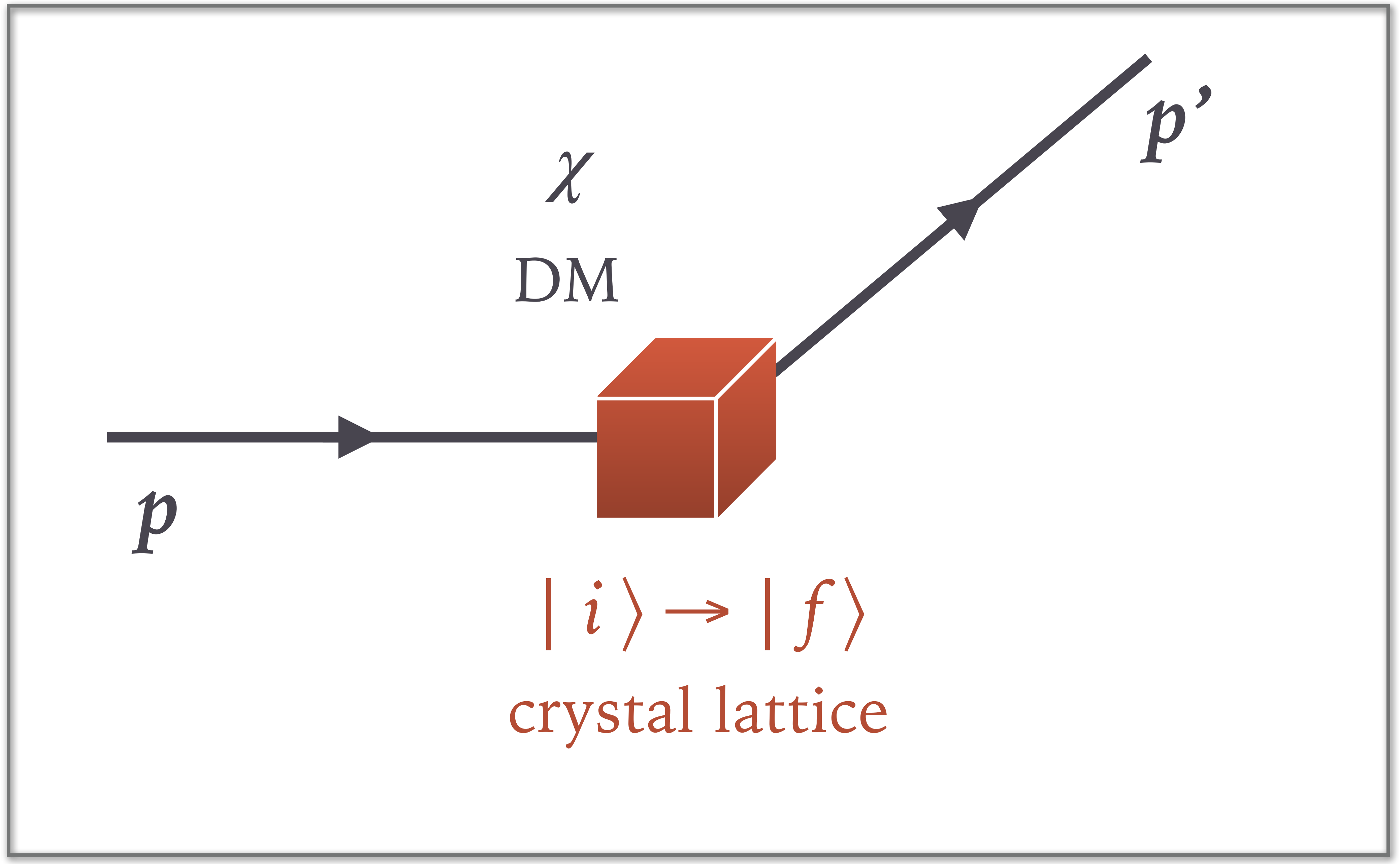
$$\Gamma(\boldsymbol{v}) = \frac{1}{V}\int \frac{d^3q}{(2\pi)^3}\sum_f \left|\langle f|\boxed{\widetilde{\mathcal{V}}(-\boldsymbol{q},\boldsymbol{v})}|i\rangle\right|^2 2\pi\,\delta\big(E_f - E_i - \boxed{\omega_{\boldsymbol{q}}}\big)$$

Fourier transform of lattice potential

[can be velocity dependent (non-static)]

energy deposition

$$\boxed{\omega_{\boldsymbol{q}} = \frac{1}{2}m_\chi v^2 - \frac{(m_\chi\boldsymbol{v}-\boldsymbol{q})^2}{2m_\chi} = \boldsymbol{q}\cdot\boldsymbol{v} - \frac{q^2}{2m_\chi}}$$



- For a periodic lattice,

$$\mathcal{V}(\boldsymbol{x},\boldsymbol{v}) = \boxed{\sum_{lj}}\mathcal{V}_{lj}(\boldsymbol{x}-\boldsymbol{x}_{lj},\boldsymbol{v})$$

$$\widetilde{\mathcal{V}}(-\boldsymbol{q},\boldsymbol{v}) = \int d^3x\, e^{i\boldsymbol{q}\cdot\boldsymbol{x}}\,\mathcal{V}(\boldsymbol{x},\boldsymbol{v}) = \boxed{\sum_{l,j}} e^{i\boldsymbol{q}\cdot\boldsymbol{x}_{lj}}\,\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v})$$

sum over ions

l labels primitive cells.
j labels ions in each cell.

Step 2: matching onto lattice d.o.f.

- By Fermi's golden rule,

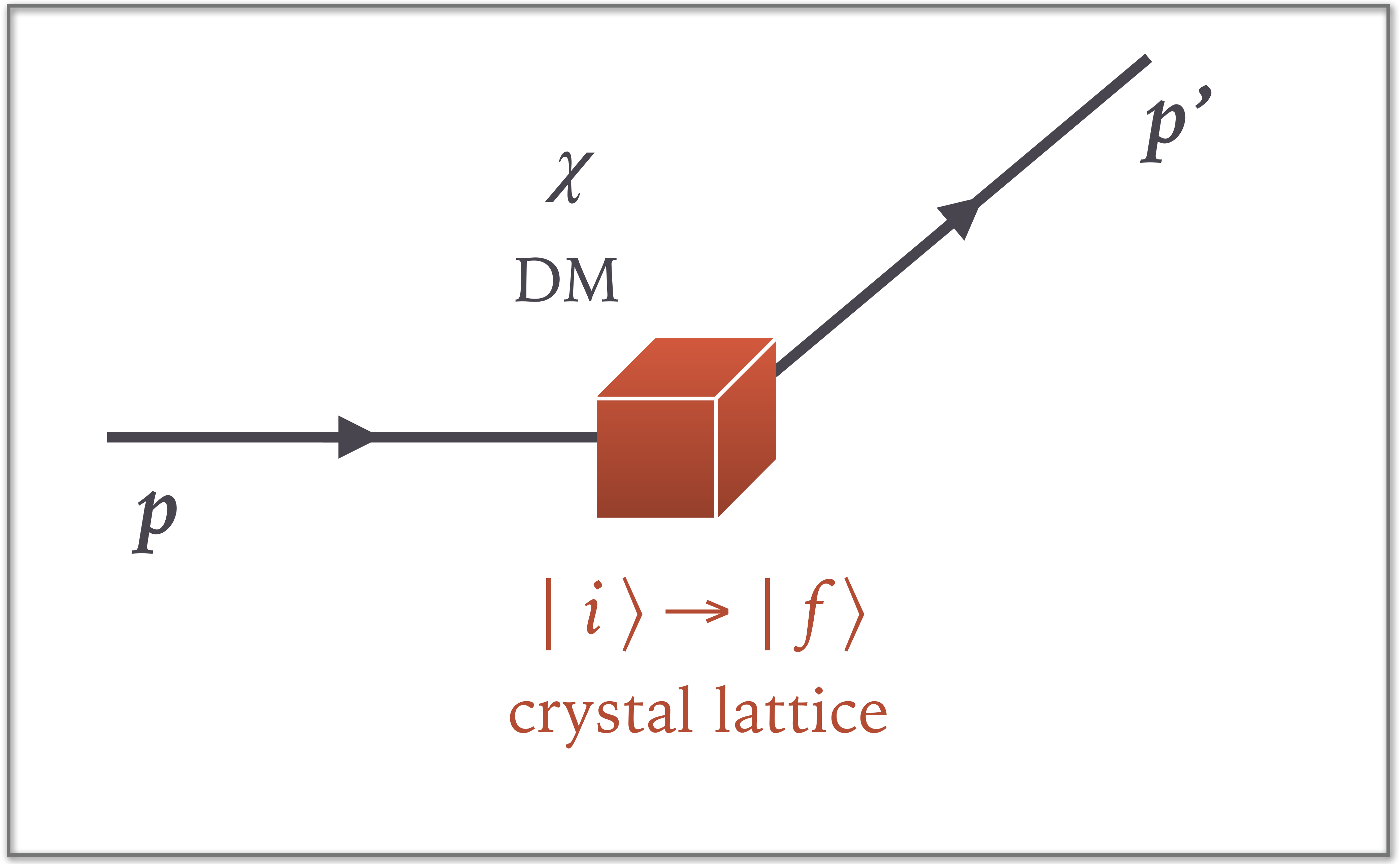
$$\Gamma(\boldsymbol{v}) = \frac{1}{V}\int \frac{d^3q}{(2\pi)^3} \sum_f \left|\langle f|\widetilde{\mathcal{V}}(-\boldsymbol{q},\boldsymbol{v})|i\rangle\right|^2 2\pi\,\delta\big(E_f - E_i - \omega_{\boldsymbol{q}}\big)$$

Fourier transform of lattice potential

[can be velocity dependent (non-static)]

energy deposition

$$\boxed{\omega_{\boldsymbol{q}} = \frac{1}{2}m_\chi v^2 - \frac{(m_\chi \boldsymbol{v} - \boldsymbol{q})^2}{2m_\chi} = \boldsymbol{q}\cdot\boldsymbol{v} - \frac{q^2}{2m_\chi}}$$



- For a periodic lattice,

$$\mathcal{V}(\boldsymbol{x},\boldsymbol{v}) = \sum_{lj} \mathcal{V}_{lj}(\boldsymbol{x}-\boldsymbol{x}_{lj},\boldsymbol{v})$$

$$\widetilde{\mathcal{V}}(-\boldsymbol{q},\boldsymbol{v}) = \int d^3x\, e^{i\boldsymbol{q}\cdot\boldsymbol{x}}\,\mathcal{V}(\boldsymbol{x},\boldsymbol{v}) = \sum_{l,j} e^{i\boldsymbol{q}\cdot\boldsymbol{x}_{lj}}\,\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v})$$

sum over ions

l labels primitive cells.
j labels ions in each cell.

key quantity to compute:
DM-ion scattering potential.

Step 2: matching onto lattice d.o.f.

➤ Goal: compute $\boxed{\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v})}$ in the NR EFT.

Interaction Type	NR Operators
Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_1^{(\psi)} = \mathbb{1}$ $\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{S}_\psi$ $\mathcal{O}_6^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)\left(\boldsymbol{S}_\psi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp$ $\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_{13}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{14}^{(\psi)} = \left(\boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{15}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$

Step 2: matching onto lattice d.o.f.

- Goal: compute $\boxed{\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v})}$ in the NR EFT.
- Pick one operator from each category.
  - Other operators are completely analogous.

$$\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v}) \supset \sum_\alpha \Big[c_1^{(\psi)} \langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}\rangle_{lj} + c_4^{(\psi)}\, \boldsymbol{S}_\chi \cdot \langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha} \boldsymbol{S}_{\psi,\alpha}\rangle_{lj} + c_8^{(\psi)}\, \boldsymbol{S}_\chi \cdot \langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha} \boldsymbol{v}_\alpha^\perp\rangle_{lj} + c_3^{(\psi)} \frac{i\boldsymbol{q}}{m_\psi} \cdot \langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}\, \boldsymbol{v}_\alpha^\perp \times \boldsymbol{S}_{\psi,\alpha}\rangle_{lj} \Big]$$

Interaction Type	NR Operators
Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_1^{(\psi)} = \mathbb{1}$ $\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{S}_\psi$ $\mathcal{O}_6^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)\left(\boldsymbol{S}_\psi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp$ $\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_{13}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{14}^{(\psi)} = \left(\boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{15}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$

Step 2: matching onto lattice d.o.f.

- Goal: compute $\boxed{\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v})}$ in the NR EFT.
- Pick one operator from each category.
  - Other operators are completely analogous.

$$\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v}) \supset \boxed{\sum_\alpha}\Big[c_1^{(\psi)}\,\langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}\rangle_{lj} + c_4^{(\psi)}\,\boldsymbol{S}_\chi\cdot\langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}\boldsymbol{S}_{\psi,\alpha}\rangle_{lj} + c_8^{(\psi)}\,\boldsymbol{S}_\chi\cdot\langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}\boldsymbol{v}_\alpha^\perp\rangle_{lj} + c_3^{(\psi)}\,\frac{i\boldsymbol{q}}{m_\psi}\cdot\langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}\,\boldsymbol{v}_\alpha^\perp\times\boldsymbol{S}_{\psi,\alpha}\rangle_{lj}\Big]$$

sum over the ion's constituent ψ $(= p, n, e)$ particles

Interaction Type	NR Operators
Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_1^{(\psi)} = \mathbb{1}$ $\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi\cdot\frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi\cdot\left(\frac{i\boldsymbol{q}}{m_\psi}\times\boldsymbol{v}^\perp\right)$ $\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi\cdot\boldsymbol{v}^\perp$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi\cdot\boldsymbol{S}_\psi$ $\mathcal{O}_6^{(\psi)} = \left(\boldsymbol{S}_\chi\cdot\frac{\boldsymbol{q}}{m_\psi}\right)\left(\boldsymbol{S}_\psi\cdot\frac{\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi\cdot\left(\boldsymbol{S}_\psi\times\frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi\cdot\frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi\cdot\left(\frac{i\boldsymbol{q}}{m_\psi}\times\boldsymbol{v}^\perp\right)$ $\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi\cdot\boldsymbol{v}^\perp$ $\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi\cdot\left(\boldsymbol{S}_\psi\times\boldsymbol{v}^\perp\right)$ $\mathcal{O}_{13}^{(\psi)} = \left(\boldsymbol{S}_\chi\cdot\boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\psi\cdot\frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{14}^{(\psi)} = \left(\boldsymbol{S}_\psi\cdot\boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\chi\cdot\frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{15}^{(\psi)} = \left(\boldsymbol{S}_\chi\cdot\left(\frac{i\boldsymbol{q}}{m_\psi}\times\boldsymbol{v}^\perp\right)\right)\left(\boldsymbol{S}_\psi\cdot\frac{i\boldsymbol{q}}{m_\psi}\right)$

Step 2: matching onto lattice d.o.f.

- Goal: compute $\boxed{\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v})}$ in the NR EFT.
- Pick one operator from each category.
  - Other operators are completely analogous.

$$\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v}) \supset \boxed{\sum_\alpha}\Big[c_1^{(\psi)}\langle \boxed{e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}}\rangle_{lj} + c_4^{(\psi)}\,\boldsymbol{S}_\chi\cdot\langle \boxed{e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}}\boldsymbol{S}_{\psi,\alpha}\rangle_{lj} + c_8^{(\psi)}\,\boldsymbol{S}_\chi\cdot\langle \boxed{e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}}\boldsymbol{v}_\alpha^\perp\rangle_{lj} + c_3^{(\psi)}\,\frac{i\boldsymbol{q}}{m_\psi}\cdot\langle \boxed{e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}}\boldsymbol{v}_\alpha^\perp\times\boldsymbol{S}_{\psi,\alpha}\rangle_{lj}\Big]$$

sum over the ion's constituent ψ $(= p, n, e)$ particles

expand in the long wavelength limit $\Rightarrow 1 + i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha + \ldots$

  - For DM lighter than $\sim$10MeV, momentum transfer is small compared to $1/r_{\text{ion}}$. The expansion is justified.

Interaction Type	NR Operators
Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_1^{(\psi)} = \mathbb{1}$ $\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi\cdot\frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi\cdot\left(\frac{i\boldsymbol{q}}{m_\psi}\times\boldsymbol{v}^\perp\right)$ $\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi\cdot\boldsymbol{v}^\perp$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi\cdot\boldsymbol{S}_\psi$ $\mathcal{O}_6^{(\psi)} = \left(\boldsymbol{S}_\chi\cdot\frac{\boldsymbol{q}}{m_\psi}\right)\left(\boldsymbol{S}_\psi\cdot\frac{\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi\cdot\left(\boldsymbol{S}_\psi\times\frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi\cdot\frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi\cdot\left(\frac{i\boldsymbol{q}}{m_\psi}\times\boldsymbol{v}^\perp\right)$ $\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi\cdot\boldsymbol{v}^\perp$ $\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi\cdot\left(\boldsymbol{S}_\psi\times\boldsymbol{v}^\perp\right)$ $\mathcal{O}_{13}^{(\psi)} = \left(\boldsymbol{S}_\chi\cdot\boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\psi\cdot\frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{14}^{(\psi)} = \left(\boldsymbol{S}_\psi\cdot\boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\chi\cdot\frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{15}^{(\psi)} = \left(\boldsymbol{S}_\chi\cdot\left(\frac{i\boldsymbol{q}}{m_\psi}\times\boldsymbol{v}^\perp\right)\right)\left(\boldsymbol{S}_\psi\cdot\frac{i\boldsymbol{q}}{m_\psi}\right)$

Step 2: matching onto lattice d.o.f.

$$\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v}) \supset \sum_{\alpha}\Big[\boxed{c_1^{(\psi)}\,\langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}\rangle_{lj} + c_4^{(\psi)}\,\boldsymbol{S}_\chi\cdot\langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}\boldsymbol{S}_{\psi,\alpha}\rangle_{lj}}$$
$$+\,c_8^{(\psi)}\,\boldsymbol{S}_\chi\cdot\langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}\boldsymbol{v}_\alpha^\perp\rangle_{lj} + c_3^{(\psi)}\,\frac{i\boldsymbol{q}}{m_\psi}\cdot\langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}\,\boldsymbol{v}_\alpha^\perp\times\boldsymbol{S}_{\psi,\alpha}\rangle_{lj}\Big]$$

- $\boldsymbol{v}^\perp$-independent operators.

Interaction Type	NR Operators
Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\boxed{\mathcal{O}_1^{(\psi)} = \mathbb{1}}$ $\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi\cdot\frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi\cdot\big(\frac{i\boldsymbol{q}}{m_\psi}\times\boldsymbol{v}^\perp\big)$ $\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi\cdot\boldsymbol{v}^\perp$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\boxed{\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi\cdot\boldsymbol{S}_\psi}$ $\mathcal{O}_6^{(\psi)} = \big(\boldsymbol{S}_\chi\cdot\frac{\boldsymbol{q}}{m_\psi}\big)\big(\boldsymbol{S}_\psi\cdot\frac{\boldsymbol{q}}{m_\psi}\big)$ $\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi\cdot\big(\boldsymbol{S}_\psi\times\frac{i\boldsymbol{q}}{m_\psi}\big)$ $\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi\cdot\frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi\cdot\big(\frac{i\boldsymbol{q}}{m_\psi}\times\boldsymbol{v}^\perp\big)$ $\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi\cdot\boldsymbol{v}^\perp$ $\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi\cdot\big(\boldsymbol{S}_\psi\times\boldsymbol{v}^\perp\big)$ $\mathcal{O}_{13}^{(\psi)} = \big(\boldsymbol{S}_\chi\cdot\boldsymbol{v}^\perp\big)\big(\boldsymbol{S}_\psi\cdot\frac{i\boldsymbol{q}}{m_\psi}\big)$ $\mathcal{O}_{14}^{(\psi)} = \big(\boldsymbol{S}_\psi\cdot\boldsymbol{v}^\perp\big)\big(\boldsymbol{S}_\chi\cdot\frac{i\boldsymbol{q}}{m_\psi}\big)$ $\mathcal{O}_{15}^{(\psi)} = \big(\boldsymbol{S}_\chi\cdot\big(\frac{i\boldsymbol{q}}{m_\psi}\times\boldsymbol{v}^\perp\big)\big)\big(\boldsymbol{S}_\psi\cdot\frac{i\boldsymbol{q}}{m_\psi}\big)$

Step 2: matching onto lattice d.o.f.

$$\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v}) \supset \sum_{\alpha} \Big[\boxed{c_1^{(\psi)} \langle \cancelto{1}{e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}} \rangle_{lj} + c_4^{(\psi)}\, \boldsymbol{S}_\chi \cdot \langle \cancelto{1}{e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}}\, \boldsymbol{S}_{\psi,\alpha} \rangle_{lj}} + c_8^{(\psi)}\, \boldsymbol{S}_\chi \cdot \langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha} \boldsymbol{v}_\alpha^\perp \rangle_{lj} + c_3^{(\psi)} \frac{i\boldsymbol{q}}{m_\psi} \cdot \langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}\, \boldsymbol{v}_\alpha^\perp \times \boldsymbol{S}_{\psi,\alpha} \rangle_{lj} \Big]$$

- $\boldsymbol{v}^\perp$-independent operators.

Interaction Type	NR Operators
Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\boxed{\mathcal{O}_1^{(\psi)} = \mathbb{1}}$
	$\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$
	$\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\boxed{\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{S}_\psi}$
	$\mathcal{O}_6^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)\left(\boldsymbol{S}_\psi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)$
	$\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \frac{i\boldsymbol{q}}{m_\psi}\right)$
	$\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$
	$\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp$
	$\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \boldsymbol{v}^\perp\right)$
	$\mathcal{O}_{13}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$
	$\mathcal{O}_{14}^{(\psi)} = \left(\boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$
	$\mathcal{O}_{15}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$

Step 2: matching onto lattice d.o.f.

$$\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v}) \supset \sum_{\alpha}\Big[\boxed{c_1^{(\psi)}\,\langle \cancelto{1}{e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}}\rangle_{lj} + c_4^{(\psi)}\,\boldsymbol{S}_\chi\cdot\langle \cancelto{1}{e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}}\boldsymbol{S}_{\psi,\alpha}\rangle_{lj}}$$

$$+\,c_8^{(\psi)}\,\boldsymbol{S}_\chi\cdot\langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}\boldsymbol{v}_\alpha^\perp\rangle_{lj} + c_3^{(\psi)}\,\frac{i\boldsymbol{q}}{m_\psi}\cdot\langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}\,\boldsymbol{v}_\alpha^\perp\times\boldsymbol{S}_{\psi,\alpha}\rangle_{lj}\Big]$$

- $v^\perp$-independent operators.

$$c_1^{(\psi)}\sum_\alpha\langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}\rangle_{lj} \simeq c_1^{(\psi)}\sum_\alpha\langle 1\rangle_{lj} = c_1^{(\psi)}\boxed{\langle N_\psi\rangle_{lj}},$$

$$c_4^{(\psi)}\,\boldsymbol{S}_\chi\cdot\sum_\alpha\langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}\boldsymbol{S}_{\psi,\alpha}\rangle_{lj} \simeq c_4^{(\psi)}\,\boldsymbol{S}_\chi\cdot\sum_\alpha\langle \boldsymbol{S}_{\psi,\alpha}\rangle_{lj} = c_4^{(\psi)}\,\boldsymbol{S}_\chi\cdot\boxed{\langle \boldsymbol{S}_\psi\rangle_{lj}}.$$

  - Couplings to particle number and total spin of the protons/neutrons/electrons associated with the ion at site l,j.

Interaction Type	NR Operators
Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\boxed{\mathcal{O}_1^{(\psi)} = \mathbb{1}}$ $\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi\cdot\frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi\cdot\left(\frac{i\boldsymbol{q}}{m_\psi}\times\boldsymbol{v}^\perp\right)$ $\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi\cdot\boldsymbol{v}^\perp$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\boxed{\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi\cdot\boldsymbol{S}_\psi}$ $\mathcal{O}_6^{(\psi)} = \left(\boldsymbol{S}_\chi\cdot\frac{\boldsymbol{q}}{m_\psi}\right)\left(\boldsymbol{S}_\psi\cdot\frac{\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi\cdot\left(\boldsymbol{S}_\psi\times\frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi\cdot\frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi\cdot\left(\frac{i\boldsymbol{q}}{m_\psi}\times\boldsymbol{v}^\perp\right)$ $\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi\cdot\boldsymbol{v}^\perp$ $\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi\cdot\left(\boldsymbol{S}_\psi\times\boldsymbol{v}^\perp\right)$ $\mathcal{O}_{13}^{(\psi)} = \left(\boldsymbol{S}_\chi\cdot\boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\psi\cdot\frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{14}^{(\psi)} = \left(\boldsymbol{S}_\psi\cdot\boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\chi\cdot\frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{15}^{(\psi)} = \left(\boldsymbol{S}_\chi\cdot\left(\frac{i\boldsymbol{q}}{m_\psi}\times\boldsymbol{v}^\perp\right)\right)\left(\boldsymbol{S}_\psi\cdot\frac{i\boldsymbol{q}}{m_\psi}\right)$

Step 2: matching onto lattice d.o.f.

$$\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v}) \supset \sum_{\alpha}\Big[c_1^{(\psi)}\,\langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}\rangle_{lj} + c_4^{(\psi)}\,\boldsymbol{S}_\chi\cdot\langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}\boldsymbol{S}_{\psi,\alpha}\rangle_{lj}$$

$$+\,c_8^{(\psi)}\,\boldsymbol{S}_\chi\cdot\langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}\boldsymbol{v}_\alpha^\perp\rangle_{lj} + c_3^{(\psi)}\,\frac{i\boldsymbol{q}}{m_\psi}\cdot\langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}\,\boldsymbol{v}_\alpha^\perp\times\boldsymbol{S}_{\psi,\alpha}\rangle_{lj}\Big]$$

- $\boldsymbol{v}^\perp$-dependent operators.

$$\boldsymbol{v}_\alpha^\perp = \boldsymbol{v} - \frac{\boldsymbol{q}}{2m_\chi} - \frac{(\boldsymbol{k}+\boldsymbol{k}')_\alpha}{2m_\psi} = \boldsymbol{v} - \frac{\boldsymbol{q}}{2m_\chi} + \frac{i}{2m_\psi}\overleftrightarrow{\nabla}_\alpha$$

Interaction Type	NR Operators
Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_1^{(\psi)} = \mathbb{1}$ $\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi\cdot\frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi\cdot\left(\frac{i\boldsymbol{q}}{m_\psi}\times\boldsymbol{v}^\perp\right)$ $\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi\cdot\boldsymbol{v}^\perp$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi\cdot\boldsymbol{S}_\psi$ $\mathcal{O}_6^{(\psi)} = \left(\boldsymbol{S}_\chi\cdot\frac{\boldsymbol{q}}{m_\psi}\right)\left(\boldsymbol{S}_\psi\cdot\frac{\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi\cdot\left(\boldsymbol{S}_\psi\times\frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi\cdot\frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi\cdot\left(\frac{i\boldsymbol{q}}{m_\psi}\times\boldsymbol{v}^\perp\right)$ $\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi\cdot\boldsymbol{v}^\perp$ $\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi\cdot\left(\boldsymbol{S}_\psi\times\boldsymbol{v}^\perp\right)$ $\mathcal{O}_{13}^{(\psi)} = \left(\boldsymbol{S}_\chi\cdot\boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\psi\cdot\frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{14}^{(\psi)} = \left(\boldsymbol{S}_\psi\cdot\boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\chi\cdot\frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{15}^{(\psi)} = \left(\boldsymbol{S}_\chi\cdot\left(\frac{i\boldsymbol{q}}{m_\psi}\times\boldsymbol{v}^\perp\right)\right)\left(\boldsymbol{S}_\psi\cdot\frac{i\boldsymbol{q}}{m_\psi}\right)$

Step 2: matching onto lattice d.o.f.

$$\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v}) \supset \sum_\alpha \Big[c_1^{(\psi)} \langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}\rangle_{lj} + c_4^{(\psi)} \boldsymbol{S}_\chi \cdot \langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha} \boldsymbol{S}_{\psi,\alpha}\rangle_{lj} + \boxed{c_8^{(\psi)} \boldsymbol{S}_\chi \cdot \langle \cancel{e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}} \boldsymbol{v}_\alpha^\perp\rangle_{lj} + c_3^{(\psi)} \frac{i\boldsymbol{q}}{m_\psi} \cdot \langle \cancel{e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}}^{\,1} \boldsymbol{v}_\alpha^\perp \times \boldsymbol{S}_{\psi,\alpha}\rangle_{lj}} \Big]$$

- $\boldsymbol{v}^\perp$-dependent operators.

$$\boldsymbol{v}_\alpha^\perp = \boldsymbol{v} - \frac{\boldsymbol{q}}{2m_\chi} - \frac{(\boldsymbol{k}+\boldsymbol{k}')_\alpha}{2m_\psi} = \boxed{\boldsymbol{v} - \frac{\boldsymbol{q}}{2m_\chi}} + \frac{i}{2m_\psi} \overleftrightarrow{\nabla}_\alpha$$

same treatment as before

$\Rightarrow$ total particle number & spin

Interaction Type	NR Operators
Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_1^{(\psi)} = \mathbb{1}$ $\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{S}_\psi$ $\mathcal{O}_6^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)\left(\boldsymbol{S}_\psi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp$ $\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_{13}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{14}^{(\psi)} = \left(\boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{15}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$

Step 2: matching onto lattice d.o.f.

$$\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v}) \supset \sum_\alpha \Big[c_1^{(\psi)} \langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha} \rangle_{lj} + c_4^{(\psi)} \boldsymbol{S}_\chi \cdot \langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha} \boldsymbol{S}_{\psi,\alpha} \rangle_{lj} + \boxed{c_8^{(\psi)} \boldsymbol{S}_\chi \cdot \langle e^{\cancel{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}} \boldsymbol{v}_\alpha^\perp \rangle_{lj} + c_3^{(\psi)} \frac{i\boldsymbol{q}}{m_\psi} \cdot \langle e^{\cancel{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}} \boldsymbol{v}_\alpha^\perp \times \boldsymbol{S}_{\psi,\alpha} \rangle_{lj}} \Big]$$

$iq{\cdot}x_\alpha$

- $v^\perp$-dependent operators.

$$\boldsymbol{v}_\alpha^\perp = \boldsymbol{v} - \frac{\boldsymbol{q}}{2m_\chi} - \frac{(\boldsymbol{k}+\boldsymbol{k}')_\alpha}{2m_\psi} = \boxed{\boldsymbol{v} - \frac{\boldsymbol{q}}{2m_\chi}} + \boxed{\frac{i}{2m_\psi} \overleftrightarrow{\nabla}_\alpha}$$

same treatment as before
$\Rightarrow$ total particle number & spin

probability current

Interaction Type	NR Operators
Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_1^{(\psi)} = \mathbb{1}$ $\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{S}_\psi$ $\mathcal{O}_6^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)\left(\boldsymbol{S}_\psi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp$ $\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_{13}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{14}^{(\psi)} = \left(\boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{15}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$

Step 2: matching onto lattice d.o.f.

$$\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v}) \supset \sum_\alpha \Big[c_1^{(\psi)} \langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}\rangle_{lj} + c_4^{(\psi)}\, \boldsymbol{S}_\chi \cdot \langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha} \boldsymbol{S}_{\psi,\alpha}\rangle_{lj}$$

$$\boxed{+ c_8^{(\psi)}\, \boldsymbol{S}_\chi \cdot \langle \xrightarrow{} e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha} \boldsymbol{v}_\alpha^\perp \rangle_{lj} + c_3^{(\psi)}\, \frac{i\boldsymbol{q}}{m_\psi} \cdot \langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}\, \boldsymbol{v}_\alpha^\perp \times \boldsymbol{S}_{\psi,\alpha}\rangle_{lj}} \Big]$$

$iq{\cdot}x_\alpha$

➤ $v^\perp$-dependent operators.

$$\boldsymbol{v}_\alpha^\perp = \boldsymbol{v} - \frac{\boldsymbol{q}}{2m_\chi} - \frac{(\boldsymbol{k}+\boldsymbol{k}')_\alpha}{2m_\psi} = \boxed{\boldsymbol{v} - \frac{\boldsymbol{q}}{2m_\chi}} + \boxed{\frac{i}{2m_\psi} \overleftrightarrow{\nabla}_\alpha}$$

same treatment as before
⇒ total particle number & spin

probability current

$$\sum_\alpha \langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha} \boldsymbol{j}_\alpha\rangle_{lj} \simeq \frac{i\boldsymbol{q}}{2m_\psi} \times \sum_\alpha \langle \boldsymbol{L}_{\psi,\alpha}\rangle_{lj} = \frac{i\boldsymbol{q}}{2m_\psi} \times \boxed{\langle \boldsymbol{L}_\psi\rangle_{lj}}$$

➤ Orbital angular momentum emerges (after some algebra).

Interaction Type	NR Operators
Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_1^{(\psi)} = \mathbb{1}$ $\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\boxed{\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{S}_\psi$ $\mathcal{O}_6^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)\left(\boldsymbol{S}_\psi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\boxed{\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)}$ $\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp$ $\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_{13}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{14}^{(\psi)} = \left(\boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{15}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$

Step 2: matching onto lattice d.o.f.

$$\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v}) \supset \sum_\alpha \Big[c_1^{(\psi)} \langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}\rangle_{lj} + c_4^{(\psi)}\, \boldsymbol{S}_\chi \cdot \langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha} \boldsymbol{S}_{\psi,\alpha}\rangle_{lj}$$
$$\boxed{+ c_8^{(\psi)}\, \boldsymbol{S}_\chi \cdot \langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha} \boldsymbol{v}_\alpha^\perp\rangle_{lj} + c_3^{(\psi)}\, \frac{i\boldsymbol{q}}{m_\psi} \cdot \langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}\, \boldsymbol{v}_\alpha^\perp \times \boldsymbol{S}_{\psi,\alpha}\rangle_{lj}} \Big]$$

- $v^\perp$-dependent operators.

$$c_8^{(\psi)}\, \boldsymbol{S}_\chi \cdot \sum_\alpha \langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha} \boldsymbol{v}_\alpha^\perp\rangle_{lj} = c_8^{(\psi)}\, \boldsymbol{S}_\chi \cdot \left[\left(\boldsymbol{v} - \frac{\boldsymbol{q}}{2m_\chi}\right) \langle N_\psi\rangle_{lj} + \frac{i\boldsymbol{q}}{2m_\psi} \times \langle \boldsymbol{L}_\psi\rangle_{lj} \right],$$

$$c_3^{(\psi)}\, \frac{i\boldsymbol{q}}{m_\psi} \cdot \sum_\alpha \langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha}\, \boldsymbol{v}_\alpha^\perp \times \boldsymbol{S}_{\psi,\alpha}\rangle_{lj}$$
$$= c_3^{(\psi)} \left[\left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}\right) \cdot \langle \boldsymbol{S}_\psi\rangle_{lj} + \frac{1}{2m_\psi^2} (q^2\delta^{ik} - q^i q^k) \big(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi\rangle_{lj}\big)^{ik} \right].$$

Interaction Type	NR Operators
Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_1^{(\psi)} = \mathbb{1}$ $\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi \cdot (\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp)$ $\boxed{\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{S}_\psi$ $\mathcal{O}_6^{(\psi)} = (\boldsymbol{S}_\chi \cdot \frac{\boldsymbol{q}}{m_\psi})(\boldsymbol{S}_\psi \cdot \frac{\boldsymbol{q}}{m_\psi})$ $\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi \cdot (\boldsymbol{S}_\psi \times \frac{i\boldsymbol{q}}{m_\psi})$ $\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\boxed{\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi \cdot (\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp)}$ $\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp$ $\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi \cdot (\boldsymbol{S}_\psi \times \boldsymbol{v}^\perp)$ $\mathcal{O}_{13}^{(\psi)} = (\boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp)(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi})$ $\mathcal{O}_{14}^{(\psi)} = (\boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp)(\boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi})$ $\mathcal{O}_{15}^{(\psi)} = (\boldsymbol{S}_\chi \cdot (\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp))(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi})$

Step 2: matching onto lattice d.o.f.

$$\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v}) \supset \sum_\alpha \Big[c_1^{(\psi)} \langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha} \rangle_{lj} + c_4^{(\psi)} \boldsymbol{S}_\chi \cdot \langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha} \boldsymbol{S}_{\psi,\alpha} \rangle_{lj}$$
$$+ \boxed{c_8^{(\psi)} \boldsymbol{S}_\chi \cdot \langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha} \boldsymbol{v}_\alpha^\perp \rangle_{lj} + c_3^{(\psi)} \frac{i\boldsymbol{q}}{m_\psi} \cdot \langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha} \boldsymbol{v}_\alpha^\perp \times \boldsymbol{S}_{\psi,\alpha} \rangle_{lj}} \Big]$$

- $v^\perp$-dependent operators.

$$c_8^{(\psi)} \boldsymbol{S}_\chi \cdot \sum_\alpha \langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha} \boldsymbol{v}_\alpha^\perp \rangle_{lj} = c_8^{(\psi)} \boldsymbol{S}_\chi \cdot \left[\left(\boldsymbol{v} - \frac{\boldsymbol{q}}{2m_\chi} \right) \boxed{\langle N_\psi \rangle_{lj}} + \frac{i\boldsymbol{q}}{2m_\psi} \times \boxed{\langle \boldsymbol{L}_\psi \rangle_{lj}} \right],$$

$$c_3^{(\psi)} \frac{i\boldsymbol{q}}{m_\psi} \cdot \sum_\alpha \langle e^{i\boldsymbol{q}\cdot\boldsymbol{x}_\alpha} \boldsymbol{v}_\alpha^\perp \times \boldsymbol{S}_{\psi,\alpha} \rangle_{lj}$$
$$= c_3^{(\psi)} \left[\left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v} \right) \cdot \boxed{\langle \boldsymbol{S}_\psi \rangle_{lj}} + \frac{1}{2m_\psi^2} (q^2 \delta^{ik} - q^i q^k) \Big(\boxed{\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj}} \Big)^{ik} \right].$$

  - Couplings to particle number, total spin, orbital angular momentum and spin-orbit coupling of the protons/neutrons/electrons associated with the ion at site *l,j*.

Interaction Type	NR Operators
Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_1^{(\psi)} = \mathbb{1}$ $\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\boxed{\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{S}_\psi$ $\mathcal{O}_6^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)\left(\boldsymbol{S}_\psi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\boxed{\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)}$ $\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp$ $\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_{13}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{14}^{(\psi)} = \left(\boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{15}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$

Step 2: matching onto lattice d.o.f.

crystal responses

DM-ion scattering potential

$$
\begin{aligned}
\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v}) = \sum_{\psi=p,n,e} & c_1^{(\psi)} \langle N_\psi \rangle_{lj} \\
& + c_3^{(\psi)} \left[-\frac{iq}{m_\psi} \boldsymbol{v}' \cdot \left(\hat{\boldsymbol{q}} \times \langle \boldsymbol{S}_\psi \rangle_{lj} \right) + \frac{q^2}{2m_\psi^2} \left(\delta^{ik} - \hat{q}^i \hat{q}^k \right) \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj} \right)^{ik} \right] \\
& + c_4^{(\psi)} \boldsymbol{S}_\chi \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} \\
& + c_5^{(\psi)} \left[\frac{i\boldsymbol{q}}{m_\psi} \cdot \left(\boldsymbol{v}' \times \boldsymbol{S}_\chi \right) \langle N_\psi \rangle_{lj} + \frac{q^2}{2m_\psi^2} \boldsymbol{S}_\chi \cdot \left(\mathbb{1} - \hat{\boldsymbol{q}}\hat{\boldsymbol{q}} \right) \cdot \langle \boldsymbol{L}_\psi \rangle_{lj} \right] \\
& + c_6^{(\psi)} \frac{q^2}{m_\psi^2} \left(\hat{\boldsymbol{q}} \cdot \boldsymbol{S}_\chi \right) \left(\hat{\boldsymbol{q}} \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} \right) \\
& + c_7^{(\psi)} \left[\boldsymbol{v}' \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} + \epsilon^{ikk'} \frac{iq^{k'}}{2m_\chi} \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj} \right)^{ik} \right] \\
& + c_8^{(\psi)} \left[\left(\boldsymbol{v}' \cdot \boldsymbol{S}_\chi \right) \langle N_\psi \rangle_{lj} + \frac{iq}{2m_\psi} \boldsymbol{S}_\chi \cdot \left(\hat{\boldsymbol{q}} \times \langle \boldsymbol{L}_\psi \rangle_{lj} \right) \right] \\
& + c_9^{(\psi)} \frac{iq}{m_\psi} \boldsymbol{S}_\chi \cdot \left(\langle \boldsymbol{S}_\psi \rangle_{lj} \times \hat{\boldsymbol{q}} \right) \\
& + c_{10}^{(\psi)} \frac{i\boldsymbol{q}}{m_\psi} \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} \\
& + c_{11}^{(\psi)} \frac{i\boldsymbol{q}}{m_\psi} \cdot \boldsymbol{S}_\chi \langle N_\psi \rangle_{lj} \\
& + c_{12}^{(\psi)} \left[\left(\boldsymbol{v}' \times \boldsymbol{S}_\chi \right) \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} + \frac{iq}{2m_\psi} \left((\hat{\boldsymbol{q}} \cdot \boldsymbol{S}_\chi) \delta^{ik} - \hat{q}^k S_\chi^i \right) \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj} \right)^{ik} \right] \\
& + c_{13}^{(\psi)} \left[\frac{iq}{m_\psi} \left(\boldsymbol{v}' \cdot \boldsymbol{S}_\chi \right) \left(\hat{\boldsymbol{q}} \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} \right) + \frac{q^2}{2m_\psi^2} \left(\hat{\boldsymbol{q}} \times \boldsymbol{S}_\chi \right) \cdot \langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj} \cdot \hat{\boldsymbol{q}} \right] \\
& + c_{14}^{(\psi)} \left[\frac{iq}{m_\psi} \left(\hat{\boldsymbol{q}} \cdot \boldsymbol{S}_\chi \right) \left(\boldsymbol{v}' \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} \right) - \epsilon^{ikk'} \frac{q^2}{2m_\psi^2} \hat{q}^{k'} \left(\hat{\boldsymbol{q}} \cdot \boldsymbol{S}_\chi \right) \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj} \right)^{ik} \right] \\
& + c_{15}^{(\psi)} \left[-\frac{q^2}{m_\psi^2} \left(\hat{\boldsymbol{q}} \cdot \left(\boldsymbol{v}' \times \boldsymbol{S}_\chi \right) \right) \left(\hat{\boldsymbol{q}} \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} \right) \right. \\
& \qquad \left. + \frac{iq^3}{2m_\psi^3} \boldsymbol{S}_\chi \cdot \left(\mathbb{1} - \hat{\boldsymbol{q}}\hat{\boldsymbol{q}} \right) \cdot \langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj} \cdot \hat{\boldsymbol{q}} \right],
\end{aligned}
$$

Interaction Type	NR Operators	Point-like Response	Composite Response
Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\boxed{\mathcal{O}_1^{(\psi)} = \mathbb{1}}$ $\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}$	N	-
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp \right)$ $\boxed{\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp}$	N	L
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\boxed{\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{S}_\psi}$ $\mathcal{O}_6^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \frac{\boldsymbol{q}}{m_\psi} \right) \left(\boldsymbol{S}_\psi \cdot \frac{\boldsymbol{q}}{m_\psi} \right)$ $\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \frac{i\boldsymbol{q}}{m_\psi} \right)$ $\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}$	S	-
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\boxed{\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp \right)}$ $\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp$ $\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \boldsymbol{v}^\perp \right)$ $\mathcal{O}_{13}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp \right) \left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi} \right)$ $\mathcal{O}_{14}^{(\psi)} = \left(\boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp \right) \left(\boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi} \right)$ $\mathcal{O}_{15}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp \right) \right) \left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi} \right)$	S	$L \otimes S$

Step 2: matching onto lattice d.o.f.

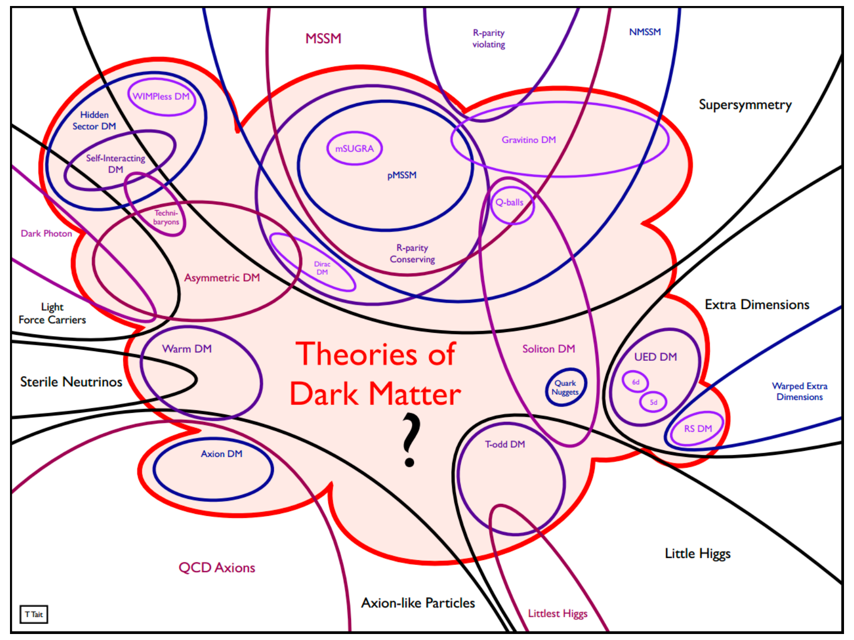
crystal responses

DM-ion scattering potential

$$
\begin{aligned}
\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v}) = \sum_{\psi=p,n,e} & c_1^{(\psi)} \langle N_\psi\rangle_{lj} \\
&+ c_3^{(\psi)}\left[-\frac{iq}{m_\psi}\boldsymbol{v}'\cdot\left(\hat{\boldsymbol{q}}\times\langle \boldsymbol{S}_\psi\rangle_{lj}\right) + \frac{q^2}{2m_\psi^2}\left(\delta^{ik}-\hat{q}^i\hat{q}^k\right)\left(\langle \boldsymbol{L}_\psi\otimes \boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
&+ c_4^{(\psi)}\,\boldsymbol{S}_\chi\cdot\langle \boldsymbol{S}_\psi\rangle_{lj} \\
&+ c_5^{(\psi)}\left[\frac{i\boldsymbol{q}}{m_\psi}\cdot\left(\boldsymbol{v}'\times \boldsymbol{S}_\chi\right)\langle N_\psi\rangle_{lj} + \frac{q^2}{2m_\psi^2}\boldsymbol{S}_\chi\cdot\left(\mathbb{1}-\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}\right)\cdot\langle \boldsymbol{L}_\psi\rangle_{lj}\right] \\
&+ c_6^{(\psi)}\frac{q^2}{m_\psi^2}\left(\hat{\boldsymbol{q}}\cdot \boldsymbol{S}_\chi\right)\left(\hat{\boldsymbol{q}}\cdot\langle \boldsymbol{S}_\psi\rangle_{lj}\right) \\
&+ c_7^{(\psi)}\left[\boldsymbol{v}'\cdot\langle \boldsymbol{S}_\psi\rangle_{lj} + \epsilon^{ikk'}\frac{iq^{k'}}{2m_\chi}\left(\langle \boldsymbol{L}_\psi\otimes \boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
&+ c_8^{(\psi)}\left[\left(\boldsymbol{v}'\cdot \boldsymbol{S}_\chi\right)\langle N_\psi\rangle_{lj} + \frac{iq}{2m_\psi}\boldsymbol{S}_\chi\cdot\left(\hat{\boldsymbol{q}}\times\langle \boldsymbol{L}_\psi\rangle_{lj}\right)\right] \\
&+ c_9^{(\psi)}\frac{iq}{m_\psi}\boldsymbol{S}_\chi\cdot\left(\langle \boldsymbol{S}_\psi\rangle_{lj}\times\hat{\boldsymbol{q}}\right) \\
&+ c_{10}^{(\psi)}\frac{i\boldsymbol{q}}{m_\psi}\cdot\langle \boldsymbol{S}_\psi\rangle_{lj} \\
&+ c_{11}^{(\psi)}\frac{i\boldsymbol{q}}{m_\psi}\cdot \boldsymbol{S}_\chi\,\langle N_\psi\rangle_{lj} \\
&+ c_{12}^{(\psi)}\left[\left(\boldsymbol{v}'\times \boldsymbol{S}_\chi\right)\cdot\langle \boldsymbol{S}_\psi\rangle_{lj} + \frac{iq}{2m_\psi}\left(\left(\hat{\boldsymbol{q}}\cdot \boldsymbol{S}_\chi\right)\delta^{ik}-\hat{q}^k S_\chi^i\right)\left(\langle \boldsymbol{L}_\psi\otimes \boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
&+ c_{13}^{(\psi)}\left[\frac{iq}{m_\psi}\left(\boldsymbol{v}'\cdot \boldsymbol{S}_\chi\right)\left(\hat{\boldsymbol{q}}\cdot\langle \boldsymbol{S}_\psi\rangle_{lj}\right) + \frac{q^2}{2m_\psi^2}\left(\hat{\boldsymbol{q}}\times \boldsymbol{S}_\chi\right)\cdot\langle \boldsymbol{L}_\psi\otimes \boldsymbol{S}_\psi\rangle_{lj}\cdot\hat{\boldsymbol{q}}\right] \\
&+ c_{14}^{(\psi)}\left[\frac{iq}{m_\psi}\left(\hat{\boldsymbol{q}}\cdot \boldsymbol{S}_\chi\right)\left(\boldsymbol{v}'\cdot\langle \boldsymbol{S}_\psi\rangle_{lj}\right) - \epsilon^{ikk'}\frac{q^2}{2m_\psi^2}\hat{q}^{k'}\left(\hat{\boldsymbol{q}}\cdot \boldsymbol{S}_\chi\right)\left(\langle \boldsymbol{L}_\psi\otimes \boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
&+ c_{15}^{(\psi)}\left[-\frac{q^2}{m_\psi^2}\left(\hat{\boldsymbol{q}}\cdot\left(\boldsymbol{v}'\times \boldsymbol{S}_\chi\right)\right)\left(\hat{\boldsymbol{q}}\cdot\langle \boldsymbol{S}_\psi\rangle_{lj}\right)\right. \\
&\qquad\left. + \frac{iq^3}{2m_\psi^3}\boldsymbol{S}_\chi\cdot\left(\mathbb{1}-\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}\right)\cdot\langle \boldsymbol{L}_\psi\otimes \boldsymbol{S}_\psi\rangle_{lj}\cdot\hat{\boldsymbol{q}}\right],
\end{aligned}
$$

Interaction Type	NR Operators	Point-like Response	Composite Response
Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\boxed{\mathcal{O}_1^{(\psi)} = \mathbb{1}}$ $\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi\cdot\frac{i\boldsymbol{q}}{m_\psi}$	N	-
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi\cdot\left(\frac{i\boldsymbol{q}}{m_\psi}\times\boldsymbol{v}^\perp\right)$ $\boxed{\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi\cdot\boldsymbol{v}^\perp}$	N	L
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\boxed{\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi\cdot\boldsymbol{S}_\psi}$ $\mathcal{O}_6^{(\psi)} = \left(\boldsymbol{S}_\chi\cdot\frac{\boldsymbol{q}}{m_\psi}\right)\left(\boldsymbol{S}_\psi\cdot\frac{\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi\cdot\left(\boldsymbol{S}_\psi\times\frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi\cdot\frac{i\boldsymbol{q}}{m_\psi}$	S	-
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\boxed{\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi\cdot\left(\frac{i\boldsymbol{q}}{m_\psi}\times\boldsymbol{v}^\perp\right)}$ $\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi\cdot\boldsymbol{v}^\perp$ $\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi\cdot\left(\boldsymbol{S}_\psi\times\boldsymbol{v}^\perp\right)$ $\mathcal{O}_{13}^{(\psi)} = \left(\boldsymbol{S}_\chi\cdot\boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\psi\cdot\frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{14}^{(\psi)} = \left(\boldsymbol{S}_\psi\cdot\boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\chi\cdot\frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{15}^{(\psi)} = \left(\boldsymbol{S}_\chi\cdot\left(\frac{i\boldsymbol{q}}{m_\psi}\times\boldsymbol{v}^\perp\right)\right)\left(\boldsymbol{S}_\psi\cdot\frac{i\boldsymbol{q}}{m_\psi}\right)$	S	$L\otimes S$

EFT of DM direct detection: preview



1 ✅

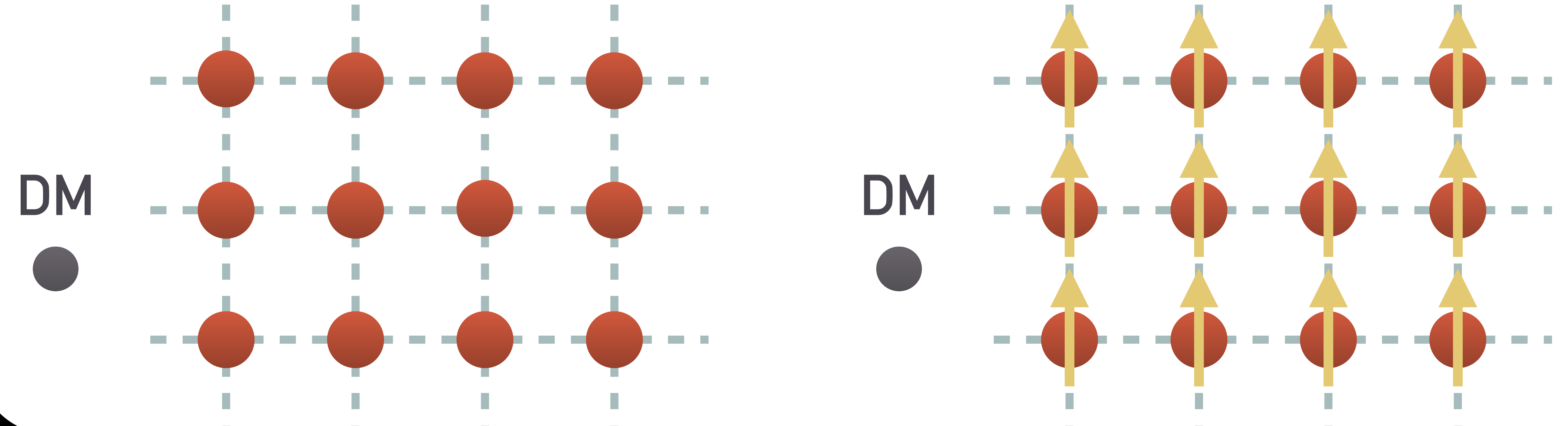
Nonrelativistic (NR) EFT of DM-SM interactions

2

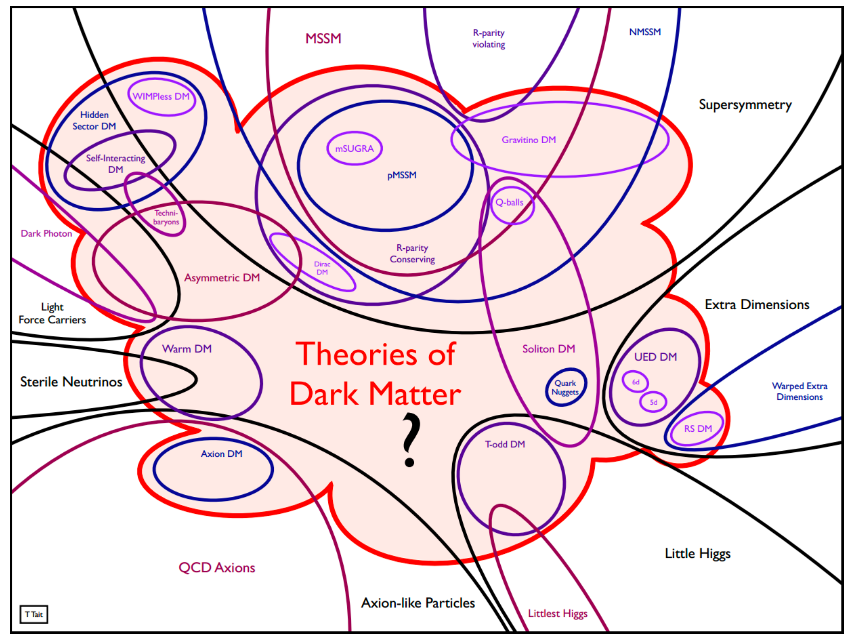
Crystal responses

3

Phonon & magnon excitation rates



EFT of DM direct detection: preview



1

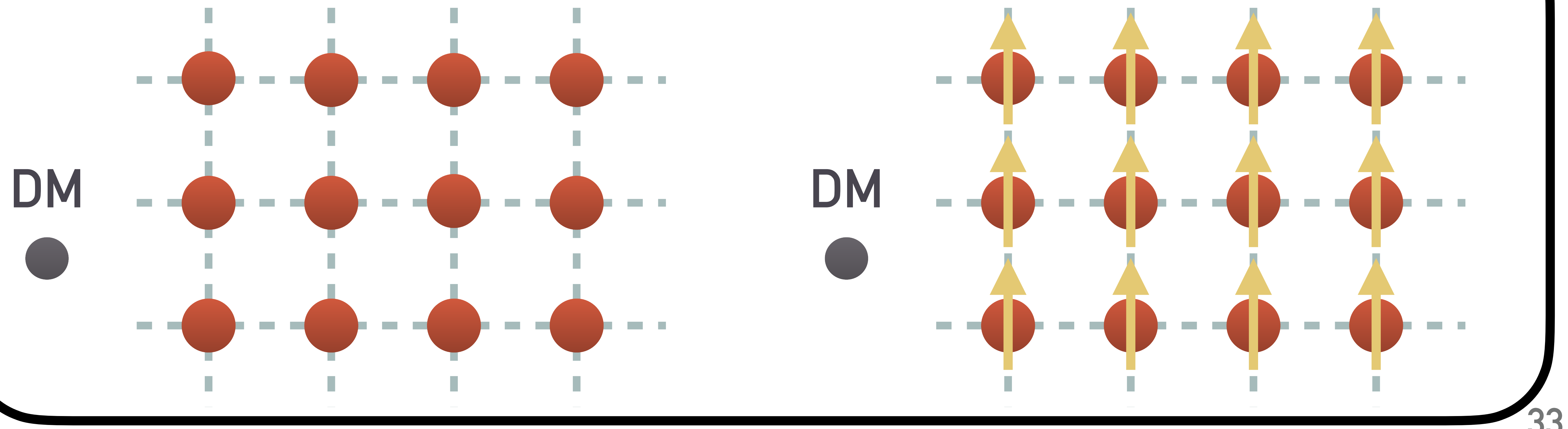
2

Crystal responses

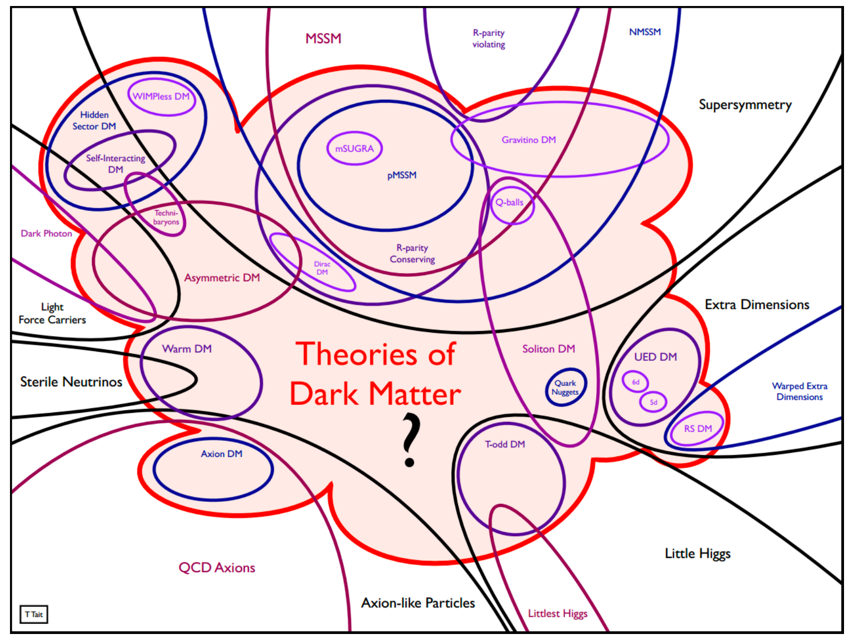
3

Nonrelativistic (NR) EFT of DM-SM interactions

Phonon & magnon excitation rates



EFT of DM direct detection: preview



1 ✓

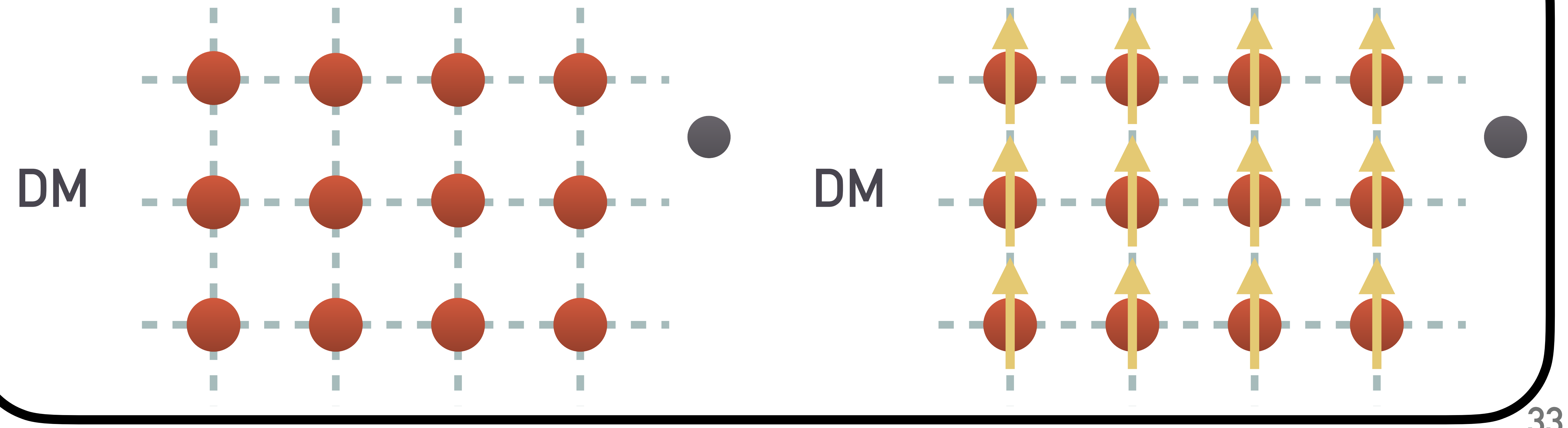
Nonrelativistic (NR) EFT of DM-SM interactions

2 ✓

Crystal responses

3

Phonon & magnon excitation rates



Step 3: quantizing lattice potential for phonons & magnons

- Big picture.
  - All 4 responses can excite phonons.
    - Any coupling can shake the lattice.
    - But it needs to be **"coherent"** to excite phonons.

Crystal responses

<u>DM couplings to lattice d.o.f.</u>

$\boldsymbol{N}$ (particle number)	$\boldsymbol{S}$ (spin)
$\boldsymbol{L}$ (orbital angular momentum)	$\boldsymbol{L} \otimes \boldsymbol{S}$ (spin-orbit coupling)

Step 3: quantizing lattice potential for phonons & magnons

- Big picture.
  - All 4 responses can excite phonons.
    - Any coupling can shake the lattice.
    - But it needs to be **"coherent"** to excite phonons.
    - Scalar couplings (N, $\boldsymbol{L}\cdot\boldsymbol{S}$) are trivially coherent.

Crystal responses

DM couplings to lattice d.o.f.

$\boxed{N}$ (particle number)	$\boldsymbol{S}$ (spin)
$\boldsymbol{L}$ (orbital angular momentum)	$\boxed{\boldsymbol{L}\otimes\boldsymbol{S}}$ (spin-orbit coupling)

Step 3: quantizing lattice potential for phonons & magnons

- Big picture.
  - All 4 responses can excite phonons.
    - Any coupling can shake the lattice.
    - But it needs to be **"coherent"** to excite phonons.
    - Scalar couplings (N, $\boldsymbol{L}\cdot\boldsymbol{S}$) are trivially coherent.
    - Vector/tensor couplings are not coherent unless ordered (spontaneously or by external fields).

Crystal responses

DM couplings to lattice d.o.f.

N (particle number)	$\boxed{\boldsymbol{S}}$ (spin)
$\boxed{\boldsymbol{L}}$ (orbital angular momentum)	$\boxed{\boldsymbol{L}\otimes\boldsymbol{S}}$ (spin-orbit coupling)

Step 3: quantizing lattice potential for phonons & magnons

- Big picture.
  - All 4 responses can excite phonons.
    - Any coupling can shake the lattice.
    - But it needs to be **"coherent"** to excite phonons.
    - Scalar couplings (N, $\boldsymbol{L}\cdot\boldsymbol{S}$) are trivially coherent.
    - Vector/tensor couplings are not coherent unless ordered (spontaneously or by external fields).
  - Need to couple to magnetic ions' spins to excite magnons.

Crystal responses

<u>DM couplings to lattice d.o.f.</u>

N (particle number)	$\boldsymbol{S}$ (spin)
$\boldsymbol{L}$ (orbital angular momentum)	$\boldsymbol{L}\otimes\boldsymbol{S}$ (spin-orbit coupling)

Step 3: quantizing lattice potential for phonons & magnons

- Big picture.
  - All 4 responses can excite phonons.
    - Any coupling can shake the lattice.
    - But it needs to be **"coherent"** to excite phonons.
    - Scalar couplings (N, $\boldsymbol{L}\cdot\boldsymbol{S}$) are trivially coherent.
    - Vector/tensor couplings are not coherent unless ordered (spontaneously or by external fields).
  - Need to couple to magnetic ions' spins to excite magnons.
    - Ionic spins come from electrons. In many cases, they are $\boldsymbol{S}_e$.

Crystal responses

DM couplings to lattice d.o.f.

N (particle number)	$\boxed{\boldsymbol{S}}$ (spin)
$\boldsymbol{L}$ (orbital angular momentum)	$\boldsymbol{L}\otimes\boldsymbol{S}$ (spin-orbit coupling)

Step 3: quantizing lattice potential for phonons & magnons

- Big picture.
  - All 4 responses can excite phonons.
    - Any coupling can shake the lattice.
    - But it needs to be "coherent" to excite phonons.
    - Scalar couplings (N, $\boldsymbol{L}\cdot\boldsymbol{S}$) are trivially coherent.
    - Vector/tensor couplings are not coherent unless ordered (spontaneously or by external fields).
  - Need to couple to magnetic ions' spins to excite magnons.
    - Ionic spins come from electrons. In many cases, they are $\boldsymbol{S}_e$.
    - They may also have orbital components $\boldsymbol{L}_e$.

Crystal responses

DM couplings to lattice d.o.f.

N (particle number)	$\boxed{\boldsymbol{S}}$ (spin)
$\boxed{\boldsymbol{L}}$ (orbital angular momentum)	$\boldsymbol{L}\otimes\boldsymbol{S}$ (spin-orbit coupling)

Step 3: quantizing lattice potential for phonons & magnons

➤ Upshot of the calculation.

$$\boxed{\langle \nu, \boldsymbol{k}|}\widetilde{\mathcal{V}}(-\boldsymbol{q}, \boldsymbol{v})|0\rangle = \sum_{l,j} \langle \nu, \boldsymbol{k}|\, e^{i\boldsymbol{q}\cdot\boldsymbol{x}_{lj}} \boxed{\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q}, \boldsymbol{v})}|0\rangle$$

single phonon/magnon states

DM-ion scattering potential

$$\begin{aligned}
\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q}, \boldsymbol{v}) = \sum_{\psi=p,n,e} & c_1^{(\psi)} \langle N_\psi \rangle_{lj} \\
& + c_3^{(\psi)} \left[-\frac{iq}{m_\psi} \boldsymbol{v}' \cdot \left(\hat{\boldsymbol{q}} \times \langle \boldsymbol{S}_\psi \rangle_{lj}\right) + \frac{q^2}{2m_\psi^2} \left(\delta^{ik} - \hat{q}^i \hat{q}^k\right) \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj}\right)^{ik} \right] \\
& + c_4^{(\psi)} \boldsymbol{S}_\chi \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} \\
& + c_5^{(\psi)} \left[\frac{i\boldsymbol{q}}{m_\psi} \cdot \left(\boldsymbol{v}' \times \boldsymbol{S}_\chi\right) \langle N_\psi \rangle_{lj} + \frac{q^2}{2m_\psi^2} \boldsymbol{S}_\chi \cdot \left(\mathbb{1} - \hat{\boldsymbol{q}}\hat{\boldsymbol{q}}\right) \cdot \langle \boldsymbol{L}_\psi \rangle_{lj} \right] \\
& + c_6^{(\psi)} \frac{q^2}{m_\psi^2} \left(\hat{\boldsymbol{q}} \cdot \boldsymbol{S}_\chi\right) \left(\hat{\boldsymbol{q}} \cdot \langle \boldsymbol{S}_\psi \rangle_{lj}\right) \\
& + c_7^{(\psi)} \left[\boldsymbol{v}' \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} + \epsilon^{ikk'} \frac{iq^{k'}}{2m_\chi} \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj}\right)^{ik} \right] \\
& + c_8^{(\psi)} \left[\left(\boldsymbol{v}' \cdot \boldsymbol{S}_\chi\right) \langle N_\psi \rangle_{lj} + \frac{iq}{2m_\psi} \boldsymbol{S}_\chi \cdot \left(\hat{\boldsymbol{q}} \times \langle \boldsymbol{L}_\psi \rangle_{lj}\right) \right] \\
& + c_9^{(\psi)} \frac{iq}{m_\psi} \boldsymbol{S}_\chi \cdot \left(\langle \boldsymbol{S}_\psi \rangle_{lj} \times \hat{\boldsymbol{q}}\right) \\
& + c_{10}^{(\psi)} \frac{i\boldsymbol{q}}{m_\psi} \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} \\
& + c_{11}^{(\psi)} \frac{i\boldsymbol{q}}{m_\psi} \cdot \boldsymbol{S}_\chi \langle N_\psi \rangle_{lj} \\
& + c_{12}^{(\psi)} \left[\left(\boldsymbol{v}' \times \boldsymbol{S}_\chi\right) \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} + \frac{iq}{2m_\psi} \left(\left(\hat{\boldsymbol{q}} \cdot \boldsymbol{S}_\chi\right)\delta^{ik} - \hat{q}^k S_\chi^i\right) \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj}\right)^{ik} \right] \\
& + c_{13}^{(\psi)} \left[\frac{iq}{m_\psi} \left(\boldsymbol{v}' \cdot \boldsymbol{S}_\chi\right) \left(\hat{\boldsymbol{q}} \cdot \langle \boldsymbol{S}_\psi \rangle_{lj}\right) + \frac{q^2}{2m_\psi^2} \left(\hat{\boldsymbol{q}} \times \boldsymbol{S}_\chi\right) \cdot \langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj} \cdot \hat{\boldsymbol{q}} \right] \\
& + c_{14}^{(\psi)} \left[\frac{iq}{m_\psi} \left(\hat{\boldsymbol{q}} \cdot \boldsymbol{S}_\chi\right) \left(\boldsymbol{v}' \cdot \langle \boldsymbol{S}_\psi \rangle_{lj}\right) - \epsilon^{ikk'} \frac{q^2}{2m_\psi^2} \hat{q}^{k'} \left(\hat{\boldsymbol{q}} \cdot \boldsymbol{S}_\chi\right) \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj}\right)^{ik} \right] \\
& + c_{15}^{(\psi)} \left[-\frac{q^2}{m_\psi^2} \left(\hat{\boldsymbol{q}} \cdot \left(\boldsymbol{v}' \times \boldsymbol{S}_\chi\right)\right) \left(\hat{\boldsymbol{q}} \cdot \langle \boldsymbol{S}_\psi \rangle_{lj}\right) \right. \\
& \qquad \left. + \frac{iq^3}{2m_\psi^3} \boldsymbol{S}_\chi \cdot \left(\mathbb{1} - \hat{\boldsymbol{q}}\hat{\boldsymbol{q}}\right) \cdot \langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj} \cdot \hat{\boldsymbol{q}} \right],
\end{aligned}$$

Step 3: quantizing lattice potential for phonons & magnons

➤ Upshot of the calculation.

$$\boxed{\langle \nu, \boldsymbol{k}|}\widetilde{\mathcal{V}}(-\boldsymbol{q},\boldsymbol{v})|0\rangle = \sum_{l,j} \langle \nu, \boldsymbol{k}|\boxed{e^{i\boldsymbol{q}\cdot\boldsymbol{x}_{lj}}}\boxed{\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v})}|0\rangle$$

single phonon/magnon states

phonons

Leading dependence on lattice displacements comes from

$$\boldsymbol{u}_{lj} = \boldsymbol{x}_{lj} - \boldsymbol{x}^0_{lj} = \sum_{\nu=1}^{3n} \sum_{\boldsymbol{k}\in 1\mathrm{BZ}} \frac{1}{\sqrt{2N m_j \omega_{\nu,\boldsymbol{k}}}} \left(\hat{a}_{\nu,\boldsymbol{k}}\, \boldsymbol{\epsilon}_{\nu,\boldsymbol{k},j}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}^0_{lj}} + \hat{a}^\dagger_{\nu,\boldsymbol{k}}\, \boldsymbol{\epsilon}^*_{\nu,\boldsymbol{k},j}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}^0_{lj}} \right)$$

➤ Compute crystal Hamiltonian by density function theory (DFT).

- Done in previous works in collaboration with Griffin group at LBL.

  Griffin, Inzani, Trickle, ZZ, Zurek, 1910.10716.

- There are also online databases (e.g. phonondb@kyoto-u).

➤ Then solve eigensystem using the phonopy program.

DM-ion scattering potential

$$\begin{aligned}
\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v}) = \sum_{\psi=p,n,e} & c_1^{(\psi)} \langle N_\psi \rangle_{lj} \\
&+ c_3^{(\psi)} \left[-\frac{iq}{m_\psi} \boldsymbol{v}' \cdot \left(\hat{\boldsymbol{q}} \times \langle \boldsymbol{S}_\psi \rangle_{lj} \right) + \frac{q^2}{2m_\psi^2} (\delta^{ik} - \hat{q}^i \hat{q}^k) \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj} \right)^{ik} \right] \\
&+ c_4^{(\psi)} \boldsymbol{S}_\chi \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} \\
&+ c_5^{(\psi)} \left[\frac{i\boldsymbol{q}}{m_\psi} \cdot (\boldsymbol{v}' \times \boldsymbol{S}_\chi) \langle N_\psi \rangle_{lj} + \frac{q^2}{2m_\psi^2} \boldsymbol{S}_\chi \cdot (\mathbb{1} - \hat{\boldsymbol{q}}\hat{\boldsymbol{q}}) \cdot \langle \boldsymbol{L}_\psi \rangle_{lj} \right] \\
&+ c_6^{(\psi)} \frac{q^2}{m_\psi^2} (\hat{\boldsymbol{q}} \cdot \boldsymbol{S}_\chi)(\hat{\boldsymbol{q}} \cdot \langle \boldsymbol{S}_\psi \rangle_{lj}) \\
&+ c_7^{(\psi)} \left[\boldsymbol{v}' \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} + \epsilon^{ikk'} \frac{iq^{k'}}{2m_\chi} \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj} \right)^{ik} \right] \\
&+ c_8^{(\psi)} \left[(\boldsymbol{v}' \cdot \boldsymbol{S}_\chi) \langle N_\psi \rangle_{lj} + \frac{iq}{2m_\psi} \boldsymbol{S}_\chi \cdot \left(\hat{\boldsymbol{q}} \times \langle \boldsymbol{L}_\psi \rangle_{lj} \right) \right] \\
&+ c_9^{(\psi)} \frac{iq}{m_\psi} \boldsymbol{S}_\chi \cdot \left(\langle \boldsymbol{S}_\psi \rangle_{lj} \times \hat{\boldsymbol{q}} \right) \\
&+ c_{10}^{(\psi)} \frac{i\boldsymbol{q}}{m_\psi} \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} \\
&+ c_{11}^{(\psi)} \frac{i\boldsymbol{q}}{m_\psi} \cdot \boldsymbol{S}_\chi \langle N_\psi \rangle_{lj} \\
&+ c_{12}^{(\psi)} \left[(\boldsymbol{v}' \times \boldsymbol{S}_\chi) \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} + \frac{iq}{2m_\psi} \left((\hat{\boldsymbol{q}} \cdot \boldsymbol{S}_\chi) \delta^{ik} - \hat{q}^k S_\chi^i \right) \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj} \right)^{ik} \right] \\
&+ c_{13}^{(\psi)} \left[\frac{iq}{m_\psi} (\boldsymbol{v}' \cdot \boldsymbol{S}_\chi)(\hat{\boldsymbol{q}} \cdot \langle \boldsymbol{S}_\psi \rangle_{lj}) + \frac{q^2}{2m_\psi^2} (\hat{\boldsymbol{q}} \times \boldsymbol{S}_\chi) \cdot \langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj} \cdot \hat{\boldsymbol{q}} \right] \\
&+ c_{14}^{(\psi)} \left[\frac{iq}{m_\psi} (\hat{\boldsymbol{q}} \cdot \boldsymbol{S}_\chi)(\boldsymbol{v}' \cdot \langle \boldsymbol{S}_\psi \rangle_{lj}) - \epsilon^{ikk'} \frac{q^2}{2m_\psi^2} \hat{q}^{k'} (\hat{\boldsymbol{q}} \cdot \boldsymbol{S}_\chi) \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj} \right)^{ik} \right] \\
&+ c_{15}^{(\psi)} \left[-\frac{q^2}{m_\psi^2} \left(\hat{\boldsymbol{q}} \cdot (\boldsymbol{v}' \times \boldsymbol{S}_\chi) \right) (\hat{\boldsymbol{q}} \cdot \langle \boldsymbol{S}_\psi \rangle_{lj}) \right. \\
&\qquad \left. + \frac{iq^3}{2m_\psi^3} \boldsymbol{S}_\chi \cdot (\mathbb{1} - \hat{\boldsymbol{q}}\hat{\boldsymbol{q}}) \cdot \langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj} \cdot \hat{\boldsymbol{q}} \right],
\end{aligned}$$

Step 3: quantizing lattice potential for phonons & magnons

➤ Upshot of the calculation.

$$\boxed{\langle \nu, \boldsymbol{k}|}\widetilde{\mathcal{V}}(-\boldsymbol{q}, \boldsymbol{v})|0\rangle = \sum_{l,j} \langle \nu, \boldsymbol{k}|\boxed{e^{i\boldsymbol{q}\cdot\boldsymbol{x}_{lj}}}\,\boxed{\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q}, \boldsymbol{v})}|0\rangle$$

single phonon/magnon states

phonons

Leading dependence on lattice displacements comes from

$$\boldsymbol{u}_{lj} = \boldsymbol{x}_{lj} - \boldsymbol{x}_{lj}^0 = \sum_{\nu=1}^{3n} \sum_{\boldsymbol{k}\in 1\mathrm{BZ}} \frac{1}{\sqrt{2N m_j \omega_{\nu,\boldsymbol{k}}}} \left(\hat{a}_{\nu,\boldsymbol{k}}\, \boldsymbol{\epsilon}_{\nu,\boldsymbol{k},j}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}_{lj}^0} + \hat{a}_{\nu,\boldsymbol{k}}^\dagger\, \boldsymbol{\epsilon}_{\nu,\boldsymbol{k},j}^*\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}_{lj}^0} \right)$$

DM-ion scattering potential

$$\begin{aligned}
\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q}, \boldsymbol{v}) = \sum_{\psi=p,n,e} & c_1^{(\psi)} \langle N_\psi \rangle_{lj} \\
& + c_3^{(\psi)} \left[-\frac{iq}{m_\psi} \boldsymbol{v}' \cdot \left(\hat{\boldsymbol{q}} \times \langle \boldsymbol{S}_\psi \rangle_{lj} \right) + \frac{q^2}{2m_\psi^2} \left(\delta^{ik} - \hat{q}^i \hat{q}^k \right) \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj} \right)^{ik} \right] \\
& + c_4^{(\psi)} \boldsymbol{S}_\chi \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} \\
& + c_5^{(\psi)} \left[\frac{i\boldsymbol{q}}{m_\psi} \cdot \left(\boldsymbol{v}' \times \boldsymbol{S}_\chi \right) \langle N_\psi \rangle_{lj} + \frac{q^2}{2m_\psi^2} \boldsymbol{S}_\chi \cdot \left(\mathbb{1} - \hat{\boldsymbol{q}}\hat{\boldsymbol{q}} \right) \cdot \langle \boldsymbol{L}_\psi \rangle_{lj} \right] \\
& + c_6^{(\psi)} \frac{q^2}{m_\psi^2} \left(\hat{\boldsymbol{q}} \cdot \boldsymbol{S}_\chi \right) \left(\hat{\boldsymbol{q}} \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} \right) \\
& + c_7^{(\psi)} \left[\boldsymbol{v}' \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} + \epsilon^{ikk'} \frac{iq^{k'}}{2m_\chi} \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj} \right)^{ik} \right] \\
& + c_8^{(\psi)} \left[\left(\boldsymbol{v}' \cdot \boldsymbol{S}_\chi \right) \langle N_\psi \rangle_{lj} + \frac{iq}{2m_\psi} \boldsymbol{S}_\chi \cdot \left(\hat{\boldsymbol{q}} \times \langle \boldsymbol{L}_\psi \rangle_{lj} \right) \right] \\
& + c_9^{(\psi)} \frac{iq}{m_\psi} \boldsymbol{S}_\chi \cdot \left(\langle \boldsymbol{S}_\psi \rangle_{lj} \times \hat{\boldsymbol{q}} \right)
\end{aligned}$$

rate formula

contains DM physics (operator coefficients)

$$\Gamma(\boldsymbol{v}) = \frac{1}{\Omega} \int \frac{d^3 q}{(2\pi)^3} \sum_{\nu=1}^{3n} 2\pi\, \delta\left(\omega_{\nu,\boldsymbol{k}} - \omega_{\boldsymbol{q}}\right) \frac{1}{2\omega_{\nu,\boldsymbol{k}}} \left| \sum_j e^{-W_j(\boldsymbol{q})} e^{i\boldsymbol{G}\cdot\boldsymbol{x}_j^0} \frac{\boldsymbol{q}\cdot\boldsymbol{\epsilon}_{\nu,\boldsymbol{k},j}^*}{\sqrt{m_j}} \boxed{\widetilde{\mathcal{V}}_j(-\boldsymbol{q}, \boldsymbol{v})} \right|^2$$

Step 3: quantizing lattice potential for phonons & magnons

- Upshot of the calculation.

$$\boxed{\langle \nu, \boldsymbol{k}|}\widetilde{\mathcal{V}}(-\boldsymbol{q}, \boldsymbol{v})|0\rangle = \sum_{l,j} \langle \nu, \boldsymbol{k}|\boxed{e^{i\boldsymbol{q}\cdot\boldsymbol{x}_{lj}}}\boxed{\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q}, \boldsymbol{v})}|0\rangle$$

single phonon/magnon states

phonons

Leading dependence on lattice displacements comes from

$$\boldsymbol{u}_{lj} = \boldsymbol{x}_{lj} - \boldsymbol{x}_{lj}^0 = \sum_{\nu=1}^{3n} \sum_{\boldsymbol{k}\in 1\mathrm{BZ}} \frac{1}{\sqrt{2N m_j \omega_{\nu,\boldsymbol{k}}}} \left(\hat{a}_{\nu,\boldsymbol{k}}\, \boldsymbol{\epsilon}_{\nu,\boldsymbol{k},j}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}_{lj}^0} + \hat{a}_{\nu,\boldsymbol{k}}^\dagger\, \boldsymbol{\epsilon}_{\nu,\boldsymbol{k},j}^*\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}_{lj}^0} \right)$$

DM-ion scattering potential

$$\begin{aligned}
\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q}, \boldsymbol{v}) = \sum_{\psi=p,n,e} & c_1^{(\psi)} \langle N_\psi \rangle_{lj} \\
& + c_3^{(\psi)} \left[-\frac{iq}{m_\psi} \boldsymbol{v}' \cdot (\hat{\boldsymbol{q}} \times \langle \boldsymbol{S}_\psi \rangle_{lj}) + \frac{q^2}{2m_\psi^2} (\delta^{ik} - \hat{q}^i \hat{q}^k) \big(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj} \big)^{ik} \right] \\
& + c_4^{(\psi)} \boldsymbol{S}_\chi \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} \\
& + c_5^{(\psi)} \left[\frac{i\boldsymbol{q}}{m_\psi} \cdot (\boldsymbol{v}' \times \boldsymbol{S}_\chi) \langle N_\psi \rangle_{lj} + \frac{q^2}{2m_\psi^2} \boldsymbol{S}_\chi \cdot (\mathbb{1} - \hat{\boldsymbol{q}}\hat{\boldsymbol{q}}) \cdot \langle \boldsymbol{L}_\psi \rangle_{lj} \right] \\
& + c_6^{(\psi)} \frac{q^2}{m_\psi^2} (\hat{\boldsymbol{q}} \cdot \boldsymbol{S}_\chi)(\hat{\boldsymbol{q}} \cdot \langle \boldsymbol{S}_\psi \rangle_{lj}) \\
& + c_7^{(\psi)} \left[\boldsymbol{v}' \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} + \epsilon^{ikk'} \frac{iq^{k'}}{2m_\chi} \big(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj} \big)^{ik} \right] \\
& + c_8^{(\psi)} \left[(\boldsymbol{v}' \cdot \boldsymbol{S}_\chi) \langle N_\psi \rangle_{lj} + \frac{iq}{2m_\psi} \boldsymbol{S}_\chi \cdot (\hat{\boldsymbol{q}} \times \langle \boldsymbol{L}_\psi \rangle_{lj}) \right] \\
& + c_9^{(\psi)} \frac{iq}{m_\psi} \boldsymbol{S}_\chi \cdot (\langle \boldsymbol{S}_\psi \rangle_{lj} \times \hat{\boldsymbol{q}})
\end{aligned}$$

rate formula

contains DM physics (operator coefficients)

$$\Gamma(\boldsymbol{v}) = \frac{1}{\Omega} \int \frac{d^3q}{(2\pi)^3} \boxed{\sum_{\nu=1}^{3n}} 2\pi\, \delta(\omega_{\nu,\boldsymbol{k}} - \omega_{\boldsymbol{q}}) \frac{1}{2\omega_{\nu,\boldsymbol{k}}} \left| \sum_j e^{-W_j(\boldsymbol{q})} e^{i\boldsymbol{G}\cdot\boldsymbol{x}_j^0} \frac{\boldsymbol{q}\cdot\boldsymbol{\epsilon}_{\nu,\boldsymbol{k},j}^*}{\sqrt{m_j}} \boxed{\widetilde{\mathcal{V}}_j(-\boldsymbol{q}, \boldsymbol{v})} \right|^2$$

sum over phonon branches

Step 3: quantizing lattice potential for phonons & magnons

➤ Upshot of the calculation.

$$\boxed{\langle \nu, \boldsymbol{k}|}\widetilde{\mathcal{V}}(-\boldsymbol{q}, \boldsymbol{v})|0\rangle = \sum_{l,j} \langle \nu, \boldsymbol{k}|\boxed{e^{i\boldsymbol{q}\cdot\boldsymbol{x}_{lj}}}\boxed{\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q}, \boldsymbol{v})}|0\rangle$$

single phonon/magnon states

phonons

Leading dependence on lattice displacements comes from

$$\boldsymbol{u}_{lj} = \boldsymbol{x}_{lj} - \boldsymbol{x}_{lj}^0 = \sum_{\nu=1}^{3n} \sum_{\boldsymbol{k}\in 1\mathrm{BZ}} \frac{1}{\sqrt{2Nm_j\omega_{\nu,\boldsymbol{k}}}} \left(\hat{a}_{\nu,\boldsymbol{k}}\, \boldsymbol{\epsilon}_{\nu,\boldsymbol{k},j}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}_{lj}^0} + \hat{a}_{\nu,\boldsymbol{k}}^\dagger\, \boldsymbol{\epsilon}_{\nu,\boldsymbol{k},j}^*\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}_{lj}^0} \right)$$

DM-ion scattering potential

$$\begin{aligned}
\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q}, \boldsymbol{v}) = \sum_{\psi=p,n,e} & c_1^{(\psi)} \langle N_\psi \rangle_{lj} \\
& + c_3^{(\psi)} \left[-\frac{iq}{m_\psi} \boldsymbol{v}' \cdot (\hat{\boldsymbol{q}} \times \langle \boldsymbol{S}_\psi \rangle_{lj}) + \frac{q^2}{2m_\psi^2} (\delta^{ik} - \hat{q}^i \hat{q}^k) \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj} \right)^{ik} \right] \\
& + c_4^{(\psi)} \boldsymbol{S}_\chi \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} \\
& + c_5^{(\psi)} \left[\frac{i\boldsymbol{q}}{m_\psi} \cdot (\boldsymbol{v}' \times \boldsymbol{S}_\chi) \langle N_\psi \rangle_{lj} + \frac{q^2}{2m_\psi^2} \boldsymbol{S}_\chi \cdot (\mathbb{1} - \hat{\boldsymbol{q}}\hat{\boldsymbol{q}}) \cdot \langle \boldsymbol{L}_\psi \rangle_{lj} \right] \\
& + c_6^{(\psi)} \frac{q^2}{m_\psi^2} (\hat{\boldsymbol{q}} \cdot \boldsymbol{S}_\chi)(\hat{\boldsymbol{q}} \cdot \langle \boldsymbol{S}_\psi \rangle_{lj}) \\
& + c_7^{(\psi)} \left[\boldsymbol{v}' \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} + \epsilon^{ikk'} \frac{iq^{k'}}{2m_\chi} \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj} \right)^{ik} \right] \\
& + c_8^{(\psi)} \left[(\boldsymbol{v}' \cdot \boldsymbol{S}_\chi) \langle N_\psi \rangle_{lj} + \frac{iq}{2m_\psi} \boldsymbol{S}_\chi \cdot (\hat{\boldsymbol{q}} \times \langle \boldsymbol{L}_\psi \rangle_{lj}) \right] \\
& + c_9^{(\psi)} \frac{iq}{m_\psi} \boldsymbol{S}_\chi \cdot (\langle \boldsymbol{S}_\psi \rangle_{lj} \times \hat{\boldsymbol{q}})
\end{aligned}$$

rate formula

contains DM physics (operator coefficients)

$$\Gamma(\boldsymbol{v}) = \frac{1}{\Omega} \int \frac{d^3q}{(2\pi)^3} \sum_{\nu=1}^{3n} 2\pi\, \delta(\omega_{\nu,\boldsymbol{k}} - \omega_{\boldsymbol{q}}) \frac{1}{2\omega_{\nu,\boldsymbol{k}}} \left| \boxed{\sum_j} e^{-W_j(\boldsymbol{q})} e^{i\boldsymbol{G}\cdot\boldsymbol{x}_j^0} \frac{\boldsymbol{q}\cdot\boldsymbol{\epsilon}_{\nu,\boldsymbol{k},j}^*}{\sqrt{m_j}} \boxed{\widetilde{\mathcal{V}}_j(-\boldsymbol{q}, \boldsymbol{v})} \right|^2$$

sum over individual ion amplitudes

Step 3: quantizing lattice potential for phonons & magnons

➤ Upshot of the calculation.

$$\langle \nu, \boldsymbol{k}| \widetilde{\mathcal{V}}(-\boldsymbol{q}, \boldsymbol{v})|0\rangle = \sum_{l,j} \langle \nu, \boldsymbol{k}| e^{i\boldsymbol{q}\cdot\boldsymbol{x}_{lj}} \widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q}, \boldsymbol{v})|0\rangle$$

single phonon/magnon states

phonons

Leading dependence on lattice displacements comes from

$$\boldsymbol{u}_{lj} = \boldsymbol{x}_{lj} - \boldsymbol{x}_{lj}^0 = \sum_{\nu=1}^{3n} \sum_{\boldsymbol{k}\in 1\mathrm{BZ}} \frac{1}{\sqrt{2N m_j \omega_{\nu,\boldsymbol{k}}}} \left(\hat{a}_{\nu,\boldsymbol{k}}\, \boldsymbol{\epsilon}_{\nu,\boldsymbol{k},j}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}_{lj}^0} + \hat{a}_{\nu,\boldsymbol{k}}^\dagger\, \boldsymbol{\epsilon}_{\nu,\boldsymbol{k},j}^*\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}_{lj}^0} \right)$$

DM-ion scattering potential

$$\begin{aligned}
\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q}, \boldsymbol{v}) = \sum_{\psi=p,n,e} & c_1^{(\psi)} \langle N_\psi \rangle_{lj} \\
& + c_3^{(\psi)} \left[-\frac{iq}{m_\psi} \boldsymbol{v}' \cdot (\hat{\boldsymbol{q}} \times \langle \boldsymbol{S}_\psi \rangle_{lj}) + \frac{q^2}{2m_\psi^2} (\delta^{ik} - \hat{q}^i \hat{q}^k) \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj} \right)^{ik} \right] \\
& + c_4^{(\psi)} \boldsymbol{S}_\chi \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} \\
& + c_5^{(\psi)} \left[\frac{i\boldsymbol{q}}{m_\psi} \cdot (\boldsymbol{v}' \times \boldsymbol{S}_\chi) \langle N_\psi \rangle_{lj} + \frac{q^2}{2m_\psi^2} \boldsymbol{S}_\chi \cdot (\mathbb{1} - \hat{\boldsymbol{q}}\hat{\boldsymbol{q}}) \cdot \langle \boldsymbol{L}_\psi \rangle_{lj} \right] \\
& + c_6^{(\psi)} \frac{q^2}{m_\psi^2} (\hat{\boldsymbol{q}} \cdot \boldsymbol{S}_\chi)(\hat{\boldsymbol{q}} \cdot \langle \boldsymbol{S}_\psi \rangle_{lj}) \\
& + c_7^{(\psi)} \left[\boldsymbol{v}' \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} + \epsilon^{ikk'} \frac{iq^{k'}}{2m_\chi} \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj} \right)^{ik} \right] \\
& + c_8^{(\psi)} \left[(\boldsymbol{v}' \cdot \boldsymbol{S}_\chi) \langle N_\psi \rangle_{lj} + \frac{iq}{2m_\psi} \boldsymbol{S}_\chi \cdot (\hat{\boldsymbol{q}} \times \langle \boldsymbol{L}_\psi \rangle_{lj}) \right] \\
& + c_9^{(\psi)} \frac{iq}{m_\psi} \boldsymbol{S}_\chi \cdot (\langle \boldsymbol{S}_\psi \rangle_{lj} \times \hat{\boldsymbol{q}})
\end{aligned}$$

rate formula

contains DM physics (operator coefficients)

$$\Gamma(\boldsymbol{v}) = \frac{1}{\Omega} \int \frac{d^3 q}{(2\pi)^3} \sum_{\nu=1}^{3n} 2\pi\, \delta(\omega_{\nu,\boldsymbol{k}} - \omega_{\boldsymbol{q}}) \frac{1}{2\omega_{\nu,\boldsymbol{k}}} \left| \sum_j e^{-W_j(\boldsymbol{q})} e^{i\boldsymbol{G}\cdot\boldsymbol{x}_j^0} \frac{\boldsymbol{q}\cdot\boldsymbol{\epsilon}_{\nu,\boldsymbol{k},j}^*}{\sqrt{m_j}} \widetilde{\mathcal{V}}_j(-\boldsymbol{q}, \boldsymbol{v}) \right|^2$$

phonon eigenenergies & eigenvectors

Step 3: quantizing lattice potential for phonons & magnons

➤ Upshot of the calculation.

$$\langle \nu, \boldsymbol{k} | \widetilde{\mathcal{V}}(-\boldsymbol{q}, \boldsymbol{v}) | 0 \rangle = \sum_{l,j} \langle \nu, \boldsymbol{k} | e^{i\boldsymbol{q}\cdot\boldsymbol{x}_{lj}} \widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q}, \boldsymbol{v}) | 0 \rangle$$

single phonon/magnon states

phonons

Leading dependence on lattice displacements comes from

$$\boldsymbol{u}_{lj} = \boldsymbol{x}_{lj} - \boldsymbol{x}^0_{lj} = \sum_{\nu=1}^{3n} \sum_{\boldsymbol{k}\in 1\mathrm{BZ}} \frac{1}{\sqrt{2N m_j \omega_{\nu,\boldsymbol{k}}}} \left(\hat{a}_{\nu,\boldsymbol{k}}\, \boldsymbol{\epsilon}_{\nu,\boldsymbol{k},j}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}^0_{lj}} + \hat{a}^\dagger_{\nu,\boldsymbol{k}}\, \boldsymbol{\epsilon}^*_{\nu,\boldsymbol{k},j}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}^0_{lj}} \right)$$

rate formula

contains DM physics (operator coefficients)

$$\Gamma(\boldsymbol{v}) = \frac{1}{\Omega} \int \frac{d^3q}{(2\pi)^3} \sum_{\nu=1}^{3n} 2\pi\, \delta\left(\omega_{\nu,\boldsymbol{k}} - \omega_{\boldsymbol{q}}\right) \frac{1}{2\omega_{\nu,\boldsymbol{k}}} \left| \sum_j e^{-W_j(\boldsymbol{q})} e^{i\boldsymbol{G}\cdot\boldsymbol{x}^0_j} \frac{\boldsymbol{q}\cdot\boldsymbol{\epsilon}^*_{\nu,\boldsymbol{k},j}}{\sqrt{m_j}} \widetilde{\mathcal{V}}_j(-\boldsymbol{q}, \boldsymbol{v}) \right|^2$$

reminiscent of a harmonic oscillator

DM-ion scattering potential

$$\begin{aligned}
\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q}, \boldsymbol{v}) = \sum_{\psi=p,n,e} & c_1^{(\psi)} \langle N_\psi \rangle_{lj} \\
& + c_3^{(\psi)} \left[-\frac{iq}{m_\psi} \boldsymbol{v}' \cdot \left(\hat{\boldsymbol{q}} \times \langle \boldsymbol{S}_\psi \rangle_{lj}\right) + \frac{q^2}{2m_\psi^2} \left(\delta^{ik} - \hat{q}^i \hat{q}^k\right) \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj}\right)^{ik} \right] \\
& + c_4^{(\psi)} \boldsymbol{S}_\chi \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} \\
& + c_5^{(\psi)} \left[\frac{i\boldsymbol{q}}{m_\psi} \cdot \left(\boldsymbol{v}' \times \boldsymbol{S}_\chi\right) \langle N_\psi \rangle_{lj} + \frac{q^2}{2m_\psi^2} \boldsymbol{S}_\chi \cdot \left(\mathbb{1} - \hat{\boldsymbol{q}}\hat{\boldsymbol{q}}\right) \cdot \langle \boldsymbol{L}_\psi \rangle_{lj} \right] \\
& + c_6^{(\psi)} \frac{q^2}{m_\psi^2} \left(\hat{\boldsymbol{q}} \cdot \boldsymbol{S}_\chi\right) \left(\hat{\boldsymbol{q}} \cdot \langle \boldsymbol{S}_\psi \rangle_{lj}\right) \\
& + c_7^{(\psi)} \left[\boldsymbol{v}' \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} + \epsilon^{ikk'} \frac{iq^{k'}}{2m_\chi} \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj}\right)^{ik} \right] \\
& + c_8^{(\psi)} \left[\left(\boldsymbol{v}' \cdot \boldsymbol{S}_\chi\right) \langle N_\psi \rangle_{lj} + \frac{iq}{2m_\psi} \boldsymbol{S}_\chi \cdot \left(\hat{\boldsymbol{q}} \times \langle \boldsymbol{L}_\psi \rangle_{lj}\right) \right] \\
& + c_9^{(\psi)} \frac{iq}{m_\psi} \boldsymbol{S}_\chi \cdot \left(\langle \boldsymbol{S}_\psi \rangle_{lj} \times \hat{\boldsymbol{q}}\right)
\end{aligned}$$

Step 3: quantizing lattice potential for phonons & magnons

➤ Upshot of the calculation.

$$\boxed{\langle \nu, \boldsymbol{k}|}\widetilde{\mathcal{V}}(-\boldsymbol{q}, \boldsymbol{v})|0\rangle = \sum_{l,j} \langle \nu, \boldsymbol{k}|\, e^{i\boldsymbol{q}\cdot\boldsymbol{x}_{lj}}\, \boxed{\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q}, \boldsymbol{v})}|0\rangle$$

single phonon/magnon states

magnons

Project onto ionic spins (material-specific):

$$\langle \boldsymbol{S}_e \rangle_{lj} \to \lambda_{S,j} \boldsymbol{S}_{lj}\,, \qquad \langle \boldsymbol{L}_e \rangle_{lj} \to \lambda_{L,j} \boldsymbol{S}_{lj}$$

Then expand in Holstein-Primakoff bosons:

$$S'^{x}_{lj} = \left(2S_j - \hat{a}^\dagger_{lj}\hat{a}_{lj}\right)^{1/2} \hat{a}_{lj}\,, \qquad S'^{y}_{lj} = \hat{a}^\dagger_{lj} \left(2S_j - \hat{a}^\dagger_{lj}\hat{a}_{lj}\right)^{1/2}, \qquad S'^{z}_{lj} = S_j - \hat{a}^\dagger_{lj}\hat{a}_{lj}\,.$$

➤ Lattice spin Hamiltonian taken from literature (usually derived from experiment or DFT calculations).

➤ Then solve eigensystem using our own code (based on Toth&Lake).

Toth, Lake, 1402.6069

DM-ion scattering potential

$$\begin{aligned}
\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q}, \boldsymbol{v}) = \sum_{\psi=p,n,e} & c_1^{(\psi)} \langle N_\psi \rangle_{lj} \\
& + c_3^{(\psi)} \left[-\frac{iq}{m_\psi} \boldsymbol{v}' \cdot \left(\hat{\boldsymbol{q}} \times \langle \boldsymbol{S}_\psi \rangle_{lj}\right) + \frac{q^2}{2m_\psi^2} \left(\delta^{ik} - \hat{q}^i \hat{q}^k\right) \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj}\right)^{ik} \right] \\
& + c_4^{(\psi)} \boldsymbol{S}_\chi \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} \\
& + c_5^{(\psi)} \left[\frac{i\boldsymbol{q}}{m_\psi} \cdot \left(\boldsymbol{v}' \times \boldsymbol{S}_\chi\right) \langle N_\psi \rangle_{lj} + \frac{q^2}{2m_\psi^2} \boldsymbol{S}_\chi \cdot \left(\mathbb{1} - \hat{\boldsymbol{q}}\hat{\boldsymbol{q}}\right) \cdot \langle \boldsymbol{L}_\psi \rangle_{lj} \right] \\
& + c_6^{(\psi)} \frac{q^2}{m_\psi^2} \left(\hat{\boldsymbol{q}} \cdot \boldsymbol{S}_\chi\right) \left(\hat{\boldsymbol{q}} \cdot \langle \boldsymbol{S}_\psi \rangle_{lj}\right) \\
& + c_7^{(\psi)} \left[\boldsymbol{v}' \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} + \epsilon^{ikk'} \frac{iq^{k'}}{2m_\chi} \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj}\right)^{ik} \right] \\
& + c_8^{(\psi)} \left[\left(\boldsymbol{v}' \cdot \boldsymbol{S}_\chi\right) \langle N_\psi \rangle_{lj} + \frac{iq}{2m_\psi} \boldsymbol{S}_\chi \cdot \left(\hat{\boldsymbol{q}} \times \langle \boldsymbol{L}_\psi \rangle_{lj}\right) \right] \\
& + c_9^{(\psi)} \frac{iq}{m_\psi} \boldsymbol{S}_\chi \cdot \left(\langle \boldsymbol{S}_\psi \rangle_{lj} \times \hat{\boldsymbol{q}}\right) \\
& + c_{10}^{(\psi)} \frac{i\boldsymbol{q}}{m_\psi} \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} \\
& + c_{11}^{(\psi)} \frac{i\boldsymbol{q}}{m_\psi} \cdot \boldsymbol{S}_\chi \langle N_\psi \rangle_{lj} \\
& + c_{12}^{(\psi)} \left[\left(\boldsymbol{v}' \times \boldsymbol{S}_\chi\right) \cdot \langle \boldsymbol{S}_\psi \rangle_{lj} + \frac{iq}{2m_\psi} \left(\left(\hat{\boldsymbol{q}} \cdot \boldsymbol{S}_\chi\right)\delta^{ik} - \hat{q}^k S_\chi^i\right) \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj}\right)^{ik} \right] \\
& + c_{13}^{(\psi)} \left[\frac{iq}{m_\psi} \left(\boldsymbol{v}' \cdot \boldsymbol{S}_\chi\right) \left(\hat{\boldsymbol{q}} \cdot \langle \boldsymbol{S}_\psi \rangle_{lj}\right) + \frac{q^2}{2m_\psi^2} \left(\hat{\boldsymbol{q}} \times \boldsymbol{S}_\chi\right) \cdot \langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj} \cdot \hat{\boldsymbol{q}} \right] \\
& + c_{14}^{(\psi)} \left[\frac{iq}{m_\psi} \left(\hat{\boldsymbol{q}} \cdot \boldsymbol{S}_\chi\right) \left(\boldsymbol{v}' \cdot \langle \boldsymbol{S}_\psi \rangle_{lj}\right) - \epsilon^{ikk'} \frac{q^2}{2m_\psi^2} \hat{q}^{k'} \left(\hat{\boldsymbol{q}} \cdot \boldsymbol{S}_\chi\right) \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj}\right)^{ik} \right] \\
& + c_{15}^{(\psi)} \left[-\frac{q^2}{m_\psi^2} \left(\hat{\boldsymbol{q}} \cdot \left(\boldsymbol{v}' \times \boldsymbol{S}_\chi\right)\right) \left(\hat{\boldsymbol{q}} \cdot \langle \boldsymbol{S}_\psi \rangle_{lj}\right) \right. \\
& \qquad \left. + \frac{iq^3}{2m_\psi^3} \boldsymbol{S}_\chi \cdot \left(\mathbb{1} - \hat{\boldsymbol{q}}\hat{\boldsymbol{q}}\right) \cdot \langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj} \cdot \hat{\boldsymbol{q}} \right],
\end{aligned}$$

Step 3: quantizing lattice potential for phonons & magnons

➤ Upshot of the calculation.

$$\boxed{\langle \nu, \mathbf{k}|}\widetilde{\mathcal{V}}(-\boldsymbol{q}, \boldsymbol{v})|0\rangle = \sum_{l,j} \langle \nu, \mathbf{k}|\, e^{i\boldsymbol{q}\cdot\boldsymbol{x}_{lj}}\, \boxed{\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q}, \boldsymbol{v})}|0\rangle$$

single phonon/magnon states

magnons

Project onto ionic spins (material-specific):

$$\langle \boldsymbol{S}_e\rangle_{lj} \to \lambda_{S,j}\boldsymbol{S}_{lj}\,, \qquad \langle \boldsymbol{L}_e\rangle_{lj} \to \lambda_{L,j}\boldsymbol{S}_{lj}$$

Then expand in Holstein-Primakoff bosons:

$$S'^{x}_{lj} = \left(2S_j - \hat{a}^\dagger_{lj}\hat{a}_{lj}\right)^{1/2}\hat{a}_{lj}\,, \qquad S'^{y}_{lj} = \hat{a}^\dagger_{lj}\left(2S_j - \hat{a}^\dagger_{lj}\hat{a}_{lj}\right)^{1/2}, \qquad S'^{z}_{lj} = S_j - \hat{a}^\dagger_{lj}\hat{a}_{lj}\,.$$

➤ Lattice spin Hamiltonian taken from literature (usually derived from experiment or DFT calculations).

➤ Then solve eigensystem using our own code (based on Toth&Lake).

Toth, Lake, 1402.6069

DM-ion scattering potential

$$\begin{aligned}
\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v}) = \sum_{\psi=p,n,e} & c_1^{(\psi)}\langle N_\psi\rangle_{lj} \\
& + c_3^{(\psi)}\left[-\frac{iq}{m_\psi}\boldsymbol{v}'\cdot\left(\hat{\boldsymbol{q}}\times\langle\boldsymbol{S}_\psi\rangle_{lj}\right) + \frac{q^2}{2m_\psi^2}\left(\delta^{ik}-\hat{q}^i\hat{q}^k\right)\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
& + c_4^{(\psi)}\,\boldsymbol{S}_\chi\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} \\
& + c_5^{(\psi)}\left[\frac{i\boldsymbol{q}}{m_\psi}\cdot\left(\boldsymbol{v}'\times\boldsymbol{S}_\chi\right)\langle N_\psi\rangle_{lj} + \frac{q^2}{2m_\psi^2}\boldsymbol{S}_\chi\cdot\left(\mathbb{1}-\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}\right)\cdot\langle\boldsymbol{L}_\psi\rangle_{lj}\right] \\
& + c_6^{(\psi)}\frac{q^2}{m_\psi^2}\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\left(\hat{\boldsymbol{q}}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right) \\
& + c_7^{(\psi)}\left[\boldsymbol{v}'\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} + \epsilon^{ikk'}\frac{iq^{k'}}{2m_\chi}\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
& + c_8^{(\psi)}\left[\left(\boldsymbol{v}'\cdot\boldsymbol{S}_\chi\right)\langle N_\psi\rangle_{lj} + \frac{iq}{2m_\psi}\boldsymbol{S}_\chi\cdot\left(\hat{\boldsymbol{q}}\times\langle\boldsymbol{L}_\psi\rangle_{lj}\right)\right] \\
& + c_9^{(\psi)}\frac{iq}{m_\psi}\boldsymbol{S}_\chi\cdot\left(\langle\boldsymbol{S}_\psi\rangle_{lj}\times\hat{\boldsymbol{q}}\right) \\
& + c_{10}^{(\psi)}\frac{i\boldsymbol{q}}{m_\psi}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} \\
& + c_{11}^{(\psi)}\frac{i\boldsymbol{q}}{m_\psi}\cdot\boldsymbol{S}_\chi\langle N_\psi\rangle_{lj} \\
& + c_{12}^{(\psi)}\left[\left(\boldsymbol{v}'\times\boldsymbol{S}_\chi\right)\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} + \frac{iq}{2m_\psi}\left(\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\delta^{ik}-\hat{q}^k S_\chi^i\right)\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
& + c_{13}^{(\psi)}\left[\frac{iq}{m_\psi}\left(\boldsymbol{v}'\cdot\boldsymbol{S}_\chi\right)\left(\hat{\boldsymbol{q}}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right) + \frac{q^2}{2m_\psi^2}\left(\hat{\boldsymbol{q}}\times\boldsymbol{S}_\chi\right)\cdot\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\cdot\hat{\boldsymbol{q}}\right] \\
& + c_{14}^{(\psi)}\left[\frac{iq}{m_\psi}\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\left(\boldsymbol{v}'\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right) - \epsilon^{ikk'}\frac{q^2}{2m_\psi^2}\hat{q}^{k'}\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
& + c_{15}^{(\psi)}\left[-\frac{q^2}{m_\psi^2}\left(\hat{\boldsymbol{q}}\cdot\left(\boldsymbol{v}'\times\boldsymbol{S}_\chi\right)\right)\left(\hat{\boldsymbol{q}}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right)\right. \\
& \qquad \left. + \frac{iq^3}{2m_\psi^3}\boldsymbol{S}_\chi\cdot\left(\mathbb{1}-\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}\right)\cdot\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\cdot\hat{\boldsymbol{q}}\right],
\end{aligned}$$

Step 3: quantizing lattice potential for phonons & magnons

➤ Upshot of the calculation.

$$\boxed{\langle \nu, \boldsymbol{k}|}\widetilde{\mathcal{V}}(-\boldsymbol{q}, \boldsymbol{v})|0\rangle = \sum_{l,j} \langle \nu, \boldsymbol{k}|\, e^{i\boldsymbol{q}\cdot\boldsymbol{x}_{lj}} \boxed{\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q}, \boldsymbol{v})}|0\rangle$$

single phonon/magnon states

magnons

Project onto ionic spins (material-specific):

$$\langle \boldsymbol{S}_e\rangle_{lj} \to \lambda_{S,j} \boldsymbol{S}_{lj}\,, \qquad \langle \boldsymbol{L}_e\rangle_{lj} \to \lambda_{L,j} \boldsymbol{S}_{lj}$$

Then expand in Holstein-Primakoff bosons:

DM-ion scattering potential

$$\begin{aligned}\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q}, \boldsymbol{v}) = \sum_{\psi=p,n,e} & c_1^{(\psi)} \langle N_\psi\rangle_{lj} \\ & + c_3^{(\psi)} \left[-\frac{iq}{m_\psi} \boldsymbol{v}' \cdot \left(\hat{\boldsymbol{q}} \times \boxed{\langle \boldsymbol{S}_\psi\rangle_{lj}}\right) + \frac{q^2}{2m_\psi^2} (\delta^{ik} - \hat{q}^i \hat{q}^k) \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi\rangle_{lj}\right)^{ik} \right] \\ & + c_4^{(\psi)} \boldsymbol{S}_\chi \cdot \boxed{\langle \boldsymbol{S}_\psi\rangle_{lj}} \\ & + c_5^{(\psi)} \left[\frac{i\boldsymbol{q}}{m_\psi} \cdot (\boldsymbol{v}' \times \boldsymbol{S}_\chi) \langle N_\psi\rangle_{lj} + \frac{q^2}{2m_\psi^2} \boldsymbol{S}_\chi \cdot (\mathbb{1} - \hat{\boldsymbol{q}}\hat{\boldsymbol{q}}) \cdot \boxed{\langle \boldsymbol{L}_\psi\rangle_{lj}} \right] \\ & + c_6^{(\psi)} \frac{q^2}{m_\psi^2} (\hat{\boldsymbol{q}} \cdot \boldsymbol{S}_\chi) \left(\hat{\boldsymbol{q}} \cdot \boxed{\langle \boldsymbol{S}_\psi\rangle_{lj}}\right) \\ & + c_7^{(\psi)} \left[\boldsymbol{v}' \cdot \boxed{\langle \boldsymbol{S}_\psi\rangle_{lj}} + \epsilon^{ikk'} \frac{iq^{k'}}{2m_\chi} \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi\rangle_{lj}\right)^{ik} \right] \\ & + c_8^{(\psi)} \left[(\boldsymbol{v}' \cdot \boldsymbol{S}_\chi) \langle N_\psi\rangle_{lj} + \frac{iq}{2m_\psi} \boldsymbol{S}_\chi \cdot \left(\hat{\boldsymbol{q}} \times \boxed{\langle \boldsymbol{L}_\psi\rangle_{lj}}\right) \right] \\ & + c_9^{(\psi)} \frac{iq}{m_\psi} \boldsymbol{S}_\chi \cdot \left(\boxed{\langle \boldsymbol{S}_\psi\rangle_{lj}} \times \hat{\boldsymbol{q}}\right)\end{aligned}$$

rate formula

contains DM physics (operator coefficients)

$$\Gamma(\boldsymbol{v}) = \frac{1}{\Omega} \int \frac{d^3q}{(2\pi)^3} \sum_{\nu=1}^{n} 2\pi\, \delta\left(\omega_{\nu,\boldsymbol{k}} - \omega_{\boldsymbol{q}}\right) \frac{1}{2} \left| \sum_j e^{i\boldsymbol{G}\cdot\boldsymbol{x}_j^0} \sqrt{S_j} \left(\mathbb{U}^*_{j\nu,\boldsymbol{k}} \boldsymbol{r}_j + \mathbb{V}_{j\nu,-\boldsymbol{k}} \boldsymbol{r}_j^*\right) \cdot \boxed{\boldsymbol{f}_j(-\boldsymbol{q}, \boldsymbol{v})} \right|^2$$

coefficients of $\langle \boldsymbol{S}_e\rangle$ and $\langle \boldsymbol{L}_e\rangle$ in the potential

Step 3: quantizing lattice potential for phonons & magnons

➤ Upshot of the calculation.

$$\boxed{\langle \nu, \boldsymbol{k}|}\widetilde{\mathcal{V}}(-\boldsymbol{q}, \boldsymbol{v})|0\rangle = \sum_{l,j} \langle \nu, \boldsymbol{k}|\, e^{i\boldsymbol{q}\cdot\boldsymbol{x}_{lj}} \boxed{\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q}, \boldsymbol{v})}|0\rangle$$

single phonon/magnon states

magnons

Project onto ionic spins (material-specific):

$$\langle \boldsymbol{S}_e \rangle_{lj} \to \lambda_{S,j} \boldsymbol{S}_{lj}\,, \qquad \langle \boldsymbol{L}_e \rangle_{lj} \to \lambda_{L,j} \boldsymbol{S}_{lj}$$

Then expand in Holstein-Primakoff bosons:

DM-ion scattering potential

$$\begin{aligned}
\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q}, \boldsymbol{v}) = \sum_{\psi=p,n,e} & c_1^{(\psi)} \langle N_\psi \rangle_{lj} \\
& + c_3^{(\psi)} \left[-\frac{iq}{m_\psi} \boldsymbol{v}' \cdot \left(\hat{\boldsymbol{q}} \times \boxed{\langle \boldsymbol{S}_\psi \rangle_{lj}}\right) + \frac{q^2}{2m_\psi^2} (\delta^{ik} - \hat{q}^i \hat{q}^k) \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj}\right)^{ik} \right] \\
& + c_4^{(\psi)} \boldsymbol{S}_\chi \cdot \boxed{\langle \boldsymbol{S}_\psi \rangle_{lj}} \\
& + c_5^{(\psi)} \left[\frac{i\boldsymbol{q}}{m_\psi} \cdot (\boldsymbol{v}' \times \boldsymbol{S}_\chi) \langle N_\psi \rangle_{lj} + \frac{q^2}{2m_\psi^2} \boldsymbol{S}_\chi \cdot (\mathbb{1} - \hat{\boldsymbol{q}}\hat{\boldsymbol{q}}) \cdot \boxed{\langle \boldsymbol{L}_\psi \rangle_{lj}} \right] \\
& + c_6^{(\psi)} \frac{q^2}{m_\psi^2} (\hat{\boldsymbol{q}} \cdot \boldsymbol{S}_\chi)\left(\hat{\boldsymbol{q}} \cdot \boxed{\langle \boldsymbol{S}_\psi \rangle_{lj}}\right) \\
& + c_7^{(\psi)} \left[\boldsymbol{v}' \cdot \boxed{\langle \boldsymbol{S}_\psi \rangle_{lj}} + \epsilon^{ikk'} \frac{iq^{k'}}{2m_\chi} \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi \rangle_{lj}\right)^{ik} \right] \\
& + c_8^{(\psi)} \left[(\boldsymbol{v}' \cdot \boldsymbol{S}_\chi) \langle N_\psi \rangle_{lj} + \frac{iq}{2m_\psi} \boldsymbol{S}_\chi \cdot \left(\hat{\boldsymbol{q}} \times \boxed{\langle \boldsymbol{L}_\psi \rangle_{lj}}\right) \right] \\
& + c_9^{(\psi)} \frac{iq}{m_\psi} \boldsymbol{S}_\chi \cdot \left(\boxed{\langle \boldsymbol{S}_\psi \rangle_{lj}} \times \hat{\boldsymbol{q}}\right) \\
& + \ldots
\end{aligned}$$

rate formula

contains DM physics (operator coefficients)

$$\Gamma(\boldsymbol{v}) = \frac{1}{\Omega} \int \frac{d^3q}{(2\pi)^3} \boxed{\sum_{\nu=1}^{n}} 2\pi\, \delta(\omega_{\nu,\boldsymbol{k}} - \omega_{\boldsymbol{q}}) \, \frac{1}{2} \left| \sum_j e^{i\boldsymbol{G}\cdot\boldsymbol{x}_j^0} \sqrt{S_j}\, \left(\mathbb{U}^*_{j\nu,\boldsymbol{k}} \boldsymbol{r}_j + \mathbb{V}_{j\nu,-\boldsymbol{k}} \boldsymbol{r}_j^*\right) \cdot \boxed{\boldsymbol{f}_j(-\boldsymbol{q}, \boldsymbol{v})} \right|^2$$

coefficients of $\langle \boldsymbol{S}_e \rangle$ and $\langle \boldsymbol{L}_e \rangle$ in the potential

sum over magnon branches

Step 3: quantizing lattice potential for phonons & magnons

➤ Upshot of the calculation.

$$\langle \nu, \boldsymbol{k}| \widetilde{\mathcal{V}}(-\boldsymbol{q}, \boldsymbol{v})|0\rangle = \sum_{l,j} \langle \nu, \boldsymbol{k}|\, e^{i\boldsymbol{q}\cdot\boldsymbol{x}_{lj}}\, \widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q}, \boldsymbol{v})|0\rangle$$

single phonon/magnon states

DM-ion scattering potential

$$\begin{aligned}\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q}, \boldsymbol{v}) = \sum_{\psi=p,n,e} & c_1^{(\psi)} \langle N_\psi\rangle_{lj} \\ & + c_3^{(\psi)} \left[-\frac{iq}{m_\psi} \boldsymbol{v}' \cdot \left(\hat{\boldsymbol{q}} \times \langle \boldsymbol{S}_\psi\rangle_{lj}\right) + \frac{q^2}{2m_\psi^2} (\delta^{ik} - \hat{q}^i\hat{q}^k)\left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\ & + c_4^{(\psi)} \boldsymbol{S}_\chi \cdot \langle \boldsymbol{S}_\psi\rangle_{lj} \\ & + c_5^{(\psi)} \left[\frac{i\boldsymbol{q}}{m_\psi} \cdot (\boldsymbol{v}' \times \boldsymbol{S}_\chi)\langle N_\psi\rangle_{lj} + \frac{q^2}{2m_\psi^2} \boldsymbol{S}_\chi \cdot (\mathbb{1} - \hat{\boldsymbol{q}}\hat{\boldsymbol{q}}) \cdot \langle \boldsymbol{L}_\psi\rangle_{lj}\right] \\ & + c_6^{(\psi)} \frac{q^2}{m_\psi^2} (\hat{\boldsymbol{q}} \cdot \boldsymbol{S}_\chi)\left(\hat{\boldsymbol{q}} \cdot \langle \boldsymbol{S}_\psi\rangle_{lj}\right) \\ & + c_7^{(\psi)} \left[\boldsymbol{v}' \cdot \langle \boldsymbol{S}_\psi\rangle_{lj} + \epsilon^{ikk'} \frac{iq^{k'}}{2m_\chi} \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\ & + c_8^{(\psi)} \left[(\boldsymbol{v}' \cdot \boldsymbol{S}_\chi)\langle N_\psi\rangle_{lj} + \frac{iq}{2m_\psi} \boldsymbol{S}_\chi \cdot \left(\hat{\boldsymbol{q}} \times \langle \boldsymbol{L}_\psi\rangle_{lj}\right)\right] \\ & + c_9^{(\psi)} \frac{iq}{m_\psi} \boldsymbol{S}_\chi \cdot \left(\langle \boldsymbol{S}_\psi\rangle_{lj} \times \hat{\boldsymbol{q}}\right)\end{aligned}$$

magnons

Project onto ionic spins (material-specific):

$$\langle \boldsymbol{S}_e\rangle_{lj} \to \lambda_{S,j} \boldsymbol{S}_{lj}\,, \qquad \langle \boldsymbol{L}_e\rangle_{lj} \to \lambda_{L,j} \boldsymbol{S}_{lj}$$

Then expand in Holstein-Primakoff bosons:

rate formula

contains DM physics (operator coefficients)

$$\Gamma(\boldsymbol{v}) = \frac{1}{\Omega} \int \frac{d^3q}{(2\pi)^3} \sum_{\nu=1}^{n} 2\pi\, \delta\left(\omega_{\nu,\boldsymbol{k}} - \omega_{\boldsymbol{q}}\right) \frac{1}{2} \left| \sum_j e^{i\boldsymbol{G}\cdot\boldsymbol{x}_j^0} \sqrt{S_j}\, \left(\mathbb{U}^*_{j\nu,\boldsymbol{k}} \boldsymbol{r}_j + \mathbb{V}_{j\nu,-\boldsymbol{k}} \boldsymbol{r}_j^*\right) \cdot \boldsymbol{f}_j(-\boldsymbol{q}, \boldsymbol{v}) \right|^2$$

coefficients of $\langle \boldsymbol{S}_e\rangle$ and $\langle \boldsymbol{L}_e\rangle$ in the potential

sum over individual ion amplitudes

Step 3: quantizing lattice potential for phonons & magnons

➤ Upshot of the calculation.

$$\langle \nu, \boldsymbol{k}|\widetilde{\mathcal{V}}(-\boldsymbol{q}, \boldsymbol{v})|0\rangle = \sum_{l,j} \langle \nu, \boldsymbol{k}|\, e^{i\boldsymbol{q}\cdot\boldsymbol{x}_{lj}}\, \widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q}, \boldsymbol{v})|0\rangle$$

single phonon/magnon states

DM-ion scattering potential

$$\begin{aligned}
\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q}, \boldsymbol{v}) = \sum_{\psi=p,n,e} & c_1^{(\psi)} \langle N_\psi\rangle_{lj} \\
& + c_3^{(\psi)} \left[-\frac{iq}{m_\psi} \boldsymbol{v}' \cdot \left(\hat{\boldsymbol{q}} \times \langle \boldsymbol{S}_\psi\rangle_{lj}\right) + \frac{q^2}{2m_\psi^2} \left(\delta^{ik} - \hat{q}^i \hat{q}^k\right) \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi\rangle_{lj}\right)^{ik} \right] \\
& + c_4^{(\psi)} \boldsymbol{S}_\chi \cdot \langle \boldsymbol{S}_\psi\rangle_{lj} \\
& + c_5^{(\psi)} \left[\frac{i\boldsymbol{q}}{m_\psi} \cdot \left(\boldsymbol{v}' \times \boldsymbol{S}_\chi\right) \langle N_\psi\rangle_{lj} + \frac{q^2}{2m_\psi^2} \boldsymbol{S}_\chi \cdot \left(\mathbb{1} - \hat{\boldsymbol{q}}\hat{\boldsymbol{q}}\right) \cdot \langle \boldsymbol{L}_\psi\rangle_{lj} \right] \\
& + c_6^{(\psi)} \frac{q^2}{m_\psi^2} \left(\hat{\boldsymbol{q}} \cdot \boldsymbol{S}_\chi\right) \left(\hat{\boldsymbol{q}} \cdot \langle \boldsymbol{S}_\psi\rangle_{lj}\right) \\
& + c_7^{(\psi)} \left[\boldsymbol{v}' \cdot \langle \boldsymbol{S}_\psi\rangle_{lj} + \epsilon^{ikk'} \frac{iq^{k'}}{2m_\chi} \left(\langle \boldsymbol{L}_\psi \otimes \boldsymbol{S}_\psi\rangle_{lj}\right)^{ik} \right] \\
& + c_8^{(\psi)} \left[\left(\boldsymbol{v}' \cdot \boldsymbol{S}_\chi\right) \langle N_\psi\rangle_{lj} + \frac{iq}{2m_\psi} \boldsymbol{S}_\chi \cdot \left(\hat{\boldsymbol{q}} \times \langle \boldsymbol{L}_\psi\rangle_{lj}\right) \right] \\
& + c_9^{(\psi)} \frac{iq}{m_\psi} \boldsymbol{S}_\chi \cdot \left(\langle \boldsymbol{S}_\psi\rangle_{lj} \times \hat{\boldsymbol{q}}\right)
\end{aligned}$$

magnons

Project onto ionic spins (material-specific):

$$\langle \boldsymbol{S}_e\rangle_{lj} \to \lambda_{S,j} \boldsymbol{S}_{lj}\,, \qquad \langle \boldsymbol{L}_e\rangle_{lj} \to \lambda_{L,j} \boldsymbol{S}_{lj}$$

Then expand in Holstein-Primakoff bosons:

rate formula

contains DM physics (operator coefficients)

$$\Gamma(\boldsymbol{v}) = \frac{1}{\Omega} \int \frac{d^3q}{(2\pi)^3} \sum_{\nu=1}^{n} 2\pi\, \delta\left(\omega_{\nu,\boldsymbol{k}} - \omega_{\boldsymbol{q}}\right) \frac{1}{2} \left| \sum_j e^{i\boldsymbol{G}\cdot\boldsymbol{x}_j^0} \sqrt{S_j} \left(\mathbb{U}^*_{j\nu,\boldsymbol{k}} \boldsymbol{r}_j + \mathbb{V}_{j\nu,-\boldsymbol{k}} \boldsymbol{r}_j^* \right) \cdot \boldsymbol{f}_j(-\boldsymbol{q}, \boldsymbol{v}) \right|^2$$

coefficients of $\langle \boldsymbol{S}_e\rangle$ and $\langle \boldsymbol{L}_e\rangle$ in the potential

rotation matrices to magnon eigenmodes

Step 3: quantizing lattice potential for phonons & magnons

- Upshot of the calculation.

$$\langle \nu, \boldsymbol{k}| \widetilde{\mathcal{V}}(-\boldsymbol{q}, \boldsymbol{v})|0\rangle = \sum_{l,j} \langle \nu, \boldsymbol{k}|\, e^{i\boldsymbol{q}\cdot\boldsymbol{x}_{lj}}\, \widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q}, \boldsymbol{v})|0\rangle$$

single phonon/magnon states

magnons

Project onto ionic spins (material-specific):

$$\langle \boldsymbol{S}_e\rangle_{lj} \to \lambda_{S,j}\boldsymbol{S}_{lj}\,, \qquad \langle \boldsymbol{L}_e\rangle_{lj} \to \lambda_{L,j}\boldsymbol{S}_{lj}$$

Then expand in Holstein-Primakoff bosons:

DM-ion scattering potential

$$\begin{aligned}\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q}, \boldsymbol{v}) = \sum_{\psi=p,n,e} & c_1^{(\psi)} \langle N_\psi\rangle_{lj} \\ & + c_3^{(\psi)}\left[-\frac{iq}{m_\psi}\boldsymbol{v}'\cdot\left(\hat{\boldsymbol{q}}\times\langle \boldsymbol{S}_\psi\rangle_{lj}\right) + \frac{q^2}{2m_\psi^2}\left(\delta^{ik}-\hat{q}^i\hat{q}^k\right)\left(\langle \boldsymbol{L}_\psi\otimes \boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\ & + c_4^{(\psi)}\,\boldsymbol{S}_\chi\cdot\langle \boldsymbol{S}_\psi\rangle_{lj} \\ & + c_5^{(\psi)}\left[\frac{i\boldsymbol{q}}{m_\psi}\cdot\left(\boldsymbol{v}'\times\boldsymbol{S}_\chi\right)\langle N_\psi\rangle_{lj} + \frac{q^2}{2m_\psi^2}\boldsymbol{S}_\chi\cdot\left(\mathbb{1}-\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}\right)\cdot\langle \boldsymbol{L}_\psi\rangle_{lj}\right] \\ & + c_6^{(\psi)}\frac{q^2}{m_\psi^2}\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\left(\hat{\boldsymbol{q}}\cdot\langle \boldsymbol{S}_\psi\rangle_{lj}\right) \\ & + c_7^{(\psi)}\left[\boldsymbol{v}'\cdot\langle \boldsymbol{S}_\psi\rangle_{lj} + \epsilon^{ikk'}\frac{iq^{k'}}{2m_\chi}\left(\langle \boldsymbol{L}_\psi\otimes \boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\ & + c_8^{(\psi)}\left[\left(\boldsymbol{v}'\cdot\boldsymbol{S}_\chi\right)\langle N_\psi\rangle_{lj} + \frac{iq}{2m_\psi}\boldsymbol{S}_\chi\cdot\left(\hat{\boldsymbol{q}}\times\langle \boldsymbol{L}_\psi\rangle_{lj}\right)\right] \\ & + c_9^{(\psi)}\frac{iq}{m_\psi}\boldsymbol{S}_\chi\cdot\left(\langle \boldsymbol{S}_\psi\rangle_{lj}\times\hat{\boldsymbol{q}}\right) \\ & + \dots\end{aligned}$$

rate formula

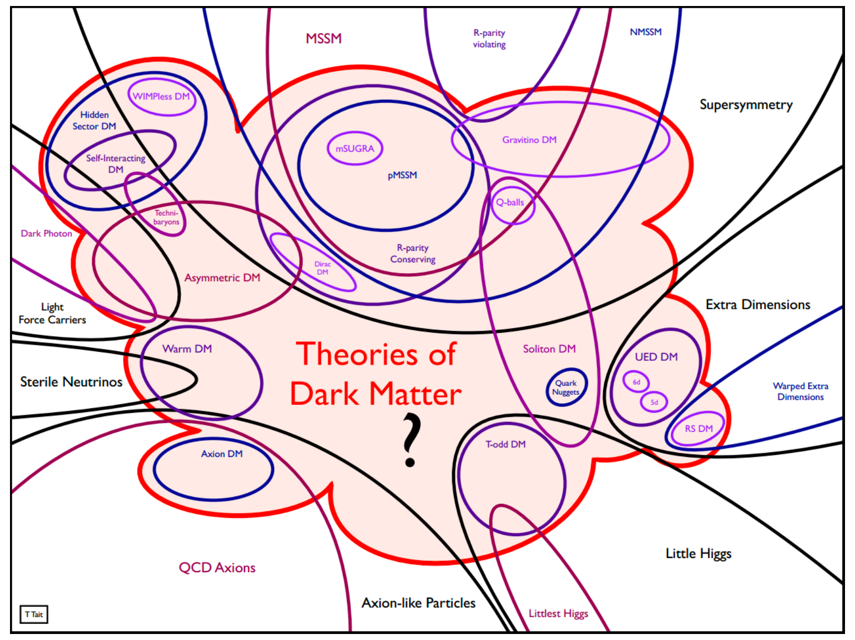
contains DM physics (operator coefficients)

$$\Gamma(\boldsymbol{v}) = \frac{1}{\Omega}\int\frac{d^3q}{(2\pi)^3}\sum_{\nu=1}^{n} 2\pi\,\delta\left(\omega_{\nu,\boldsymbol{k}}-\omega_{\boldsymbol{q}}\right)\frac{1}{2}\left|\sum_j e^{i\boldsymbol{G}\cdot\boldsymbol{x}_j^0}\sqrt{S_j}\left(\mathbb{U}_{j\nu,\boldsymbol{k}}^*\boldsymbol{r}_j + \mathbb{V}_{j\nu,-\boldsymbol{k}}\boldsymbol{r}_j^*\right)\cdot \boldsymbol{f}_j(-\boldsymbol{q}, \boldsymbol{v})\right|^2$$

coefficients of $\langle \boldsymbol{S}_e\rangle$ and $\langle \boldsymbol{L}_e\rangle$ in the potential

captures magnetic order

EFT of DM direct detection: preview



1

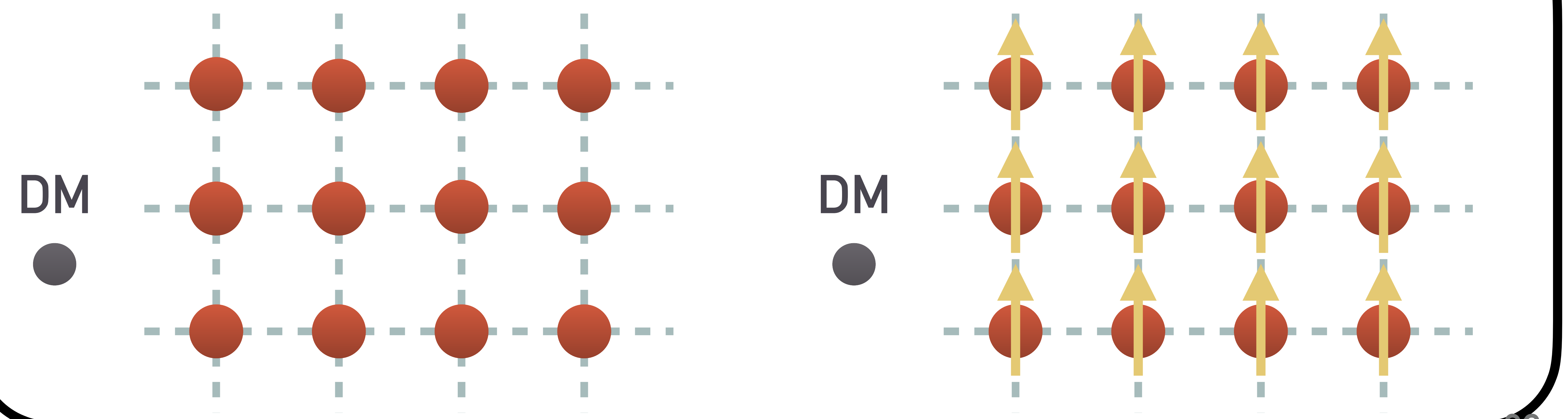
Crystal responses

2

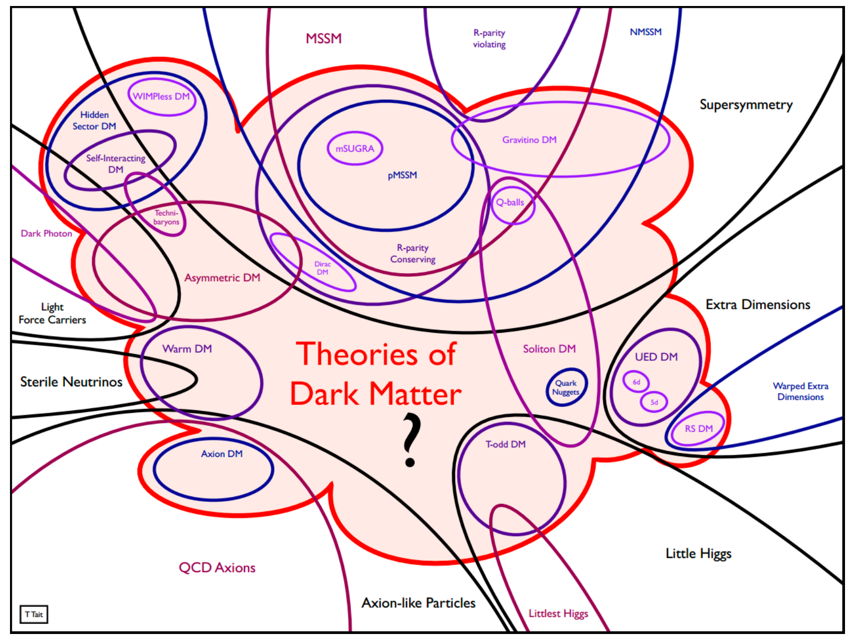
3

Nonrelativistic (NR) EFT of DM-SM interactions

Phonon & magnon excitation rates



EFT of DM direct detection: preview



1

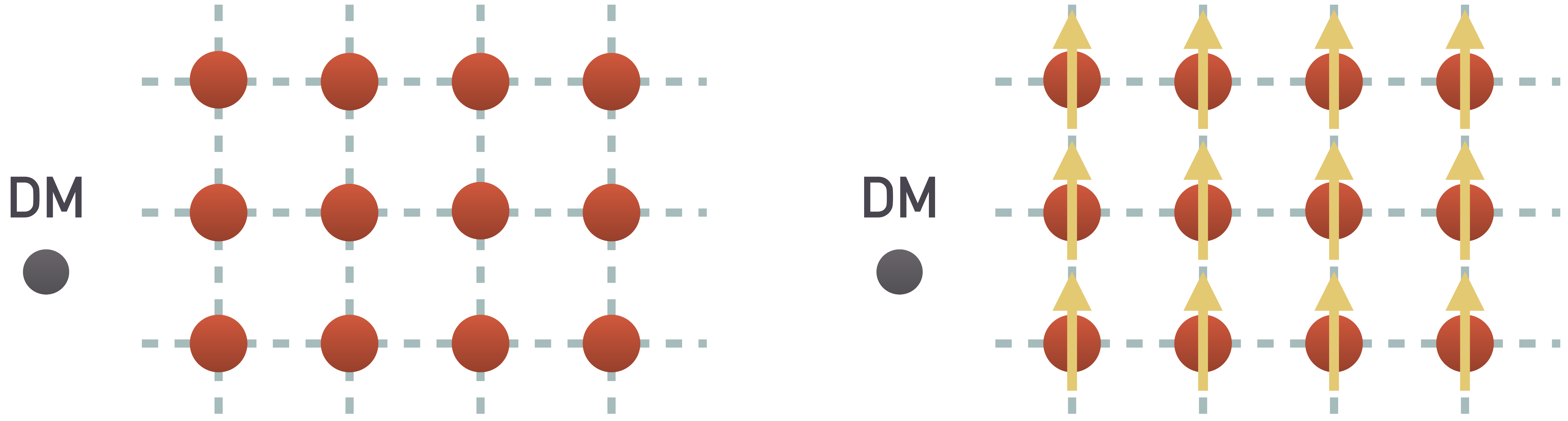
2

Crystal responses

3

Nonrelativistic (NR) EFT of DM-SM interactions

Phonon & magnon excitation rates



Example

- Dark photon mediator models.

Millicharged DM (SI)

$$c_1^{(\psi)} = -\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2+m_V^2}$$

Electric dipole DM

$$c_{11}^{(\psi)} = -\frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2+m_V^2}$$

Magnetic dipole DM

$$c_1^{(\psi)} = \frac{\boldsymbol{q}^2}{4m_\chi^2}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2+m_V^2}$$

$$c_4^{(\psi)} = \tilde{\mu}_\psi^{\rm eff}\frac{\boldsymbol{q}^2}{m_\chi m_\psi}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2+m_V^2}$$

$$c_5^{(\psi)} = \frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2+m_V^2}$$

$$c_6^{(\psi)} = -\tilde{\mu}_\psi^{\rm eff}\frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2+m_V^2}$$

Anapole DM

$$c_8^{(\psi)} = \frac{\boldsymbol{q}^2}{2m_\chi^2}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2+m_V^2}$$

$$c_9^{(\psi)} = -\tilde{\mu}_\psi^{\rm eff}\frac{\boldsymbol{q}^2}{2m_\chi^2}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2+m_V^2}$$

$$\begin{aligned}
\tilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v}) = \sum_{\psi=p,n,e} & c_1^{(\psi)}\langle N_\psi\rangle_{lj} \\
& + c_3^{(\psi)}\left[-\frac{iq}{m_\psi}\boldsymbol{v}'\cdot\left(\hat{\boldsymbol{q}}\times\langle\boldsymbol{S}_\psi\rangle_{lj}\right)+\frac{q^2}{2m_\psi^2}\left(\delta^{ik}-\hat{q}^i\hat{q}^k\right)\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
& + c_4^{(\psi)}\,\boldsymbol{S}_\chi\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} \\
& + c_5^{(\psi)}\left[\frac{i\boldsymbol{q}}{m_\psi}\cdot\left(\boldsymbol{v}'\times\boldsymbol{S}_\chi\right)\langle N_\psi\rangle_{lj}+\frac{q^2}{2m_\psi^2}\boldsymbol{S}_\chi\cdot\left(\mathbb{1}-\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}\right)\cdot\langle\boldsymbol{L}_\psi\rangle_{lj}\right] \\
& + c_6^{(\psi)}\frac{q^2}{m_\psi^2}\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\left(\hat{\boldsymbol{q}}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right) \\
& + c_7^{(\psi)}\left[\boldsymbol{v}'\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}+\epsilon^{ikk'}\frac{iq^{k'}}{2m_\chi}\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
& + c_8^{(\psi)}\left[\left(\boldsymbol{v}'\cdot\boldsymbol{S}_\chi\right)\langle N_\psi\rangle_{lj}+\frac{iq}{2m_\psi}\boldsymbol{S}_\chi\cdot\left(\hat{\boldsymbol{q}}\times\langle\boldsymbol{L}_\psi\rangle_{lj}\right)\right] \\
& + c_9^{(\psi)}\frac{iq}{m_\psi}\boldsymbol{S}_\chi\cdot\left(\langle\boldsymbol{S}_\psi\rangle_{lj}\times\hat{\boldsymbol{q}}\right) \\
& + c_{10}^{(\psi)}\frac{i\boldsymbol{q}}{m_\psi}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} \\
& + c_{11}^{(\psi)}\frac{i\boldsymbol{q}}{m_\psi}\cdot\boldsymbol{S}_\chi\langle N_\psi\rangle_{lj} \\
& + c_{12}^{(\psi)}\left[\left(\boldsymbol{v}'\times\boldsymbol{S}_\chi\right)\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}+\frac{iq}{2m_\psi}\left(\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\delta^{ik}-\hat{q}^kS_\chi^i\right)\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
& + c_{13}^{(\psi)}\left[\frac{iq}{m_\psi}\left(\boldsymbol{v}'\cdot\boldsymbol{S}_\chi\right)\left(\hat{\boldsymbol{q}}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right)+\frac{q^2}{2m_\psi^2}\left(\hat{\boldsymbol{q}}\times\boldsymbol{S}_\chi\right)\cdot\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\cdot\hat{\boldsymbol{q}}\right] \\
& + c_{14}^{(\psi)}\left[\frac{iq}{m_\psi}\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\left(\boldsymbol{v}'\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right)-\epsilon^{ikk'}\frac{q^2}{2m_\psi^2}\hat{q}^{k'}\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
& + c_{15}^{(\psi)}\left[-\frac{q^2}{m_\psi^2}\left(\hat{\boldsymbol{q}}\cdot\left(\boldsymbol{v}'\times\boldsymbol{S}_\chi\right)\right)\left(\hat{\boldsymbol{q}}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right)\right. \\
& \qquad \left.+\frac{iq^3}{2m_\psi^3}\boldsymbol{S}_\chi\cdot\left(\mathbb{1}-\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}\right)\cdot\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\cdot\hat{\boldsymbol{q}}\right],
\end{aligned}$$

Example

➤ Dark photon mediator models.

Millicharged DM (SI)

$$c_1^{(\psi)} = -\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

Electric dipole DM

$$c_{11}^{(\psi)} = -\frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

Magnetic dipole DM

$$c_1^{(\psi)} = \frac{\boldsymbol{q}^2}{4m_\chi^2}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_4^{(\psi)} = \widetilde{\mu}_\psi^{\text{eff}}\frac{\boldsymbol{q}^2}{m_\chi m_\psi}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_5^{(\psi)} = \frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_6^{(\psi)} = -\widetilde{\mu}_\psi^{\text{eff}}\frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

Anapole DM

$$c_8^{(\psi)} = \frac{\boldsymbol{q}^2}{2m_\chi^2}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_9^{(\psi)} = -\widetilde{\mu}_\psi^{\text{eff}}\frac{\boldsymbol{q}^2}{2m_\chi^2}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

$$\begin{aligned}
\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v}) = \sum_{\psi=p,n,e} & \boxed{c_1^{(\psi)}\langle N_\psi\rangle_{lj}} \\
&+ c_3^{(\psi)}\left[-\frac{iq}{m_\psi}\boldsymbol{v}'\cdot\left(\hat{\boldsymbol{q}}\times\langle\boldsymbol{S}_\psi\rangle_{lj}\right) + \frac{q^2}{2m_\psi^2}\left(\delta^{ik}-\hat{q}^i\hat{q}^k\right)\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
&+ c_4^{(\psi)}\,\boldsymbol{S}_\chi\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} \\
&+ c_5^{(\psi)}\left[\frac{i\boldsymbol{q}}{m_\psi}\cdot\left(\boldsymbol{v}'\times\boldsymbol{S}_\chi\right)\langle N_\psi\rangle_{lj} + \frac{q^2}{2m_\psi^2}\boldsymbol{S}_\chi\cdot\left(\mathbb{1}-\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}\right)\cdot\langle\boldsymbol{L}_\psi\rangle_{lj}\right] \\
&+ c_6^{(\psi)}\frac{q^2}{m_\psi^2}\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\left(\hat{\boldsymbol{q}}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right) \\
&+ c_7^{(\psi)}\left[\boldsymbol{v}'\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} + \epsilon^{ikk'}\frac{iq^{k'}}{2m_\chi}\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
&+ c_8^{(\psi)}\left[\left(\boldsymbol{v}'\cdot\boldsymbol{S}_\chi\right)\langle N_\psi\rangle_{lj} + \frac{iq}{2m_\psi}\boldsymbol{S}_\chi\cdot\left(\hat{\boldsymbol{q}}\times\langle\boldsymbol{L}_\psi\rangle_{lj}\right)\right] \\
&+ c_9^{(\psi)}\frac{iq}{m_\psi}\boldsymbol{S}_\chi\cdot\left(\langle\boldsymbol{S}_\psi\rangle_{lj}\times\hat{\boldsymbol{q}}\right) \\
&+ c_{10}^{(\psi)}\frac{i\boldsymbol{q}}{m_\psi}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} \\
&\boxed{+ c_{11}^{(\psi)}\frac{i\boldsymbol{q}}{m_\psi}\cdot\boldsymbol{S}_\chi\langle N_\psi\rangle_{lj}} \\
&+ c_{12}^{(\psi)}\left[\left(\boldsymbol{v}'\times\boldsymbol{S}_\chi\right)\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} + \frac{iq}{2m_\psi}\left(\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\delta^{ik}-\hat{q}^k S_\chi^i\right)\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
&+ c_{13}^{(\psi)}\left[\frac{iq}{m_\psi}\left(\boldsymbol{v}'\cdot\boldsymbol{S}_\chi\right)\left(\hat{\boldsymbol{q}}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right) + \frac{q^2}{2m_\psi^2}\left(\hat{\boldsymbol{q}}\times\boldsymbol{S}_\chi\right)\cdot\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\cdot\hat{\boldsymbol{q}}\right] \\
&+ c_{14}^{(\psi)}\left[\frac{iq}{m_\psi}\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\left(\boldsymbol{v}'\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right) - \epsilon^{ikk'}\frac{q^2}{2m_\psi^2}\hat{q}^{k'}\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
&+ c_{15}^{(\psi)}\left[-\frac{q^2}{m_\psi^2}\left(\hat{\boldsymbol{q}}\cdot\left(\boldsymbol{v}'\times\boldsymbol{S}_\chi\right)\right)\left(\hat{\boldsymbol{q}}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right)\right. \\
&\qquad\left. + \frac{iq^3}{2m_\psi^3}\boldsymbol{S}_\chi\cdot\left(\mathbb{1}-\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}\right)\cdot\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\cdot\hat{\boldsymbol{q}}\right],
\end{aligned}$$

Example

- Dark photon mediator models.

Millicharged DM (SI)

$$c_1^{(\psi)} = -\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

Electric dipole DM

$$c_{11}^{(\psi)} = -\frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

N response only $\Rightarrow$ probed by phonons.

Magnetic dipole DM

$$c_1^{(\psi)} = \frac{\boldsymbol{q}^2}{4m_\chi^2}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_4^{(\psi)} = \tilde{\mu}_\psi^{\text{eff}}\frac{\boldsymbol{q}^2}{m_\chi m_\psi}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_5^{(\psi)} = \frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_6^{(\psi)} = -\tilde{\mu}_\psi^{\text{eff}}\frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

Anapole DM

$$c_8^{(\psi)} = \frac{\boldsymbol{q}^2}{2m_\chi^2}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_9^{(\psi)} = -\tilde{\mu}_\psi^{\text{eff}}\frac{\boldsymbol{q}^2}{2m_\chi^2}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

$$\begin{aligned}
\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v}) = \sum_{\psi=p,n,e} & \boxed{c_1^{(\psi)}\langle N_\psi\rangle_{lj}} \\
& + c_3^{(\psi)}\left[-\frac{iq}{m_\psi}\boldsymbol{v}'\cdot\left(\hat{\boldsymbol{q}}\times\langle\boldsymbol{S}_\psi\rangle_{lj}\right) + \frac{q^2}{2m_\psi^2}\left(\delta^{ik}-\hat{q}^i\hat{q}^k\right)\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
& + c_4^{(\psi)}\,\boldsymbol{S}_\chi\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} \\
& + c_5^{(\psi)}\left[\frac{i\boldsymbol{q}}{m_\psi}\cdot\left(\boldsymbol{v}'\times\boldsymbol{S}_\chi\right)\langle N_\psi\rangle_{lj} + \frac{q^2}{2m_\psi^2}\boldsymbol{S}_\chi\cdot\left(\mathbb{1}-\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}\right)\cdot\langle\boldsymbol{L}_\psi\rangle_{lj}\right] \\
& + c_6^{(\psi)}\frac{q^2}{m_\psi^2}\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\left(\hat{\boldsymbol{q}}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right) \\
& + c_7^{(\psi)}\left[\boldsymbol{v}'\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} + \epsilon^{ikk'}\frac{iq^{k'}}{2m_\chi}\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
& + c_8^{(\psi)}\left[\left(\boldsymbol{v}'\cdot\boldsymbol{S}_\chi\right)\langle N_\psi\rangle_{lj} + \frac{iq}{2m_\psi}\boldsymbol{S}_\chi\cdot\left(\hat{\boldsymbol{q}}\times\langle\boldsymbol{L}_\psi\rangle_{lj}\right)\right] \\
& + c_9^{(\psi)}\frac{iq}{m_\psi}\boldsymbol{S}_\chi\cdot\left(\langle\boldsymbol{S}_\psi\rangle_{lj}\times\hat{\boldsymbol{q}}\right) \\
& + c_{10}^{(\psi)}\frac{i\boldsymbol{q}}{m_\psi}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} \\
& \boxed{+ c_{11}^{(\psi)}\frac{i\boldsymbol{q}}{m_\psi}\cdot\boldsymbol{S}_\chi\langle N_\psi\rangle_{lj}} \\
& + c_{12}^{(\psi)}\left[\left(\boldsymbol{v}'\times\boldsymbol{S}_\chi\right)\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} + \frac{iq}{2m_\psi}\left(\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\delta^{ik}-\hat{q}^k S_\chi^i\right)\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
& + c_{13}^{(\psi)}\left[\frac{iq}{m_\psi}\left(\boldsymbol{v}'\cdot\boldsymbol{S}_\chi\right)\left(\hat{\boldsymbol{q}}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right) + \frac{q^2}{2m_\psi^2}\left(\hat{\boldsymbol{q}}\times\boldsymbol{S}_\chi\right)\cdot\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\cdot\hat{\boldsymbol{q}}\right] \\
& + c_{14}^{(\psi)}\left[\frac{iq}{m_\psi}\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\left(\boldsymbol{v}'\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right) - \epsilon^{ikk'}\frac{q^2}{2m_\psi^2}\hat{q}^{k'}\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
& + c_{15}^{(\psi)}\left[-\frac{q^2}{m_\psi^2}\left(\hat{\boldsymbol{q}}\cdot\left(\boldsymbol{v}'\times\boldsymbol{S}_\chi\right)\right)\left(\hat{\boldsymbol{q}}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right)\right. \\
& \qquad \left. + \frac{iq^3}{2m_\psi^3}\boldsymbol{S}_\chi\cdot\left(\mathbb{1}-\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}\right)\cdot\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\cdot\hat{\boldsymbol{q}}\right],
\end{aligned}$$

Example

- Dark photon mediator models.

Millicharged DM (SI)

$$c_1^{(\psi)} = -\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2+m_V^2}$$

Electric dipole DM

$$c_{11}^{(\psi)} = -\frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2+m_V^2}$$

N response only $\Rightarrow$ probed by phonons.

Magnetic dipole DM

$$c_1^{(\psi)} = \frac{\boldsymbol{q}^2}{4m_\chi^2}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2+m_V^2}$$

$$c_4^{(\psi)} = \tilde{\mu}_\psi^{\rm eff}\frac{\boldsymbol{q}^2}{m_\chi m_\psi}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2+m_V^2}$$

$$c_5^{(\psi)} = \frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2+m_V^2}$$

$$c_6^{(\psi)} = -\tilde{\mu}_\psi^{\rm eff}\frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2+m_V^2}$$

Anapole DM

$$c_8^{(\psi)} = \frac{\boldsymbol{q}^2}{2m_\chi^2}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2+m_V^2}$$

$$c_9^{(\psi)} = -\tilde{\mu}_\psi^{\rm eff}\frac{\boldsymbol{q}^2}{2m_\chi^2}\frac{g_\chi g_\psi^{\rm eff}}{\boldsymbol{q}^2+m_V^2}$$

$$\begin{aligned}
\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v}) = \sum_{\psi=p,n,e} & \boxed{c_1^{(\psi)}\langle N_\psi\rangle_{lj}} \\
&+ c_3^{(\psi)}\Big[-\frac{iq}{m_\psi}\boldsymbol{v}'\cdot\big(\hat{\boldsymbol{q}}\times\langle\boldsymbol{S}_\psi\rangle_{lj}\big) + \frac{q^2}{2m_\psi^2}\big(\delta^{ik}-\hat{q}^i\hat{q}^k\big)\big(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\big)^{ik}\Big] \\
&+ c_4^{(\psi)}\,\boldsymbol{S}_\chi\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} \\
&+ c_5^{(\psi)}\Big[\frac{iq}{m_\psi}\cdot(\boldsymbol{v}'\times\boldsymbol{S}_\chi)\langle N_\psi\rangle_{lj} + \frac{q^2}{2m_\psi^2}\boldsymbol{S}_\chi\cdot(\mathbb{1}-\hat{\boldsymbol{q}}\hat{\boldsymbol{q}})\cdot\langle\boldsymbol{L}_\psi\rangle_{lj}\Big] \\
&+ c_6^{(\psi)}\frac{q^2}{m_\psi^2}(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi)(\hat{\boldsymbol{q}}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}) \\
&+ c_7^{(\psi)}\Big[\boldsymbol{v}'\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} + \epsilon^{ikk'}\frac{iq^{k'}}{2m_\chi}\big(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\big)^{ik}\Big] \\
&+ c_8^{(\psi)}\Big[(\boldsymbol{v}'\cdot\boldsymbol{S}_\chi)\langle N_\psi\rangle_{lj} + \frac{iq}{2m_\psi}\boldsymbol{S}_\chi\cdot(\hat{\boldsymbol{q}}\times\langle\boldsymbol{L}_\psi\rangle_{lj})\Big] \\
&+ c_9^{(\psi)}\frac{iq}{m_\psi}\boldsymbol{S}_\chi\cdot\big(\langle\boldsymbol{S}_\psi\rangle_{lj}\times\hat{\boldsymbol{q}}\big) \\
&+ c_{10}^{(\psi)}\frac{i\boldsymbol{q}}{m_\psi}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} \\
&\boxed{+ c_{11}^{(\psi)}\frac{i\boldsymbol{q}}{m_\psi}\cdot\boldsymbol{S}_\chi\langle N_\psi\rangle_{lj}} \\
&+ c_{12}^{(\psi)}\Big[(\boldsymbol{v}'\times\boldsymbol{S}_\chi)\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} + \frac{iq}{2m_\psi}\big((\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi)\delta^{ik}-\hat{q}^kS_\chi^i\big)\big(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\big)^{ik}\Big] \\
&+ c_{13}^{(\psi)}\Big[\frac{iq}{m_\psi}(\boldsymbol{v}'\cdot\boldsymbol{S}_\chi)(\hat{\boldsymbol{q}}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}) + \frac{q^2}{2m_\psi^2}(\hat{\boldsymbol{q}}\times\boldsymbol{S}_\chi)\cdot\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\cdot\hat{\boldsymbol{q}}\Big] \\
&+ c_{14}^{(\psi)}\Big[\frac{iq}{m_\psi}(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi)(\boldsymbol{v}'\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}) - \epsilon^{ikk'}\frac{q^2}{2m_\psi^2}\hat{q}^{k'}(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi)\big(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\big)^{ik}\Big] \\
&+ c_{15}^{(\psi)}\Big[-\frac{q^2}{m_\psi^2}\big(\hat{\boldsymbol{q}}\cdot(\boldsymbol{v}'\times\boldsymbol{S}_\chi)\big)(\hat{\boldsymbol{q}}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}) \\
&\qquad + \frac{iq^3}{2m_\psi^3}\boldsymbol{S}_\chi\cdot(\mathbb{1}-\hat{\boldsymbol{q}}\hat{\boldsymbol{q}})\cdot\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\cdot\hat{\boldsymbol{q}}\Big],
\end{aligned}$$

higher order in q

Example

- Dark photon mediator models.

Millicharged DM (SI)

$$c_1^{(\psi)} = -\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

Electric dipole DM

$$c_{11}^{(\psi)} = -\frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

N response only $\Rightarrow$ probed by phonons.

Magnetic dipole DM

$$c_1^{(\psi)} = \frac{\boldsymbol{q}^2}{4m_\chi^2}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_4^{(\psi)} = \tilde{\mu}_\psi^{\text{eff}}\frac{\boldsymbol{q}^2}{m_\chi m_\psi}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_5^{(\psi)} = \frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_6^{(\psi)} = -\tilde{\mu}_\psi^{\text{eff}}\frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

Anapole DM

$$c_8^{(\psi)} = \frac{\boldsymbol{q}^2}{2m_\chi^2}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_9^{(\psi)} = -\tilde{\mu}_\psi^{\text{eff}}\frac{\boldsymbol{q}^2}{2m_\chi^2}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

$$\begin{aligned}
\widetilde{V}_{lj}(-\boldsymbol{q},\boldsymbol{v}) = \sum_{\psi=p,n,e} & \boxed{c_1^{(\psi)}\langle N_\psi\rangle_{lj}} \\
& + c_3^{(\psi)}\left[-\frac{iq}{m_\psi}\boldsymbol{v}'\cdot\left(\hat{\boldsymbol{q}}\times\langle\boldsymbol{S}_\psi\rangle_{lj}\right) + \frac{q^2}{2m_\psi^2}\left(\delta^{ik}-\hat{q}^i\hat{q}^k\right)\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
& + c_4^{(\psi)}\,\boldsymbol{S}_\chi\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} \\
& + c_5^{(\psi)}\left[\frac{i\boldsymbol{q}}{m_\psi}\cdot\left(\boldsymbol{v}'\times\boldsymbol{S}_\chi\right)\langle N_\psi\rangle_{lj} + \frac{q^2}{2m_\psi^2}\boldsymbol{S}_\chi\cdot\left(\mathbb{1}-\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}\right)\cdot\langle\boldsymbol{L}_\psi\rangle_{lj}\right] \\
& + c_6^{(\psi)}\frac{q^2}{m_\psi^2}\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\left(\hat{\boldsymbol{q}}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right) \\
& + c_7^{(\psi)}\left[\boldsymbol{v}'\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} + \epsilon^{ikk'}\frac{iq^{k'}}{2m_\chi}\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
& + c_8^{(\psi)}\left[\left(\boldsymbol{v}'\cdot\boldsymbol{S}_\chi\right)\langle N_\psi\rangle_{lj} + \frac{iq}{2m_\psi}\boldsymbol{S}_\chi\cdot\left(\hat{\boldsymbol{q}}\times\langle\boldsymbol{L}_\psi\rangle_{lj}\right)\right] \\
& + c_9^{(\psi)}\frac{iq}{m_\psi}\boldsymbol{S}_\chi\cdot\left(\langle\boldsymbol{S}_\psi\rangle_{lj}\times\hat{\boldsymbol{q}}\right) \\
& + c_{10}^{(\psi)}\frac{i\boldsymbol{q}}{m_\psi}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} \\
& + c_{11}^{(\psi)}\frac{i\boldsymbol{q}}{m_\psi}\cdot\boldsymbol{S}_\chi\langle N_\psi\rangle_{lj} \\
& + c_{12}^{(\psi)}\left[\left(\boldsymbol{v}'\times\boldsymbol{S}_\chi\right)\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} + \frac{iq}{2m_\psi}\left(\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\delta^{ik}-\hat{q}^kS_\chi^i\right)\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
& + c_{13}^{(\psi)}\left[\frac{iq}{m_\psi}\left(\boldsymbol{v}'\cdot\boldsymbol{S}_\chi\right)\left(\hat{\boldsymbol{q}}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right) + \frac{q^2}{2m_\psi^2}\left(\hat{\boldsymbol{q}}\times\boldsymbol{S}_\chi\right)\cdot\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\cdot\hat{\boldsymbol{q}}\right] \\
& + c_{14}^{(\psi)}\left[\frac{iq}{m_\psi}\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\left(\boldsymbol{v}'\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right) - \epsilon^{ikk'}\frac{q^2}{2m_\psi^2}\hat{q}^{k'}\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
& + c_{15}^{(\psi)}\left[-\frac{q^2}{m_\psi^2}\left(\hat{\boldsymbol{q}}\cdot\left(\boldsymbol{v}'\times\boldsymbol{S}_\chi\right)\right)\left(\hat{\boldsymbol{q}}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right)\right. \\
& \qquad \left. + \frac{iq^3}{2m_\psi^3}\boldsymbol{S}_\chi\cdot\left(\mathbb{1}-\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}\right)\cdot\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\cdot\hat{\boldsymbol{q}}\right],
\end{aligned}$$

Example

- Dark photon mediator models.

Millicharged DM (SI)

$$c_1^{(\psi)} = -\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

Electric dipole DM

$$c_{11}^{(\psi)} = -\frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

N response only $\Rightarrow$ probed by phonons.

Magnetic dipole DM

$$c_1^{(\psi)} = \frac{\boldsymbol{q}^2}{4m_\chi^2}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_4^{(\psi)} = \tilde{\mu}_\psi^{\text{eff}}\frac{\boldsymbol{q}^2}{m_\chi m_\psi}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_5^{(\psi)} = \frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_6^{(\psi)} = -\tilde{\mu}_\psi^{\text{eff}}\frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

Anapole DM

$$c_8^{(\psi)} = \frac{\boldsymbol{q}^2}{2m_\chi^2}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_9^{(\psi)} = -\tilde{\mu}_\psi^{\text{eff}}\frac{\boldsymbol{q}^2}{2m_\chi^2}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

N response $\Rightarrow$ phonons.

$\boldsymbol{S}$, $\boldsymbol{L}$ responses $\Rightarrow$ magnons.

$$
\begin{aligned}
\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v}) = \sum_{\psi=p,n,e} & \boxed{c_1^{(\psi)}\langle N_\psi\rangle_{lj}} \\
& + c_3^{(\psi)}\left[-\frac{iq}{m_\psi}\boldsymbol{v}'\cdot\left(\hat{\boldsymbol{q}}\times\langle\boldsymbol{S}_\psi\rangle_{lj}\right) + \frac{q^2}{2m_\psi^2}\left(\delta^{ik}-\hat{q}^i\hat{q}^k\right)\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
& + c_4^{(\psi)}\,\boldsymbol{S}_\chi\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} \\
& + c_5^{(\psi)}\left[\frac{i\boldsymbol{q}}{m_\psi}\cdot\left(\boldsymbol{v}'\times\boldsymbol{S}_\chi\right)\langle N_\psi\rangle_{lj} + \frac{q^2}{2m_\psi^2}\boldsymbol{S}_\chi\cdot\left(\mathbb{1}-\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}\right)\cdot\langle\boldsymbol{L}_\psi\rangle_{lj}\right] \\
& + c_6^{(\psi)}\frac{q^2}{m_\psi^2}\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\left(\hat{\boldsymbol{q}}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right) \\
& + c_7^{(\psi)}\left[\boldsymbol{v}'\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} + \epsilon^{ikk'}\frac{iq^{k'}}{2m_\chi}\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
& + c_8^{(\psi)}\left[\left(\boldsymbol{v}'\cdot\boldsymbol{S}_\chi\right)\langle N_\psi\rangle_{lj} + \frac{iq}{2m_\psi}\boldsymbol{S}_\chi\cdot\left(\hat{\boldsymbol{q}}\times\langle\boldsymbol{L}_\psi\rangle_{lj}\right)\right] \\
& + c_9^{(\psi)}\frac{iq}{m_\psi}\boldsymbol{S}_\chi\cdot\left(\langle\boldsymbol{S}_\psi\rangle_{lj}\times\hat{\boldsymbol{q}}\right) \\
& + c_{10}^{(\psi)}\frac{i\boldsymbol{q}}{m_\psi}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} \\
& + c_{11}^{(\psi)}\frac{i\boldsymbol{q}}{m_\psi}\cdot\boldsymbol{S}_\chi\langle N_\psi\rangle_{lj} \\
& + c_{12}^{(\psi)}\left[\left(\boldsymbol{v}'\times\boldsymbol{S}_\chi\right)\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} + \frac{iq}{2m_\psi}\left(\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\delta^{ik}-\hat{q}^k S_\chi^i\right)\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
& + c_{13}^{(\psi)}\left[\frac{iq}{m_\psi}\left(\boldsymbol{v}'\cdot\boldsymbol{S}_\chi\right)\left(\hat{\boldsymbol{q}}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right) + \frac{q^2}{2m_\psi^2}\left(\hat{\boldsymbol{q}}\times\boldsymbol{S}_\chi\right)\cdot\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\cdot\hat{\boldsymbol{q}}\right] \\
& + c_{14}^{(\psi)}\left[\frac{iq}{m_\psi}\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\left(\boldsymbol{v}'\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right) - \epsilon^{ikk'}\frac{q^2}{2m_\psi^2}\hat{q}^{k'}\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
& + c_{15}^{(\psi)}\left[-\frac{q^2}{m_\psi^2}\left(\hat{\boldsymbol{q}}\cdot\left(\boldsymbol{v}'\times\boldsymbol{S}_\chi\right)\right)\left(\hat{\boldsymbol{q}}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right)\right. \\
& \qquad \left. + \frac{iq^3}{2m_\psi^3}\boldsymbol{S}_\chi\cdot\left(\mathbb{1}-\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}\right)\cdot\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\cdot\hat{\boldsymbol{q}}\right],
\end{aligned}
$$

Example

- Dark photon mediator models.

Millicharged DM (SI)

$$c_1^{(\psi)} = -\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2+m_V^2}$$

Electric dipole DM

$$c_{11}^{(\psi)} = -\frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2+m_V^2}$$

N response only $\Rightarrow$ probed by phonons.

Magnetic dipole DM

$$c_1^{(\psi)} = \frac{\boldsymbol{q}^2}{4m_\chi^2}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2+m_V^2}$$

$$c_4^{(\psi)} = \tilde{\mu}_\psi^{\text{eff}}\frac{\boldsymbol{q}^2}{m_\chi m_\psi}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2+m_V^2}$$

$$c_5^{(\psi)} = \frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2+m_V^2}$$

$$c_6^{(\psi)} = -\tilde{\mu}_\psi^{\text{eff}}\frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2+m_V^2}$$

Anapole DM

$$c_8^{(\psi)} = \frac{\boldsymbol{q}^2}{2m_\chi^2}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2+m_V^2}$$

$$c_9^{(\psi)} = -\tilde{\mu}_\psi^{\text{eff}}\frac{\boldsymbol{q}^2}{2m_\chi^2}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2+m_V^2}$$

N response $\Rightarrow$ phonons.

$\boldsymbol{S}$, $\boldsymbol{L}$ responses $\Rightarrow$ magnons.

$$
\begin{aligned}
\widetilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v}) = \sum_{\psi=p,n,e} & \boxed{c_1^{(\psi)}\langle N_\psi\rangle_{lj}} \\
& + c_3^{(\psi)}\left[-\frac{iq}{m_\psi}\boldsymbol{v}'\cdot\left(\hat{\boldsymbol{q}}\times\langle\boldsymbol{S}_\psi\rangle_{lj}\right)+\frac{q^2}{2m_\psi^2}\left(\delta^{ik}-\hat{q}^i\hat{q}^k\right)\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
& + c_4^{(\psi)}\boldsymbol{S}_\chi\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} \\
& + c_5^{(\psi)}\left[\frac{i\boldsymbol{q}}{m_\psi}\cdot\left(\boldsymbol{v}'\times\boldsymbol{S}_\chi\right)\langle N_\psi\rangle_{lj}+\frac{q^2}{2m_\psi^2}\boldsymbol{S}_\chi\cdot\left(\mathbb{1}-\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}\right)\cdot\langle\boldsymbol{L}_\psi\rangle_{lj}\right] \\
& + c_6^{(\psi)}\frac{q^2}{m_\psi^2}\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\left(\hat{\boldsymbol{q}}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right) \\
& + c_7^{(\psi)}\left[\boldsymbol{v}'\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}+\epsilon^{ikk'}\frac{iq^{k'}}{2m_\chi}\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
& + c_8^{(\psi)}\left[\left(\boldsymbol{v}'\cdot\boldsymbol{S}_\chi\right)\langle N_\psi\rangle_{lj}+\frac{iq}{2m_\psi}\boldsymbol{S}_\chi\cdot\left(\hat{\boldsymbol{q}}\times\langle\boldsymbol{L}_\psi\rangle_{lj}\right)\right] \\
& + c_9^{(\psi)}\frac{iq}{m_\psi}\boldsymbol{S}_\chi\cdot\left(\langle\boldsymbol{S}_\psi\rangle_{lj}\times\hat{\boldsymbol{q}}\right) \\
& + c_{10}^{(\psi)}\frac{i\boldsymbol{q}}{m_\psi}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} \\
& + c_{11}^{(\psi)}\frac{i\boldsymbol{q}}{m_\psi}\cdot\boldsymbol{S}_\chi\langle N_\psi\rangle_{lj} \\
& + c_{12}^{(\psi)}\left[\left(\boldsymbol{v}'\times\boldsymbol{S}_\chi\right)\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}+\frac{iq}{2m_\psi}\left(\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\delta^{ik}-\hat{q}^kS_\chi^i\right)\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
& + c_{13}^{(\psi)}\left[\frac{iq}{m_\psi}\left(\boldsymbol{v}'\cdot\boldsymbol{S}_\chi\right)\left(\hat{\boldsymbol{q}}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right)+\frac{q^2}{2m_\psi^2}\left(\hat{\boldsymbol{q}}\times\boldsymbol{S}_\chi\right)\cdot\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\cdot\hat{\boldsymbol{q}}\right] \\
& + c_{14}^{(\psi)}\left[\frac{iq}{m_\psi}\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\left(\boldsymbol{v}'\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right)-\epsilon^{ikk'}\frac{q^2}{2m_\psi^2}\hat{q}^{k'}\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
& + c_{15}^{(\psi)}\left[-\frac{q^2}{m_\psi^2}\left(\hat{\boldsymbol{q}}\cdot\left(\boldsymbol{v}'\times\boldsymbol{S}_\chi\right)\right)\left(\hat{\boldsymbol{q}}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right)\right. \\
& \qquad \left.+\frac{iq^3}{2m_\psi^3}\boldsymbol{S}_\chi\cdot\left(\mathbb{1}-\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}\right)\cdot\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\cdot\hat{\boldsymbol{q}}\right],
\end{aligned}
$$

higher order in q

Example

- Dark photon mediator models.

Millicharged DM (SI)

$$c_1^{(\psi)} = -\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

Electric dipole DM

$$c_{11}^{(\psi)} = -\frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

N response only $\Rightarrow$ probed by phonons.

$\times\, q^2$

Magnetic dipole DM

$$c_1^{(\psi)} = \frac{\boldsymbol{q}^2}{4m_\chi^2}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_4^{(\psi)} = \tilde{\mu}_\psi^{\text{eff}}\frac{\boldsymbol{q}^2}{m_\chi m_\psi}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_5^{(\psi)} = \frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_6^{(\psi)} = -\tilde{\mu}_\psi^{\text{eff}}\frac{m_\psi}{m_\chi}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

Anapole DM

$$c_8^{(\psi)} = \frac{\boldsymbol{q}^2}{2m_\chi^2}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

$$c_9^{(\psi)} = -\tilde{\mu}_\psi^{\text{eff}}\frac{\boldsymbol{q}^2}{2m_\chi^2}\frac{g_\chi g_\psi^{\text{eff}}}{\boldsymbol{q}^2 + m_V^2}$$

N response $\Rightarrow$ phonons.

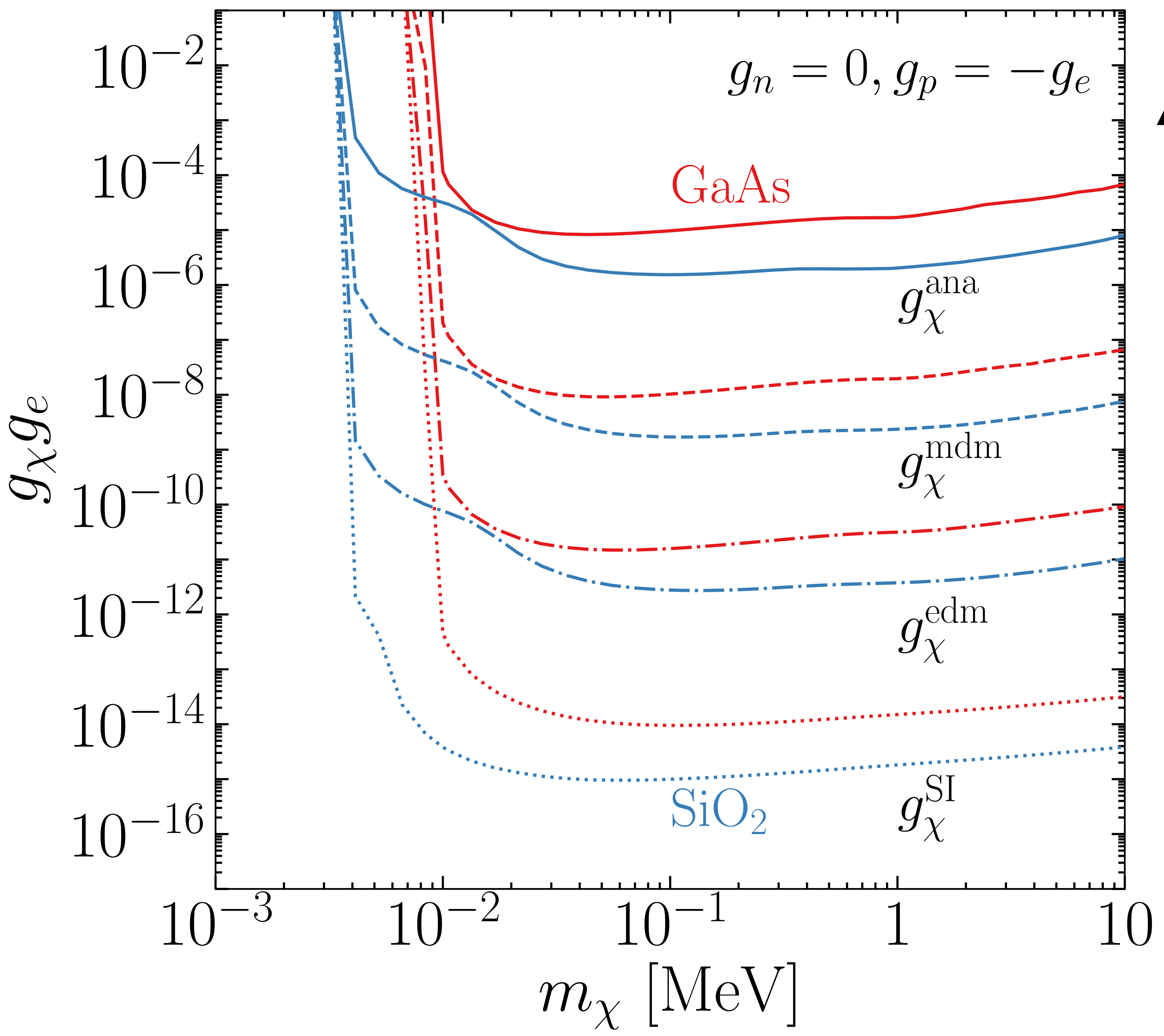
$\boldsymbol{S}$, $\boldsymbol{L}$ responses $\Rightarrow$ magnons.

$$
\begin{aligned}
\tilde{\mathcal{V}}_{lj}(-\boldsymbol{q},\boldsymbol{v}) = \sum_{\psi=p,n,e} & \boxed{c_1^{(\psi)}\langle N_\psi\rangle_{lj}} \\
& + c_3^{(\psi)}\left[-\frac{iq}{m_\psi}\boldsymbol{v}'\cdot\left(\hat{\boldsymbol{q}}\times\langle\boldsymbol{S}_\psi\rangle_{lj}\right) + \frac{q^2}{2m_\psi^2}\left(\delta^{ik}-\hat{q}^i\hat{q}^k\right)\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
& + c_4^{(\psi)}\,\boldsymbol{S}_\chi\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} \\
& + c_5^{(\psi)}\left[\frac{i\boldsymbol{q}}{m_\psi}\cdot\left(\boldsymbol{v}'\times\boldsymbol{S}_\chi\right)\langle N_\psi\rangle_{lj} + \frac{q^2}{2m_\psi^2}\boldsymbol{S}_\chi\cdot\left(\mathbb{1}-\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}\right)\cdot\langle\boldsymbol{L}_\psi\rangle_{lj}\right] \\
& + c_6^{(\psi)}\frac{q^2}{m_\psi^2}\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\left(\hat{\boldsymbol{q}}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right) \\
& + c_7^{(\psi)}\left[\boldsymbol{v}'\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} + \epsilon^{ikk'}\frac{iq^{k'}}{2m_\chi}\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
& + c_8^{(\psi)}\left[\left(\boldsymbol{v}'\cdot\boldsymbol{S}_\chi\right)\langle N_\psi\rangle_{lj} + \frac{iq}{2m_\psi}\boldsymbol{S}_\chi\cdot\left(\hat{\boldsymbol{q}}\times\langle\boldsymbol{L}_\psi\rangle_{lj}\right)\right] \\
& + c_9^{(\psi)}\frac{iq}{m_\psi}\boldsymbol{S}_\chi\cdot\left(\langle\boldsymbol{S}_\psi\rangle_{lj}\times\hat{\boldsymbol{q}}\right) \\
& + c_{10}^{(\psi)}\frac{i\boldsymbol{q}}{m_\psi}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} \\
& + c_{11}^{(\psi)}\frac{i\boldsymbol{q}}{m_\psi}\cdot\boldsymbol{S}_\chi\langle N_\psi\rangle_{lj} \\
& + c_{12}^{(\psi)}\left[\left(\boldsymbol{v}'\times\boldsymbol{S}_\chi\right)\cdot\langle\boldsymbol{S}_\psi\rangle_{lj} + \frac{iq}{2m_\psi}\left(\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\delta^{ik}-\hat{q}^kS_\chi^i\right)\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
& + c_{13}^{(\psi)}\left[\frac{iq}{m_\psi}\left(\boldsymbol{v}'\cdot\boldsymbol{S}_\chi\right)\left(\hat{\boldsymbol{q}}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right) + \frac{q^2}{2m_\psi^2}\left(\hat{\boldsymbol{q}}\times\boldsymbol{S}_\chi\right)\cdot\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\cdot\hat{\boldsymbol{q}}\right] \\
& + c_{14}^{(\psi)}\left[\frac{iq}{m_\psi}\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\left(\boldsymbol{v}'\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right) - \epsilon^{ikk'}\frac{q^2}{2m_\psi^2}\hat{q}^{k'}\left(\hat{\boldsymbol{q}}\cdot\boldsymbol{S}_\chi\right)\left(\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\right)^{ik}\right] \\
& + c_{15}^{(\psi)}\left[-\frac{q^2}{m_\psi^2}\left(\hat{\boldsymbol{q}}\cdot\left(\boldsymbol{v}'\times\boldsymbol{S}_\chi\right)\right)\left(\hat{\boldsymbol{q}}\cdot\langle\boldsymbol{S}_\psi\rangle_{lj}\right)\right. \\
& \qquad \left. + \frac{iq^3}{2m_\psi^3}\boldsymbol{S}_\chi\cdot\left(\mathbb{1}-\hat{\boldsymbol{q}}\hat{\boldsymbol{q}}\right)\cdot\langle\boldsymbol{L}_\psi\otimes\boldsymbol{S}_\psi\rangle_{lj}\cdot\hat{\boldsymbol{q}}\right],
\end{aligned}
$$

higher order in q

Example

➤ Phonon reach for kg-yr exposure, assuming background-free.



successively higher order in q

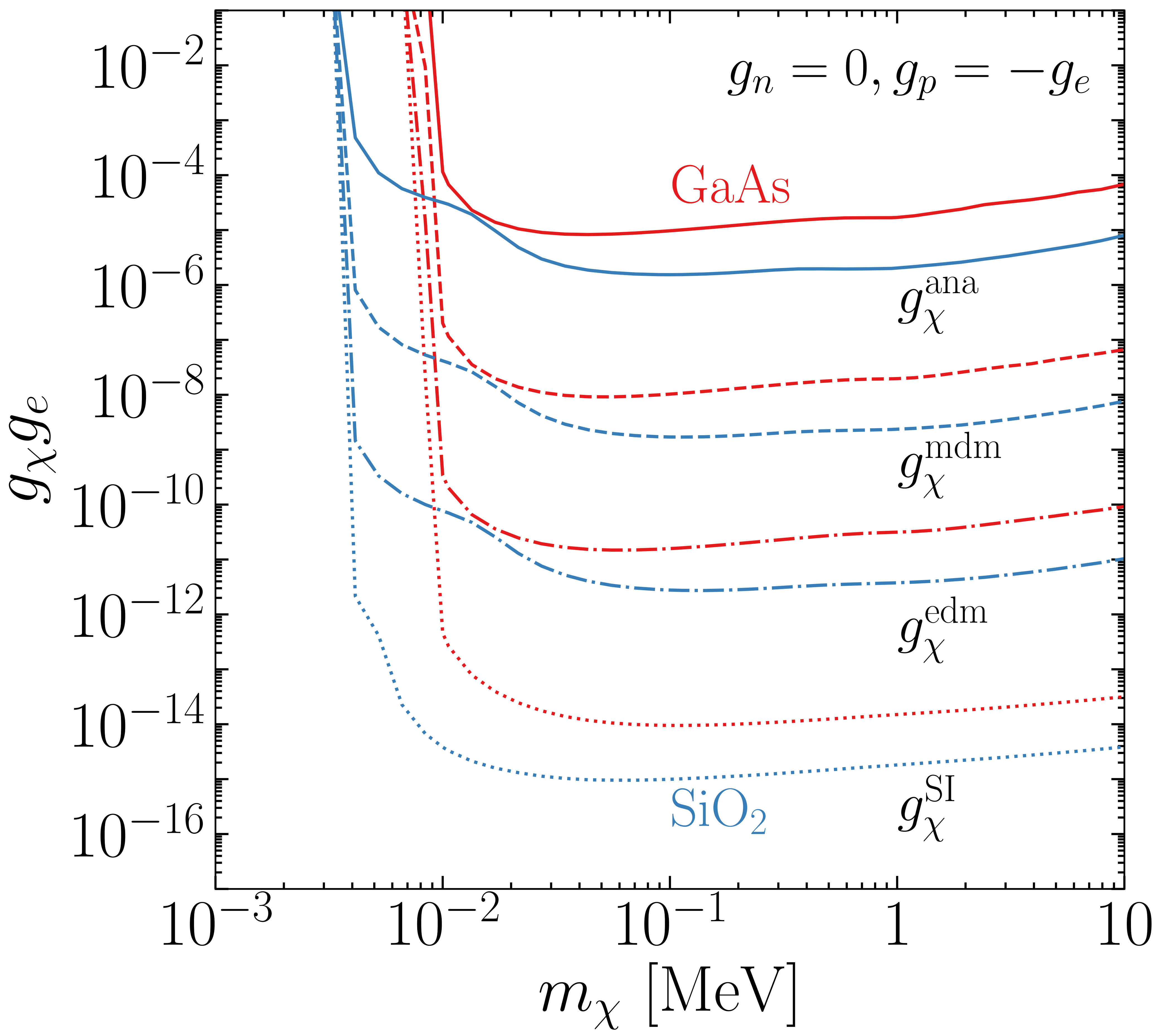
Parametrically, the rates are given by:

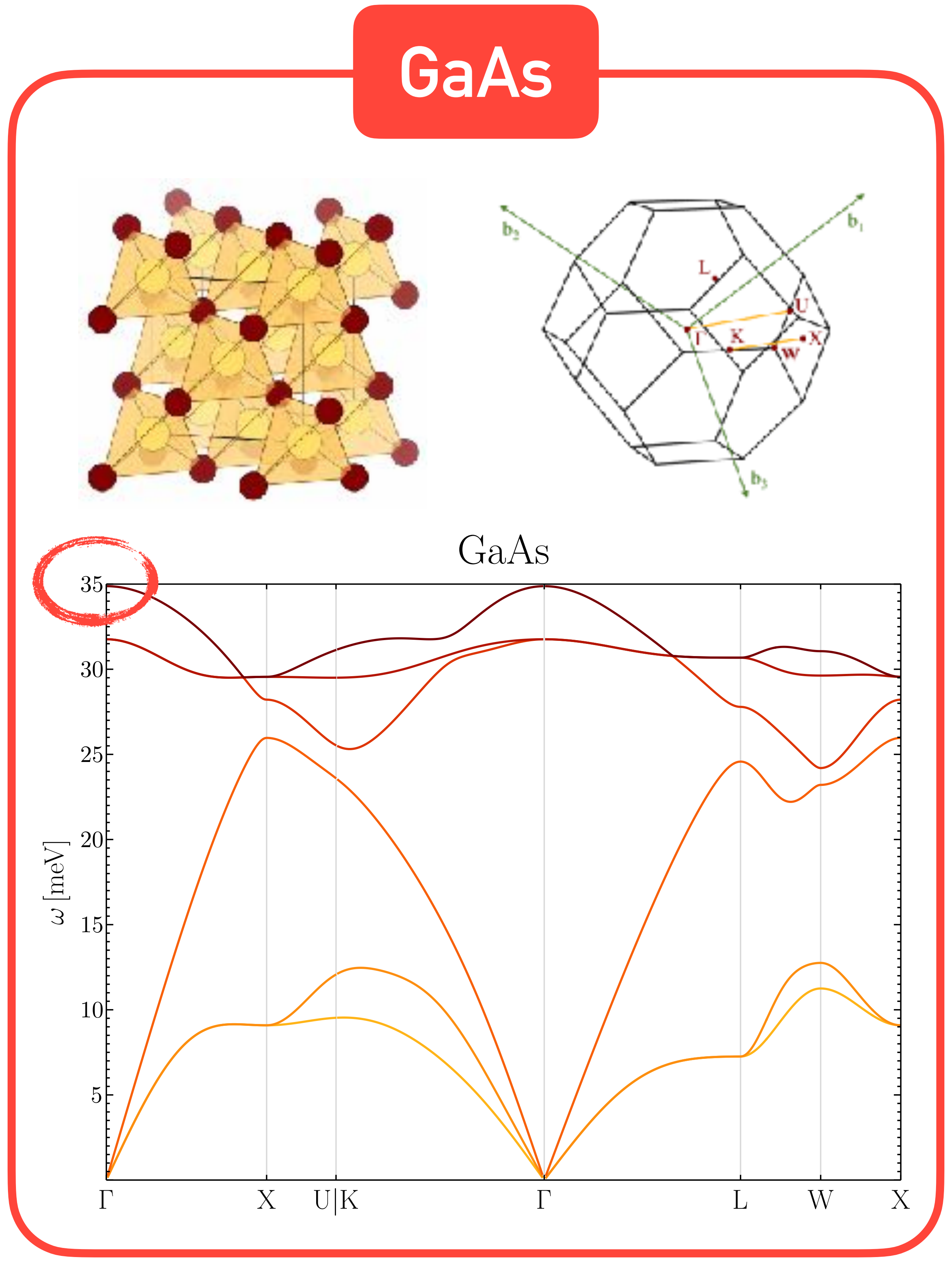
$$R^{\text{SI}}_{\text{phonon}} \sim \frac{\rho_\chi}{m_\chi}\frac{1}{m_{\text{cell}}}\frac{g_\chi^2 g_e^2}{\varepsilon_\infty^2}\frac{Q_{\text{ion}}^2}{m_{\text{ion}}\omega}\int dq\,\frac{1}{q} \sim g_\chi^2 g_e^2\,\frac{\rho_\chi}{m_\chi}\left(\frac{Q_{\text{ion}}^2}{\varepsilon_\infty^2 m_{\text{cell}} m_{\text{ion}}\omega}\right)$$

$$\frac{R^{\text{edm}}_{\text{phonon}}}{R^{\text{SI}}_{\text{phonon}}} \sim \frac{R^{\text{mdm}}_{\text{phonon}}}{R^{\text{edm}}_{\text{phonon}}} \sim \frac{R^{\text{ana}}_{\text{phonon}}}{R^{\text{mdm}}_{\text{phonon}}} \sim v^2$$

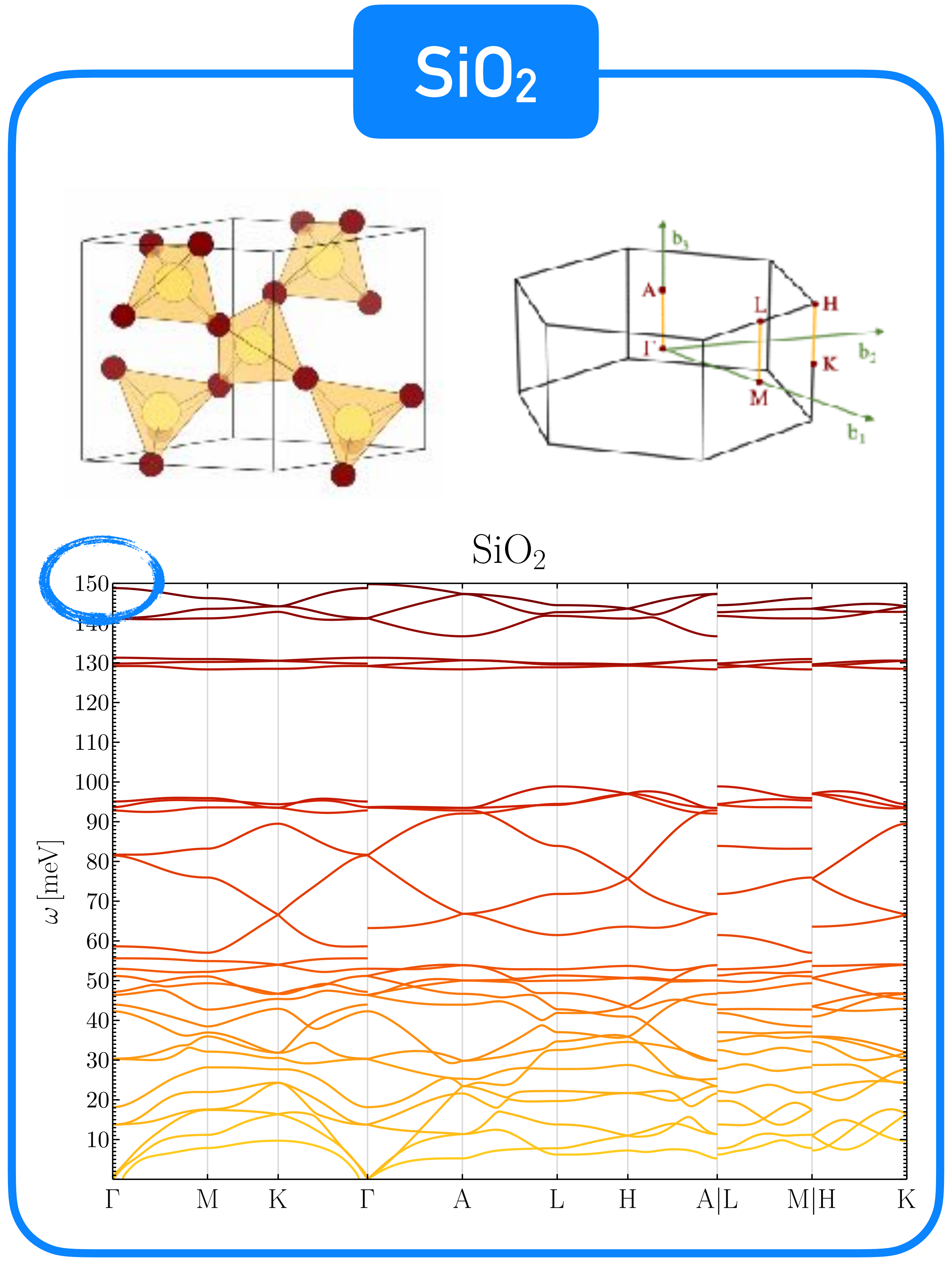
Example

➤ Phonon reach for kg-yr exposure, assuming background-free.





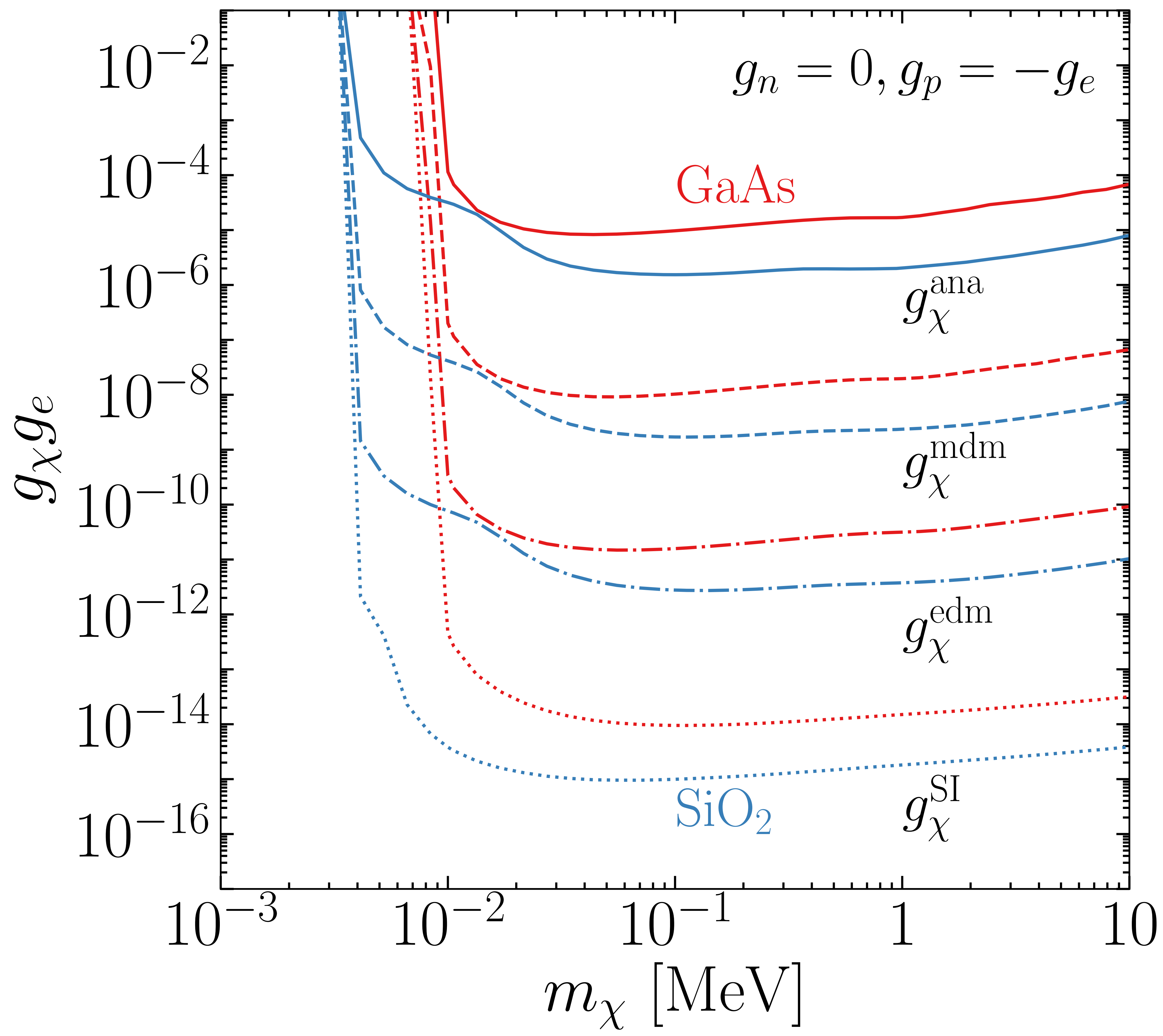
One of the targets used in SPICE
(Sub-eV Polar Interactions Cryogenic Experiment)
— part of the TESSERACT project, in R&D.

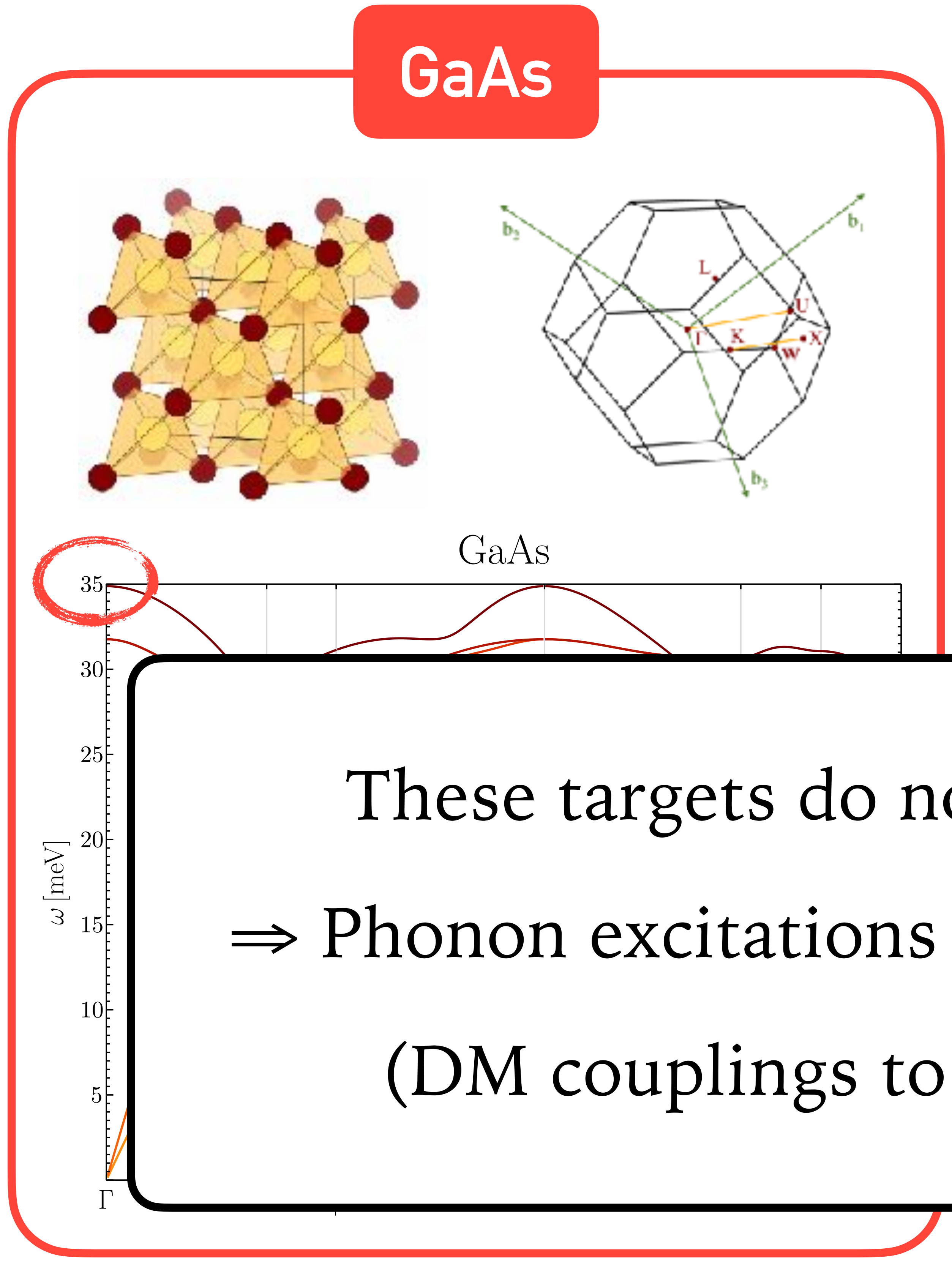


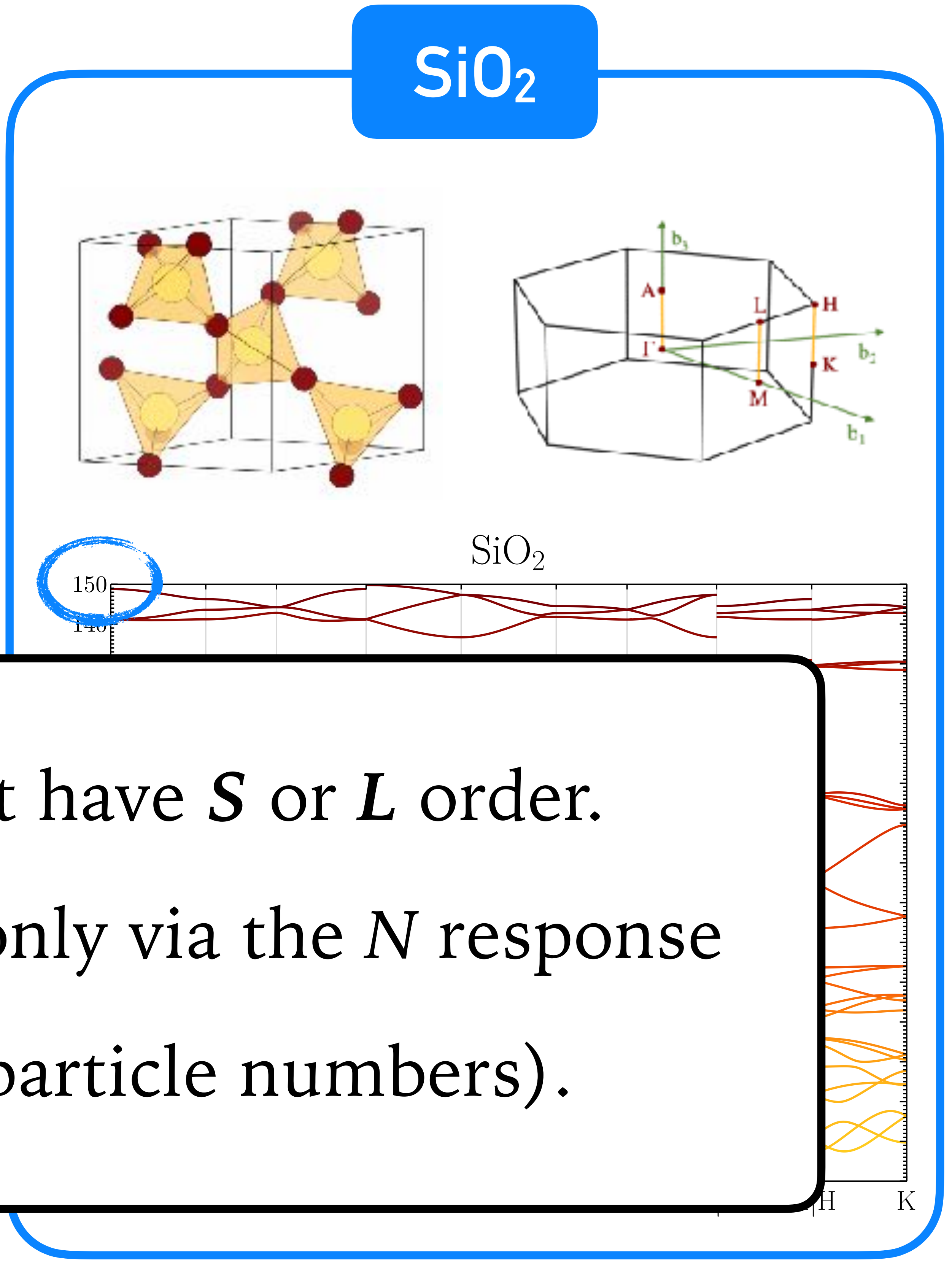
Optimal phonon target found in our theoretical study.

Example

- Phonon reach for kg-yr exposure, assuming background-free.







These targets do not have ***S*** or ***L*** order.

$\Rightarrow$ Phonon excitations only via the N response

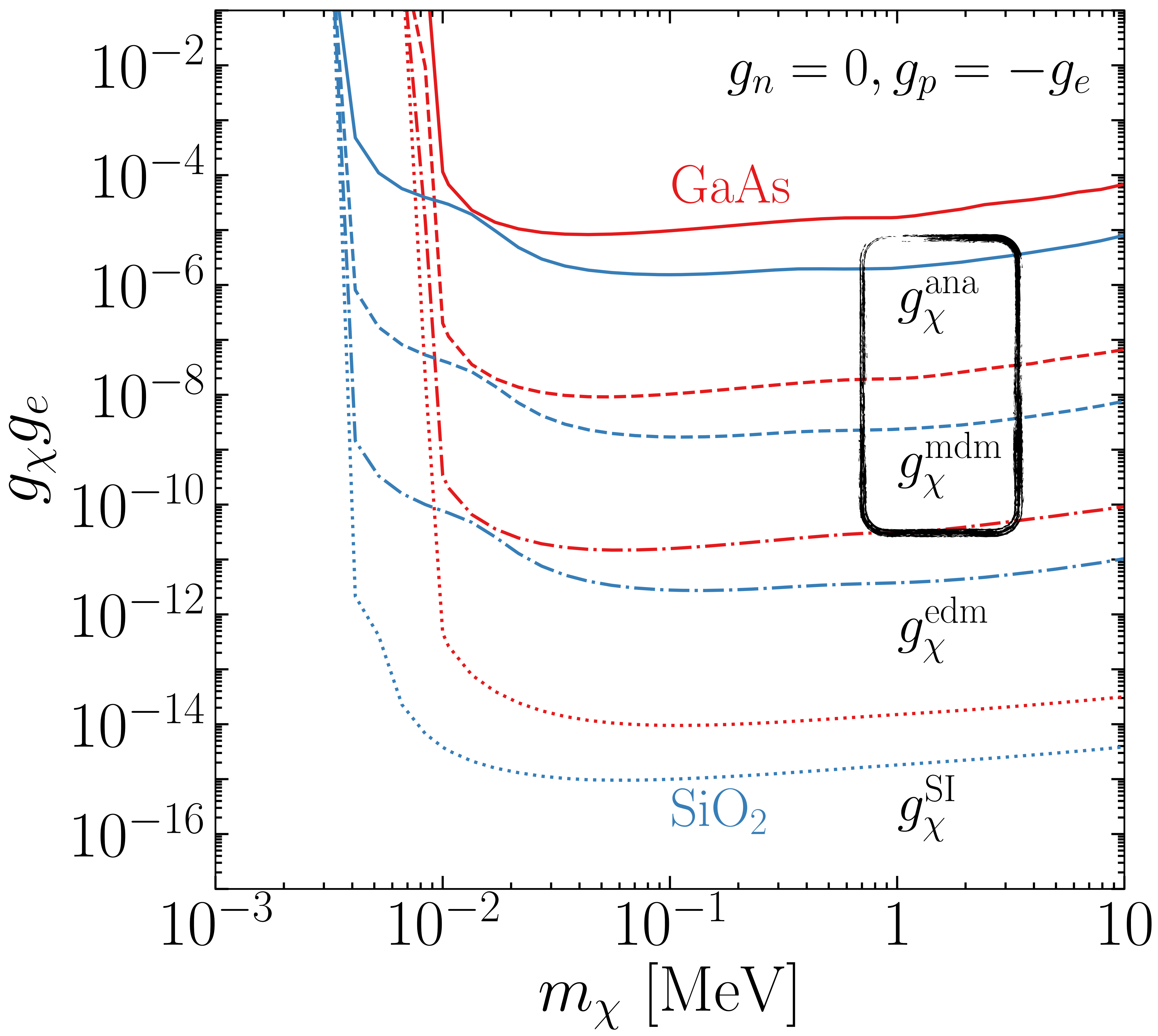
(DM couplings to particle numbers).

One of the targets used in SPICE
(Sub-eV Polar Interactions Cryogenic Experiment)
— part of the TESSERACT project, in R&D.

Optimal phonon target found in our theoretical study.

Example

➤ Phonon reach for kg-yr exposure, assuming background-free.

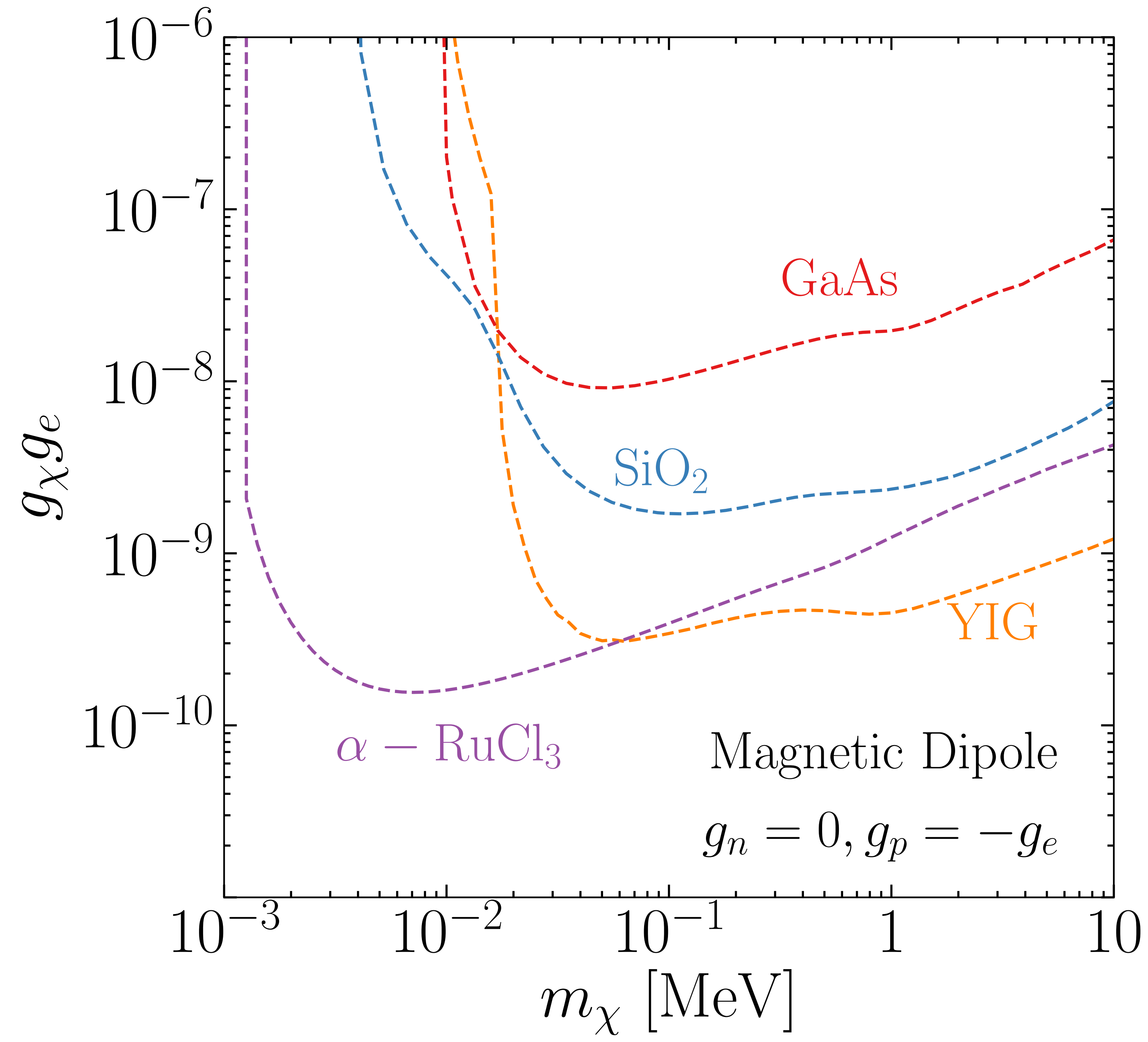


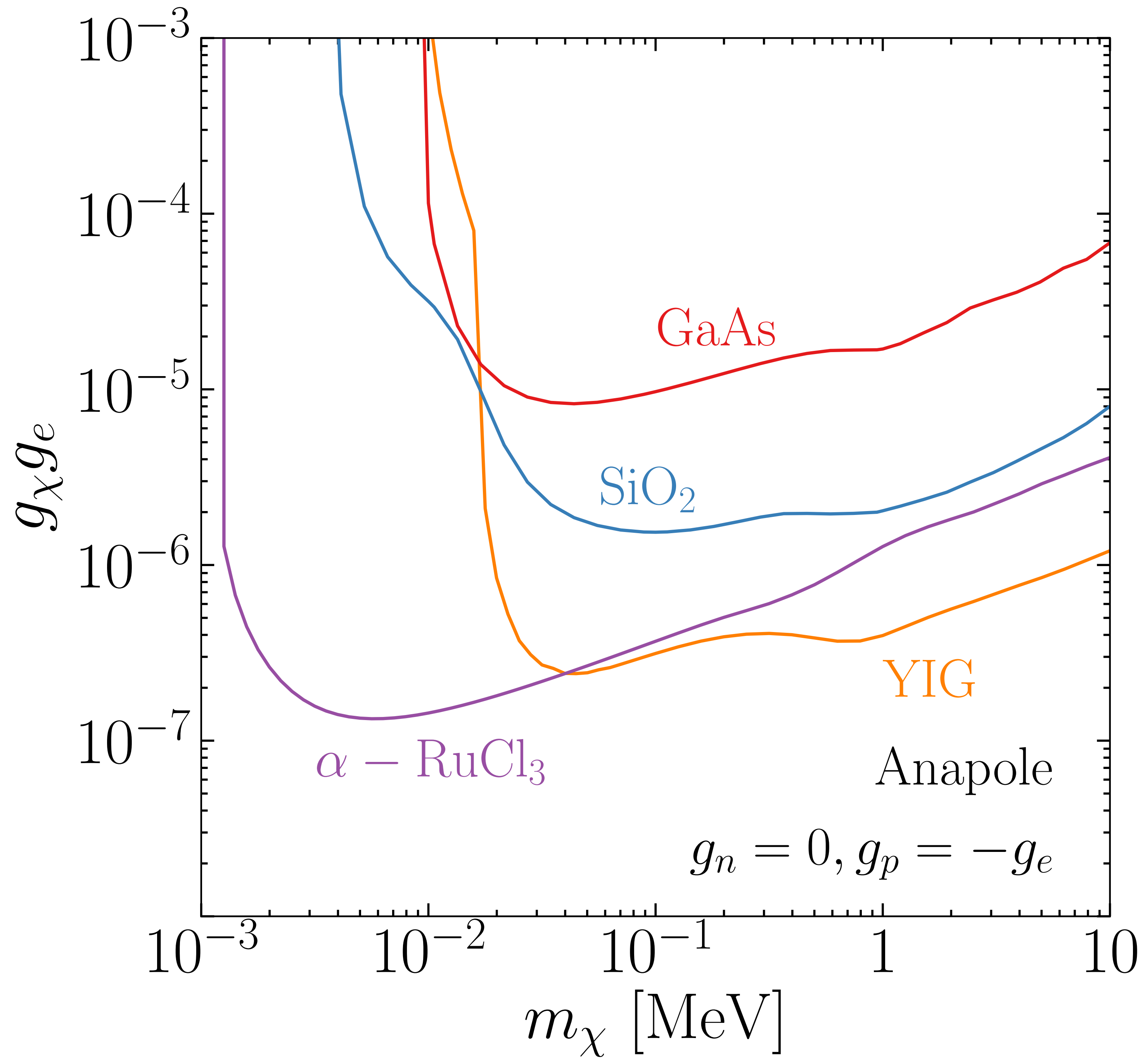
These models also generate couplings to ***S*** and ***L***.

⇒ Best probed by magnons.

Example

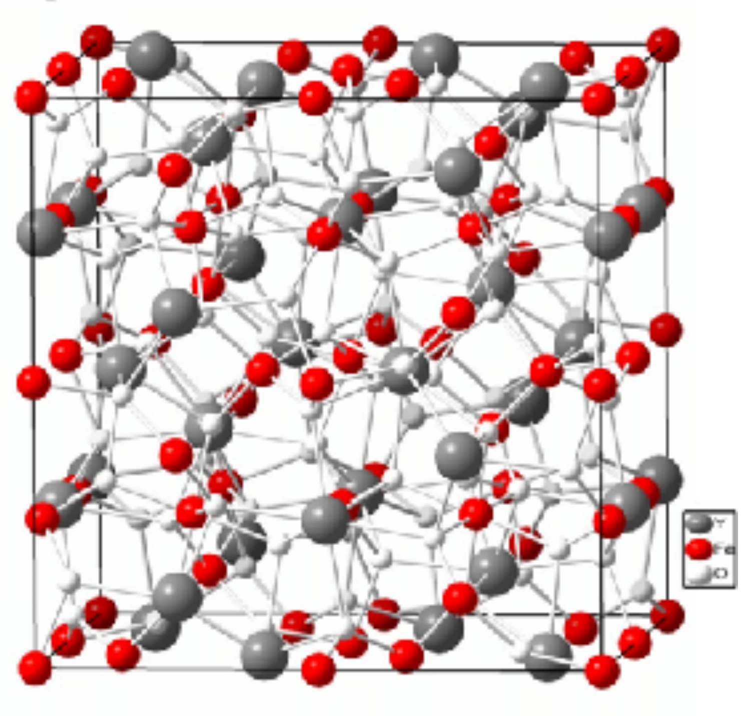
- Zoom in on these two models.
- Compare phonon reach (from previous plot) vs. magnon reach.





YIG

Yttrium iron garnet ($Y_3Fe_5O_{12}$).



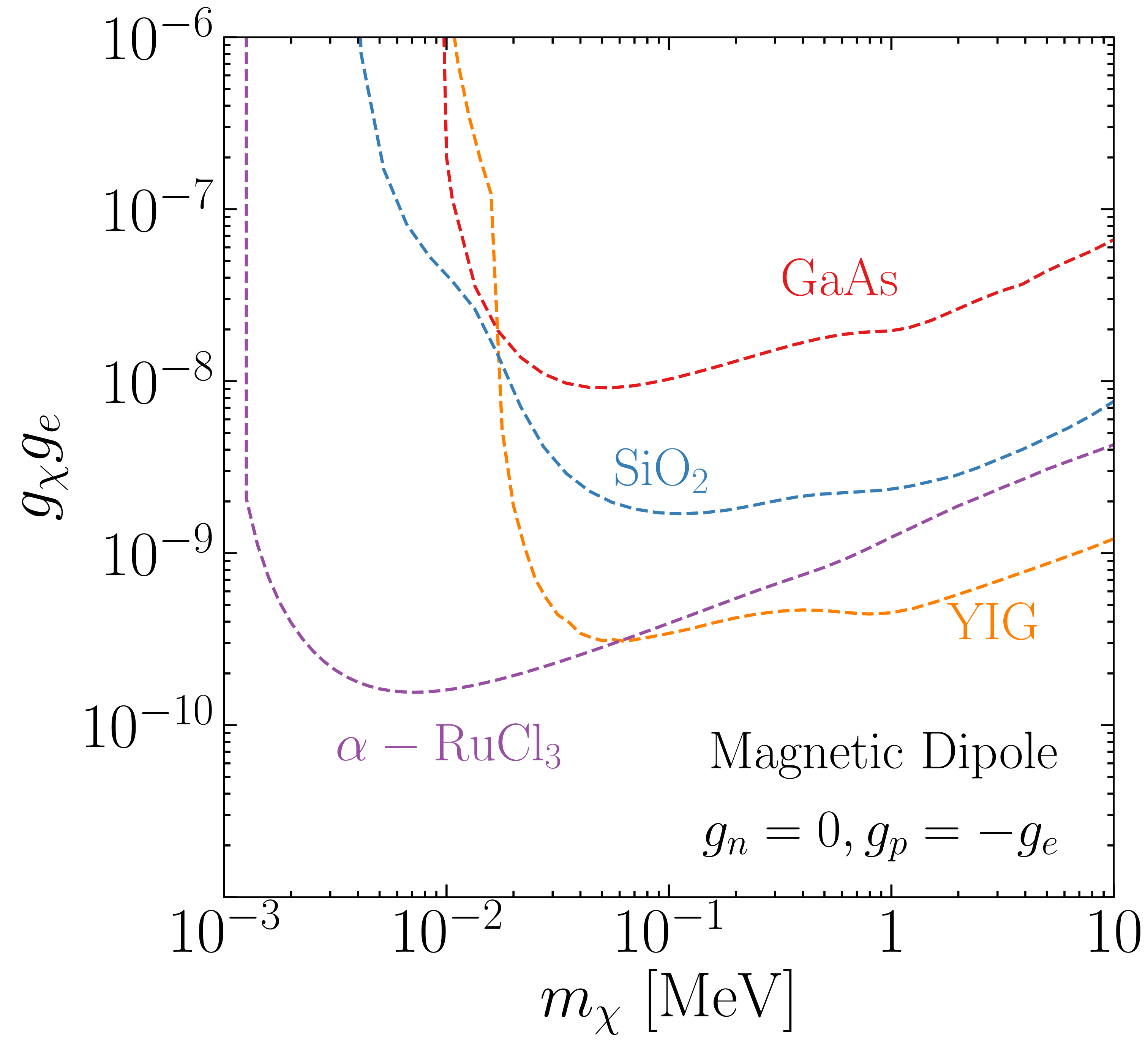
Well-studied ferrimagnetic material.

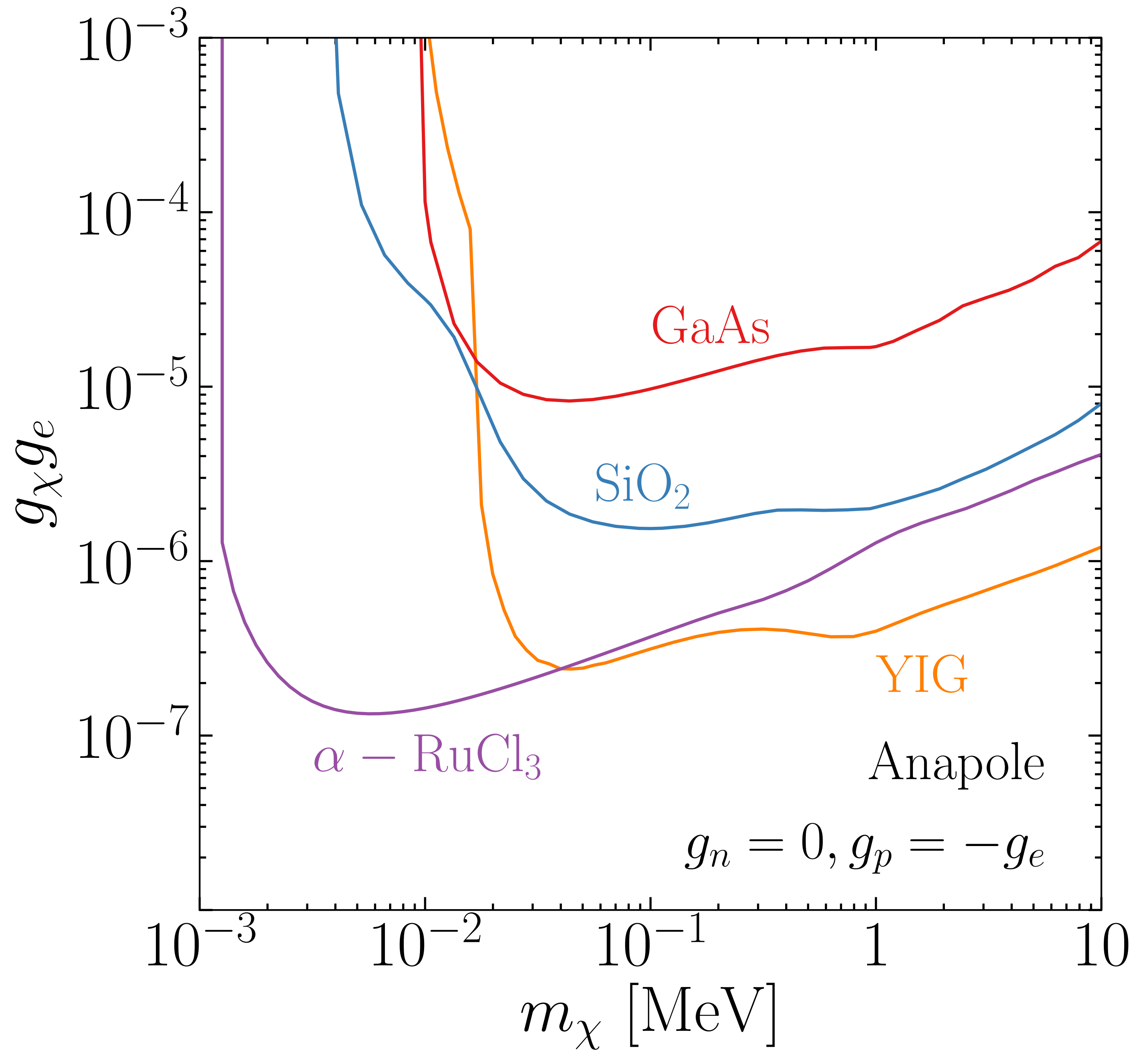
Heisenberg model involving <u>only $\boldsymbol{S}$</u> (not $\boldsymbol{L}$).

Used in QUAX (QUaere AXion) experiment for axion detection.

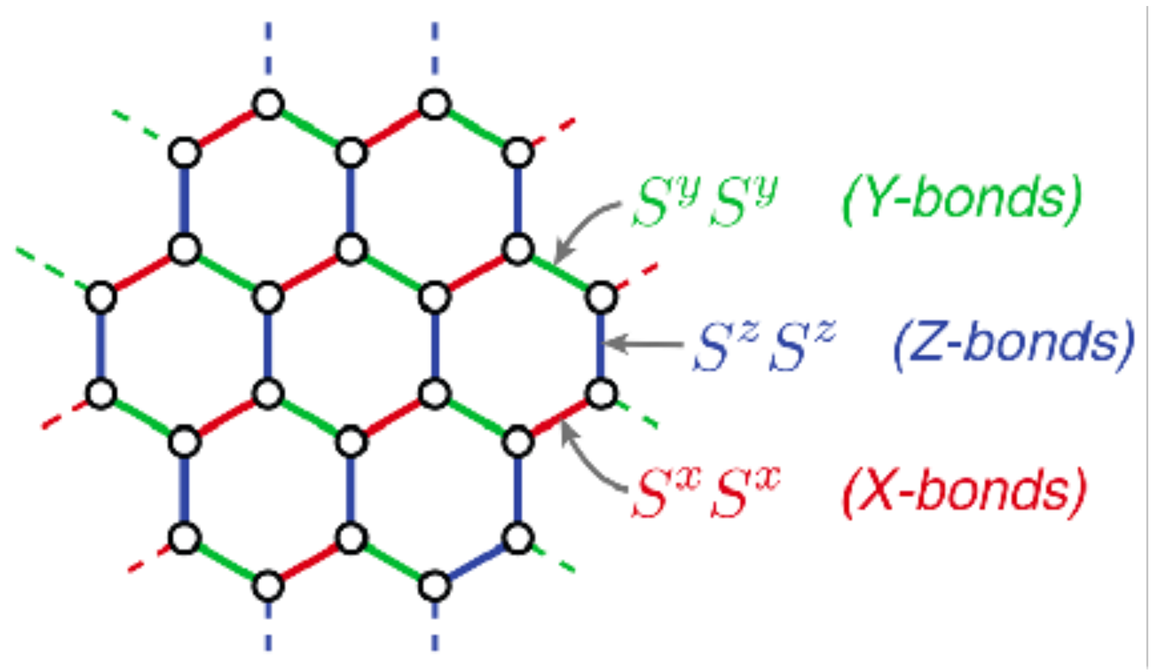
Example

- ➤ Zoom in on these two models.
- ➤ Compare phonon reach (from previous plot) vs. magnon reach.





α-RuCl$_3$



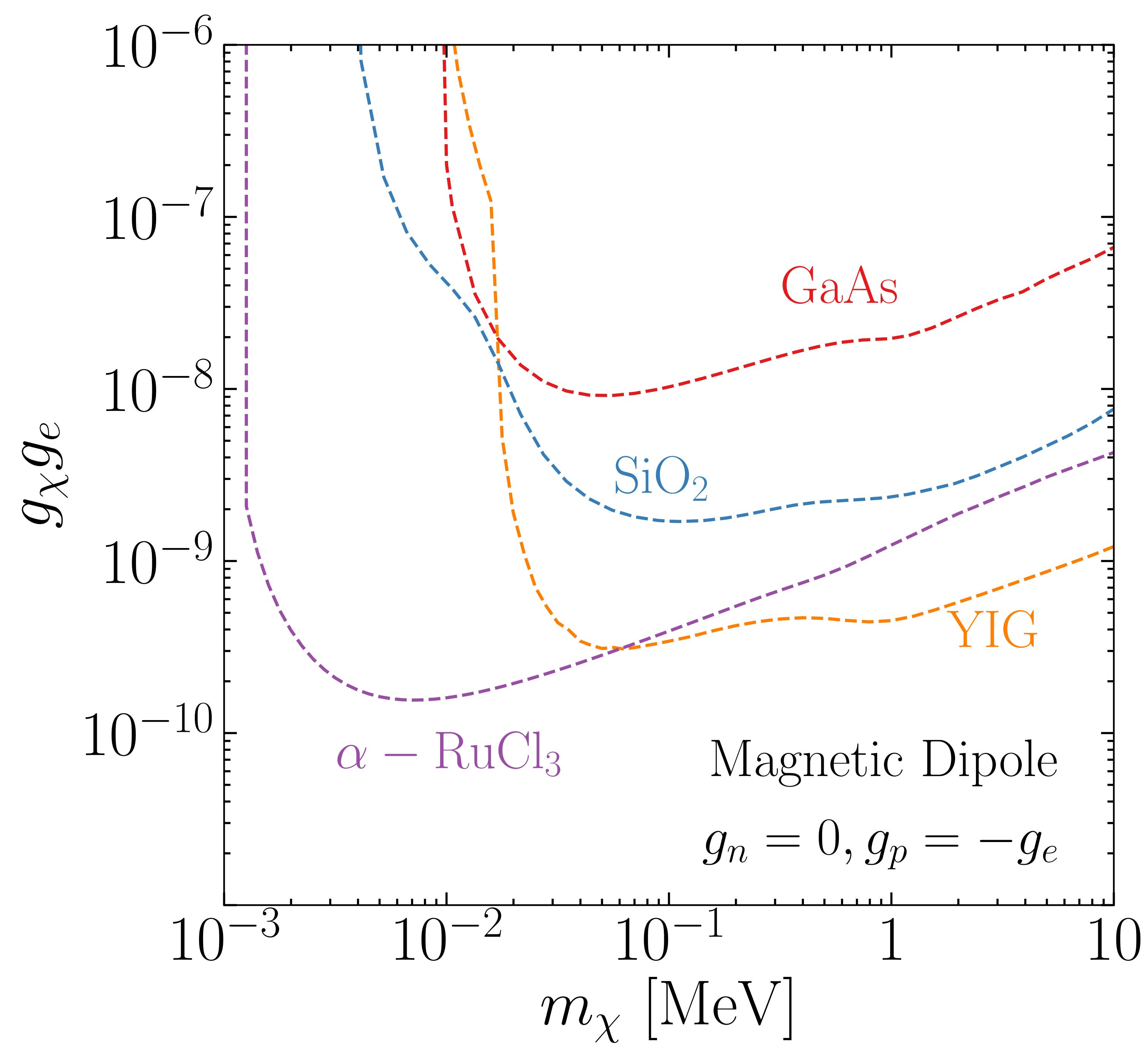
Kitaev material with bond directional coupling.

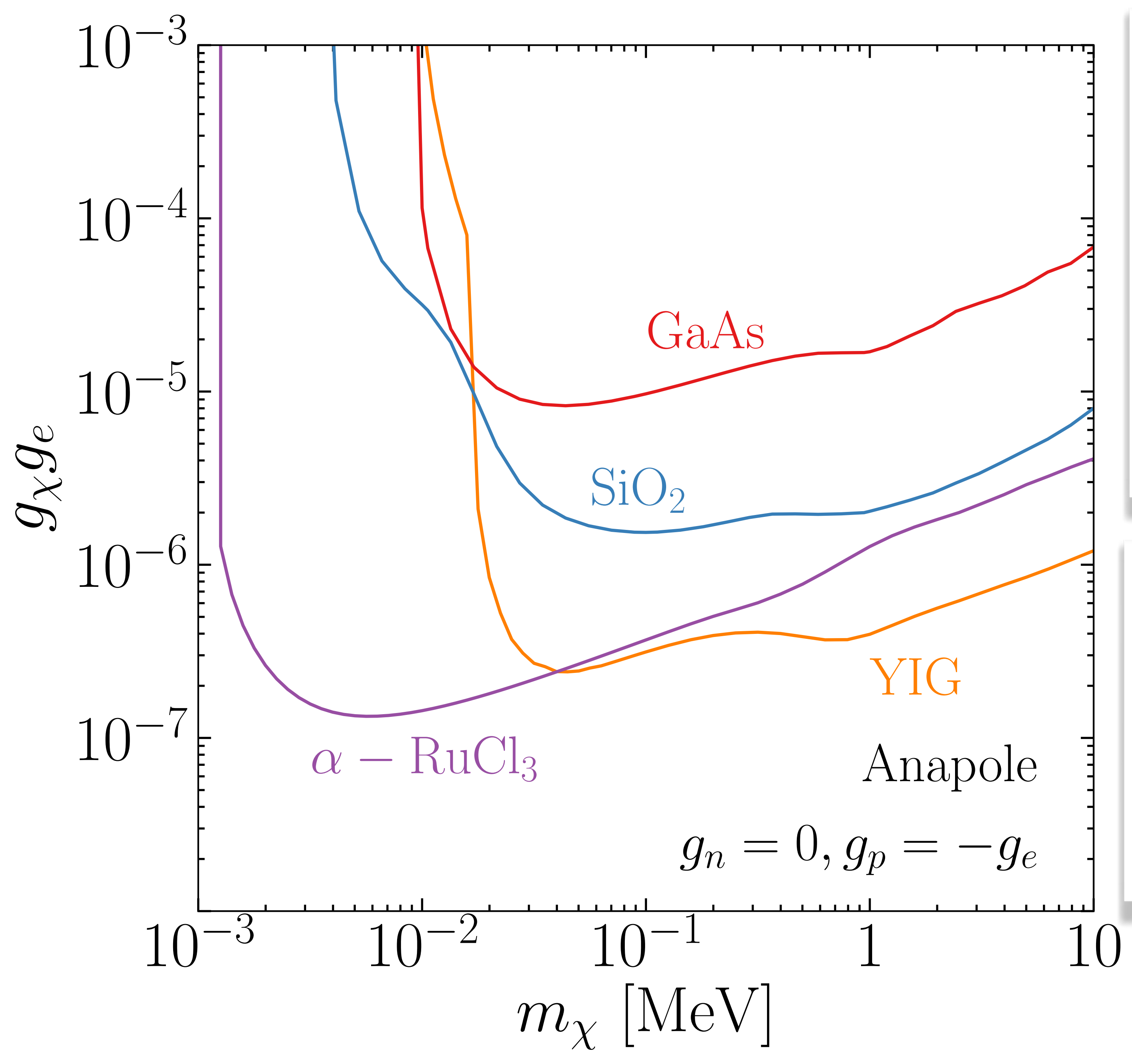
Antiferromagnetic order involves both ***S*** and ***L*** (spin-orbit entangled moments).

Very low energy (<10meV) gapped magnons $\Rightarrow$ reach lighter DM.

Example

- Zoom in on these two models.
- Compare phonon reach (from previous plot) vs. magnon reach.





Parametrically, the rates are given by:

$$R^{\rm mdm}_{\rm magnon} \sim \frac{\rho_\chi}{m_\chi}\frac{S_{\rm ion}}{m_{\rm cell}}\frac{g_\chi^2 g_e^2}{\varepsilon_\infty^2}\frac{1}{m_\chi^2 m_e^2}\int dq\, q \sim g_\chi^2 g_e^2 \frac{\rho_\chi}{m_\chi}\frac{S_{\rm ion} v^2}{\varepsilon_\infty^2 m_{\rm cell} m_e^2}$$

$$\frac{R^{\rm ana}_{\rm magnon}}{R^{\rm mdm}_{\rm magnon}} \sim v^2\,.$$

For both models,

$$\frac{R_{\rm phonon}}{R_{\rm magnon}} \sim \frac{Q_{\rm ion}^2 m_e^2 v^2}{S_{\rm ion} m_{\rm ion}\omega} \sim 10^{-3}\left(\frac{10\,\mathrm{GeV}\cdot 100\,\mathrm{meV}}{m_{\rm ion}\omega}\right)$$

- Magnon reach is parametrically better, but SiO_2 (optimal phonon target) is not too far behind.
- Encouraging for the technically more mature phonon experiments.

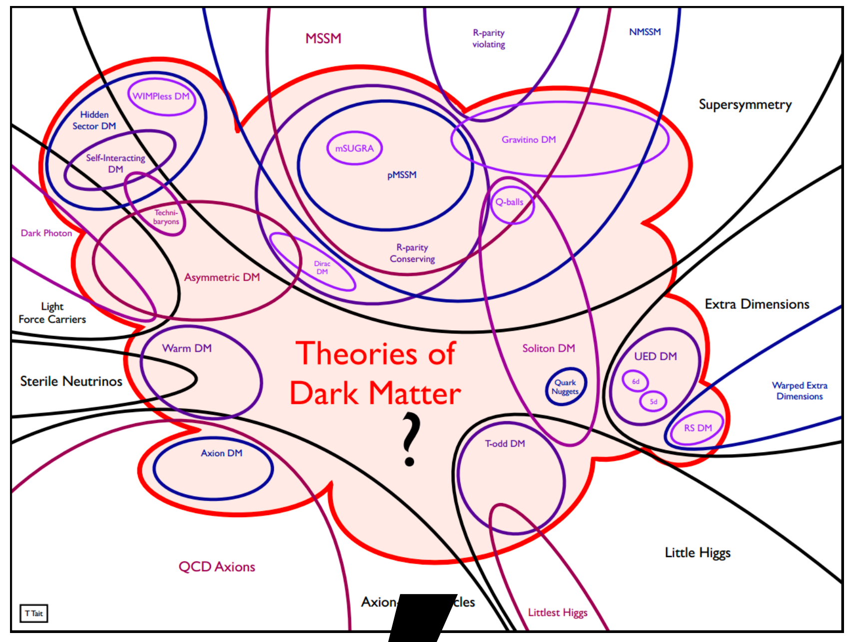
Take-home messages

Collective excitations such as phonons and magnons offer a novel path to detect light DM.

New experiments such as SPICE have broad discovery potential over the vast DM theory space.

We have developed the tools for computing detection rates for general DM models.

EFT of DM direct detection: summary



NR EFT of DM-SM interactions

Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_1^{(\psi)} = \mathbb{1}$ $\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{S}_\psi$ $\mathcal{O}_6^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)\left(\boldsymbol{S}_\psi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp$ $\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_{13}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{14}^{(\psi)} = \left(\boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{15}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$

Crystal responses

DM couplings to lattice d.o.f.

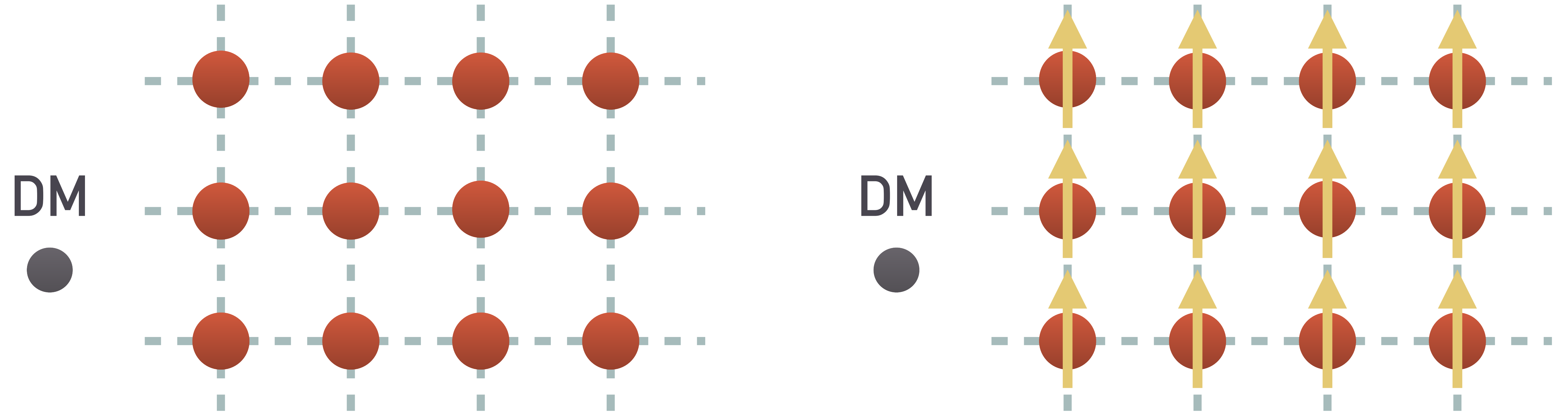
$\boldsymbol{N}$ (particle number)

$\boldsymbol{S}$ (spin)

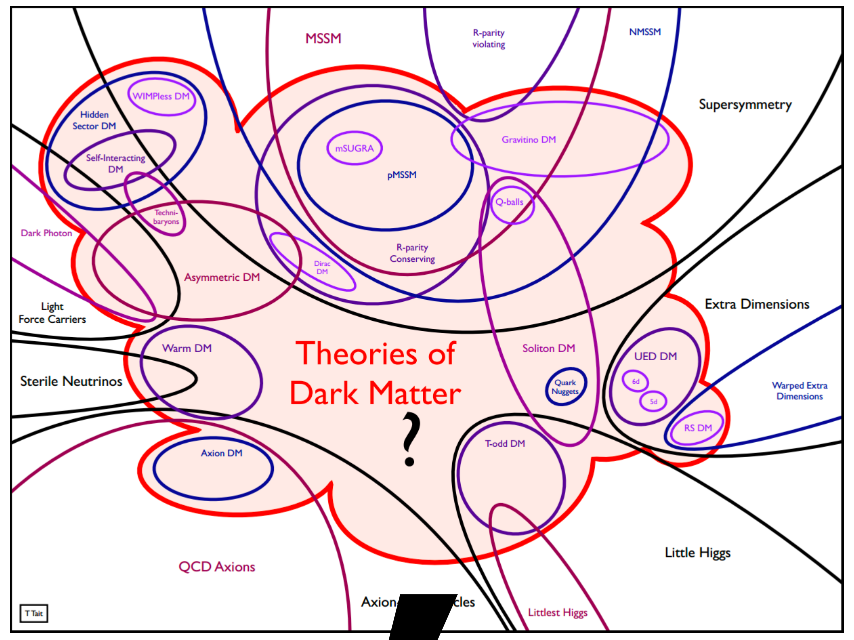
$\boldsymbol{L}$ (orbital angular momentum)

$\boldsymbol{L}\otimes\boldsymbol{S}$ (spin-orbit coupling)

Phonon & magnon excitation rates



EFT of DM direct detection: summary



NR EFT of DM-SM interactions

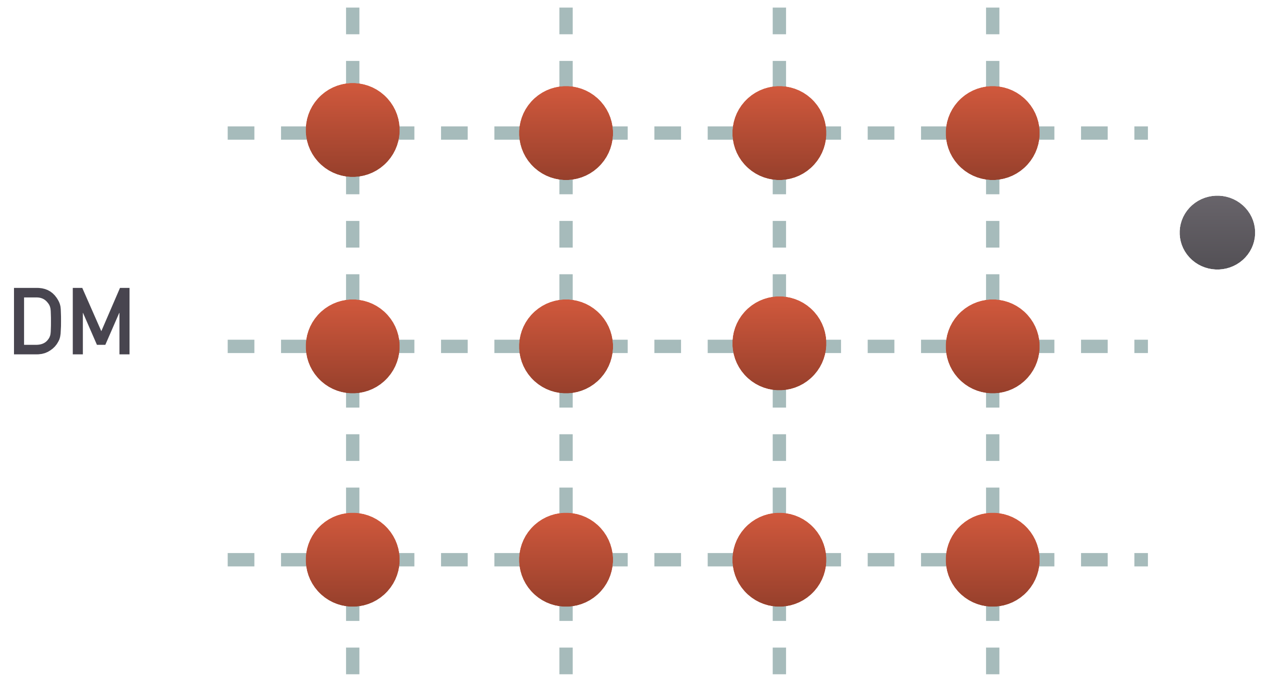
Coupling to *charge*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_1^{(\psi)} = \mathbb{1}$ $\mathcal{O}_{11}^{(\psi)} = \boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *charge*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_5^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_8^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*independent*	$\mathcal{O}_4^{(\psi)} = \boldsymbol{S}_\chi \cdot \boldsymbol{S}_\psi$ $\mathcal{O}_6^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)\left(\boldsymbol{S}_\psi \cdot \frac{\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_9^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{10}^{(\psi)} = \boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}$
Coupling to *spin*, $\boldsymbol{v}^\perp$-*dependent*	$\mathcal{O}_3^{(\psi)} = \boldsymbol{S}_\psi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_7^{(\psi)} = \boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp$ $\mathcal{O}_{12}^{(\psi)} = \boldsymbol{S}_\chi \cdot \left(\boldsymbol{S}_\psi \times \boldsymbol{v}^\perp\right)$ $\mathcal{O}_{13}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{14}^{(\psi)} = \left(\boldsymbol{S}_\psi \cdot \boldsymbol{v}^\perp\right)\left(\boldsymbol{S}_\chi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$ $\mathcal{O}_{15}^{(\psi)} = \left(\boldsymbol{S}_\chi \cdot \left(\frac{i\boldsymbol{q}}{m_\psi} \times \boldsymbol{v}^\perp\right)\right)\left(\boldsymbol{S}_\psi \cdot \frac{i\boldsymbol{q}}{m_\psi}\right)$

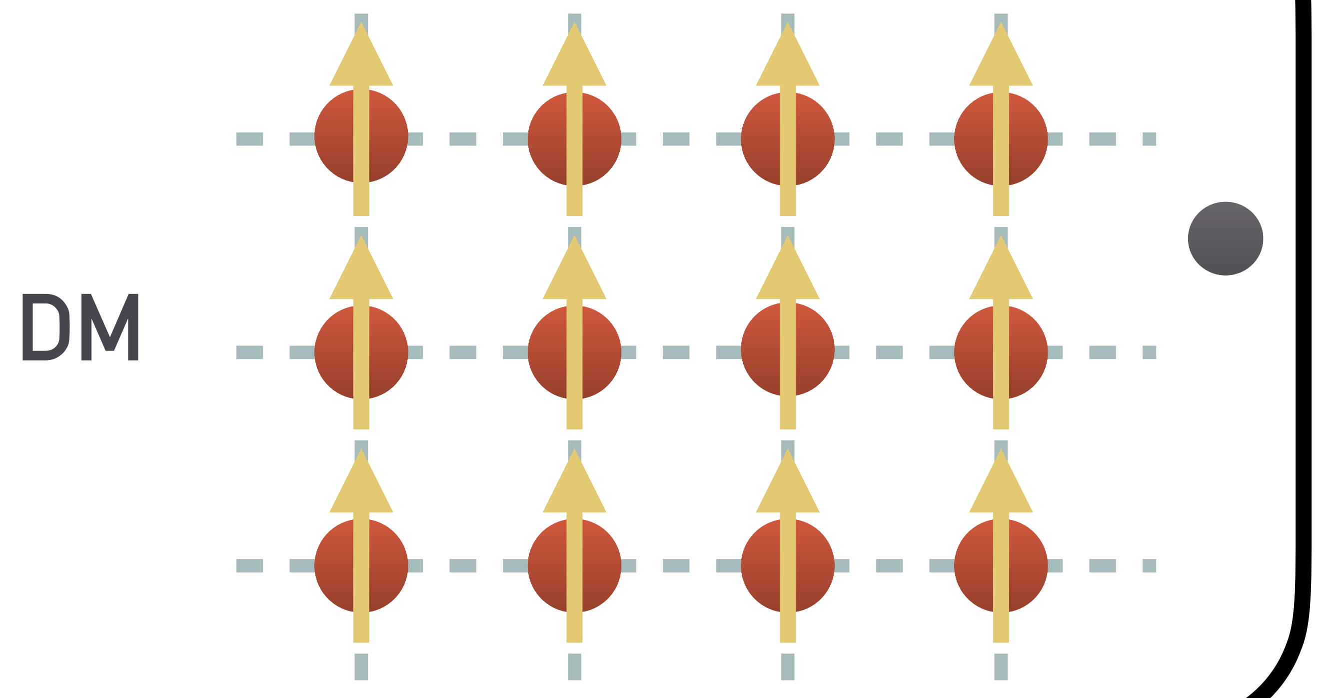
Crystal responses

<u>DM couplings to lattice d.o.f.</u>

$\boldsymbol{N}$ (particle number)	$\boldsymbol{S}$ (spin)
$\boldsymbol{L}$ (orbital angular momentum)	$\boldsymbol{L}\otimes\boldsymbol{S}$ (spin-orbit coupling)

Phonon & magnon excitation rates





THANK YOU