The missing final state puzzle in the monopole-fermion scattering

Ryutaro Matsudo

KEK Theory Center

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In collaboration with Ryuichiro Kitano

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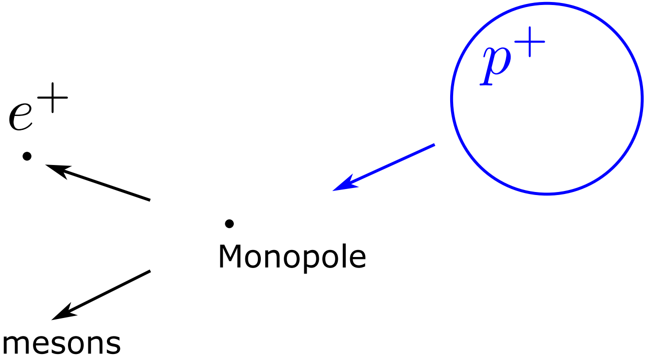
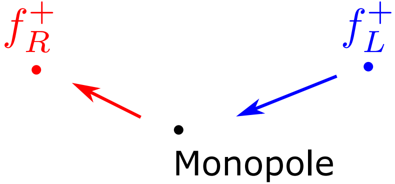
Rubakov-Callan effect



[V. A. Rubakov (1982), C. G. Callan (1982)]

- When the proton collides with a GUT monopole, it decays into a positron and mesons.
- The effect has been used to put constraint on the monopole flux in the Universe.

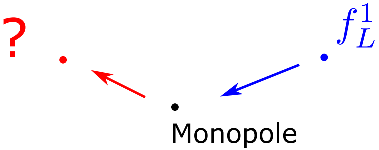
Helicity flip: One-flavor case



$$\langle \bar{f} f \rangle \propto \frac{1}{r^3}$$

- The helicity flips even in the massless QED.
- The only possible source of the fermion number violation is the chiral anomaly, which is a non-perturbative effect of QED.
- As the effect of the anomaly, the fermion condensate is nonzero.

Puzzle: Two-flavor massless case

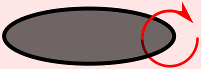


$$\langle \bar{f}^1 f^1 \rangle = 0$$

$$\langle (\bar{f}^1 f^1)(\bar{f}^2 f^2) \rangle \propto \frac{1}{r^6}$$

Any fermion cannot be the final state due to the flavor charge conservation.

? = Pancake (Wall with boundary)



Soliton of the phase of the fermion condensate

Outline

1. Missing final state puzzle
2. Review: S-wave approximation
3. Pancake soliton
4. Soliton picture of the scattering

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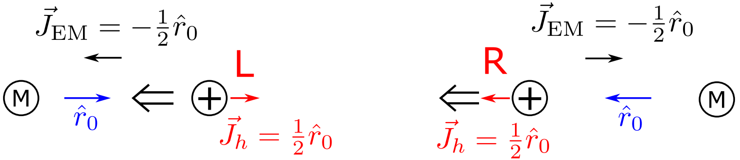
Helicity flip

- If there is a monopole and a charge, the electromagnetic field carries an angular momentum, ($\hat{r}_0$: The unit vector pointing from the monopole to the charge)

$$\vec{J}_{\mathrm{EM}} = \frac{1}{4\pi}\int d^3x\, \vec{r} \times (\vec{E} \times \vec{B}) = -\frac{1}{2}\hat{r}_0.$$

- If the incoming particle has a helicity $-1/2$ (left-handed), the total angular momentum is zero.
- After the scattering, the angular momentum from the electro magnetic field has the opposite direction to the particle momentum.
- To conserve the total angular momentum, the helicity of the particle has to flip.



Set up

- We consider an $SU(2)$ gauge theory with an adjoint Higgs and 4 flavors of Weyl fermions, where $SU(2)$ is spontaneously broken down to $U(1)$.
- The global symmetry is $SU(4)$.

$$SU(2)\text{ doublets}\left\{\overbrace{\left(\begin{pmatrix}a_1^+\\b_1^-\end{pmatrix},\begin{pmatrix}a_2^+\\b_2^-\end{pmatrix},\begin{pmatrix}a_3^+\\b_3^-\end{pmatrix},\begin{pmatrix}a_4^+\\b_4^-\end{pmatrix}\right)}^{SU(4)\text{ quadruplets (fund. rep.)}}\right.$$

- The theory can be regarded as an approximation of $SU(5)$ GUT:

$$\begin{pmatrix}a_1\\b_1\end{pmatrix}=\begin{pmatrix}e_L^+\\d_L^3\end{pmatrix},\quad \begin{pmatrix}a_2\\b_2\end{pmatrix}=\begin{pmatrix}\bar{d}_L^3\\e_L^-\end{pmatrix},\quad \begin{pmatrix}a_3\\b_3\end{pmatrix}=\begin{pmatrix}u_L^1\\\bar{u}_L^2\end{pmatrix},\quad \begin{pmatrix}a_4\\b_4\end{pmatrix}=\begin{pmatrix}\bar{u}_L^2\\\bar{u}_L^1\end{pmatrix}.$$

The low energy effective theory

- We approximate the theory as the gauge theory of the unbroken $U(1)$.

$$\begin{aligned} a &: \text{The } U(1) \text{ gauge field,} \\ a_j &: \text{The Weyl fermions with charge } +1, \\ b_j &: \text{The Weyl fermions with charge } -1. \end{aligned}$$

- The 't Hooft-Polyakov monopole is approximated by the (background) Dirac monopole.
- X, Y bosons, GUT Higgs bosons are considered to be infinitely heavy.

The missing final state puzzle

- The helicity, the $U(1)$ charge and the representation of $SU(4)$ are

$$a_j : (L, +1, \Box), \quad b_j : (L, -1, \Box), \quad \bar{a}_j : (R, -1, \bar{\Box}), \quad \bar{b}_j : (R, +1, \bar{\Box})$$

- If the initial state is a_j, the final state should be a particle with $(R, +1, \Box)$. However, there are no particle with this quantum number.
- The s-wave approximation implies that, when the initial state is a_1, the final state is something like

$$\frac{b_1}{2} + \frac{\bar{b}_2}{2} + \frac{\bar{b}_3}{2} + \frac{\bar{b}_4}{2}.$$

$b_j/2$: "Semiton", the state with halves of the $U(1)$ charge, the flavor charge and the spin of b_j.

It is hard to interpret what this "semiton state" actually is.

Interpretation of the final state

Probabilistic interpretation [C. G. Callan (1984), V. A. Rubakov (1988)]:

- In the $SU(5)$ GUT,

$$a_1 = e_L^+, \quad a_2 = \overline{d}_L^3, \quad a_3 = u_L^1, \quad a_4 = u_L^2,$$
$$b_1 = d_L^3, \quad b_2 = e_L^-, \quad b_3 = \overline{u}_L^2, \quad b_4 = \overline{u}_L^1.$$

- "The semiton state" is

$$\frac{1}{2}e_R^+ + \frac{1}{2}u_R^1 + \frac{1}{2}u_R^2 + \frac{1}{2}d_L^3$$

- The state is interpreted as

$$\frac{1}{\sqrt{2}}\left|e_R^+, \mathrm{M}\right\rangle + \frac{1}{\sqrt{2}}\left|u_R^1 u_R^2 d_L^3, \mathrm{M}\right\rangle \quad ?$$

  This is problematic because $e_R^+ = \bar{b}_2$ is in the antifundamental representation of $SU(4)$, i.e., the flavor charge is not conserved.

- If massless QED is unitary, there must be a final state.
- The non-perturbative effect of QED can make an appropriate final state.

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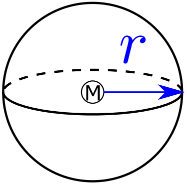
S-wave approximation and bosonization

- The final state is obtained as the soliton in the bosonized theory in the s-wave approximation, where we only consider the spherically symmetric fields.



$$S_{4d} = \int d^4x \left[-\frac{1}{4g^2} f \star f + \sum_{j=1}^{4} \left(i\bar{a}_j\, \bar{\sigma}^\mu D_\mu a_j + i\bar{b}_j\, \bar{\sigma}^\mu D_\mu b_j \right) \right],$$

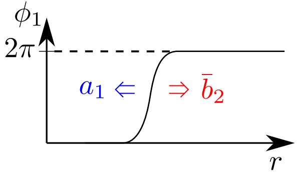
$$\xrightarrow{\text{S-wave approximation, bosonization}}$$

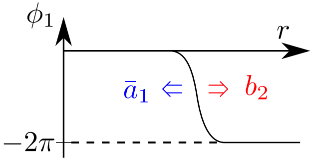
$$S_{2d} = \int_0^\infty dr \int_{-\infty}^{\infty} dt \left[\frac{1}{2\pi} \sum_{i=1}^{4} ((\partial_t \phi_i)^2 - (\partial_r \phi_i)^2) - \frac{g^2}{32\pi^3} \frac{1}{r^2} (\sum_{i=1}^{4} \phi_i)^2 \right],$$

Fermions are kinks

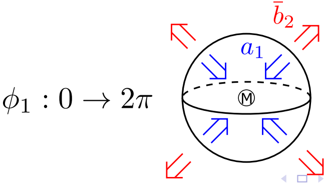
- The fermions correspond to the kink solitons.

$$\text{Kinks} = \text{Fermions with the charge } +1$$
$$\text{Anti-kinks} = \text{Fermions with the charge } -1$$
$$\text{Incoming (anti-)kinks} = a_j \ (\bar{a}_j)$$
$$\text{Outgoing (anti-)kinks} = \bar{b}_j \ (b_j)$$





- The kink corresponds to the s-wave state of the fermion in 4d.



Currents and boundary conditions

- The currents of $U(1)$ and the maximal torus of $SU(4)$ is ($\alpha, \beta = t, r$)

$$
\begin{aligned}
4\pi r^2 J^\alpha &= \frac{1}{2\pi} \sum_j \varepsilon_{\alpha\beta} \partial_\beta \phi_j, \\
4\pi r^2 J^\alpha_{(1,-1,0,0)} &= \frac{1}{2\pi} \varepsilon_{\alpha\beta} \partial_\beta (\phi_1 - \phi_2), \\
4\pi r^2 J^\alpha_{(0,0,1,-1)} &= \frac{1}{2\pi} \varepsilon_{\alpha\beta} \partial_\beta (\phi_3 - \phi_4), \\
4\pi r^2 J^\alpha_{(1,1,-1,-1)} &= \frac{1}{2\pi} \partial_\alpha (\phi_1 + \phi_2 - \phi_3 - \phi_4).
\end{aligned}
$$

  $J^\alpha_{(\dots)}$ is the current of $U(1)$ generated by $\mathrm{diag}(\dots)$.
- The boundary condition is determined so that any charge does not flow into the infinitesimal region around the monopole to prohibit the existence of the dyon:

$$
4\pi r^2 J^r = 0, \quad 4\pi r^2 J^r_{(\dots)} = 0 \text{ at } r = 0,
$$

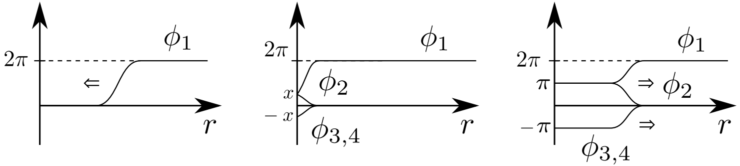
  which implies

$$
\phi_1 + \phi_2 + \phi_3 + \phi_4 = 0,\ \phi_1 = \phi_2,\ \phi_3 = \phi_4,\ \partial_r(\phi_1 + \phi_2 - \phi_3 - \phi_4) = 0.
$$

Scattering process

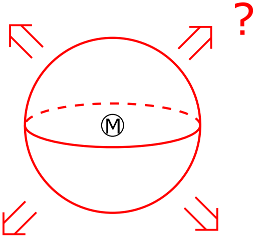
- When the initial state is a_1, the final state corresponds to the soliton:

$$\phi_1(0) = \phi_2(0) = \pi, \quad \phi_1(\infty) = 2\pi, \quad \phi_2(\infty) = 0, \quad \phi_3(0) = \phi_4(0) = -\pi,$$
$$\phi_3(\infty) = \phi_4(\infty) = 0.$$



- The final state seems to be $b_1/2 + \bar{b}_2/2 + \bar{b}_3/2 + \bar{b}_4/2$, which is hardly interpreted.

$$\begin{cases} \phi_1 : \pi \to 2\pi \\ \phi_2 : \pi \to 0 \\ \phi_3 : -\pi \to 0 \\ \phi_4 : -\pi \to 0 \end{cases}$$



Massive case: No puzzle

In the massive case, this puzzle disappears.

- When we introduce the mass term,

$$m_1(a_1 b_2 + a_2 b_1 + \text{h.c.}) + m_2(a_3 b_4 + a_4 b_3 + \text{h.c.}),$$

- The global symmetry reduces to $U(1) \times U(1)$, which is generated by

$$H_1 = \text{diag}(1, -1, 0, 0), \quad H_2 = \text{diag}(0, 0, 1, -1),$$

  and the charge correspond to

$$H_3 = \text{diag}(1, 1, -1, -1)$$

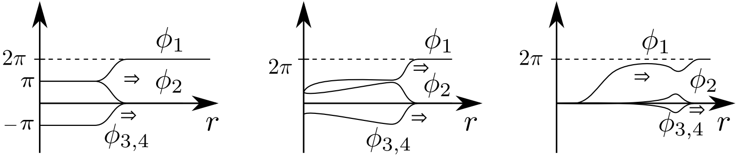
  is not conserved.

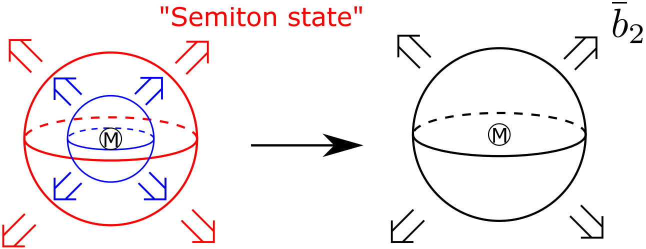
- There is a candidate of the final state

$$a_1 : (L, +1, Q_{(1,-1,0,0)} = 1, Q_{(0,0,1,-1)} = 0)$$
$$\to \bar{b}_2 : (R, +1, Q_{(1,-1,0,0)} = 1, Q_{(0,0,1,-1)} = 0).$$

Numerical result in massive case

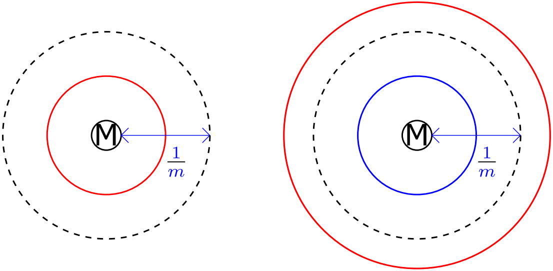
According to the numerical simulation [S. Dawson & A. N. Schellekens (1983)], the scattering process is as follows :





What happens when we take the massless limit?

- In the scattering process, when the semitons reach $r \sim 1/m$, the values of ϕ_i at the core start to change.
- This means that, in the region where $r \ll 1/m$, the theory can be regarded as the massless theory.
- In the massless limit, every point is near the monopole.



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Fermion condensates

- In the s-wave approximation, the final state is described as a soliton in the bosonized theory.
  $\Rightarrow$ To interpret this in the 4d theory, a "bosonized" picture in 4d is needed.
- The effective theory of the phases of the fermion condensates can be regarded as such a theory.
- Fact: In the monopole background, operators of the fermion fields have nonzero expectation values:

$$\left\langle (\overline{b}^i \overline{\sigma}_\mu a_j)(\overline{a}^k \overline{\sigma}^\mu b_l) \right\rangle = \frac{1}{r^6}(c_1 \delta^i_j \delta^k_l + c_2 \delta^i_l \delta^k_j),$$

$$\langle (a_{i_1} b_{i_2})(a_{i_3} b_{i_4}) \rangle = \frac{1}{r^6} c_3 \varepsilon_{i_1 i_2 i_3 i_4}.$$

- The condensation is the seed of the helicity flip.
- By "integrating in" the phases of these condensates, and integrating out the fermions fields, we obtain the effective theory of the phases.

- The variables of the effective theory is the following four:

$$\theta_A : \text{the phase of } (a_1 b_2)(a_3 b_4), \quad \theta_{1j} : \text{the phase of } (\bar{b}^1 \bar{\sigma}_\mu a_1)(\bar{a}^j \bar{\sigma}^\mu b_j)$$

- It is convenient to express them using the phases of the fermion fields. Let α_j be the phase of a_j and β_j be that of b_j, which means

$$a_1 \to e^{i\theta} a_1 \quad \text{corresponds to } \alpha_1 \to \alpha_1 + \theta.$$

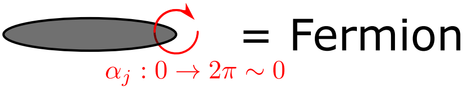
- The genuine variables are expressed as

$$\theta_A = \sum_j (\alpha_j + \beta_j), \quad \theta_{1j} = \alpha_1 - \beta_1 - \alpha_j + \beta_j \quad \text{for } j = 2, 3, 4.$$

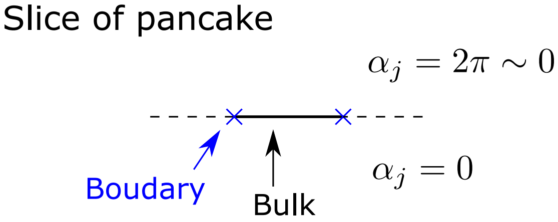
- There are configurations of α_j and β_j that represent an identical configuration of the phases of the condensates

Pancakes

- We find that pancake configurations of the phases can be fermions by determining the quantum numbers of the object.



- The pancake is a $2+1$ dimensional object.
- On the bulk of the pancake, the value of α_j (β_j) gradually changes from 0 to 2π.
- α_j (β_j) winds around the boundary of the pancake.
  $\Rightarrow$ The boundary is a string of the phase.



The effective theory of the phases of the fermion condensates

- Due to the chiral anomaly, the phase shift of the fermion causes the shift of the Lagrangian:

$$a_j \to e^{i\theta} a_j \quad \Rightarrow \quad \mathcal{L} \to \mathcal{L} + \frac{1}{8\pi^2}\theta f \wedge f$$
$$b_j \to e^{i\theta} b_j \quad \Rightarrow \quad \mathcal{L} \to \mathcal{L} + \frac{1}{8\pi^2}\theta f \wedge f$$

- To reproduce this, the effective theory of α_j, β_j has to contain the term,

$$\frac{1}{8\pi^2}\sum_j \left((\alpha_j + \beta_j)(f \wedge f)\right).$$

- The $U(1)$ current is read off as

$$J^\mu = \varepsilon^{\mu\nu\rho\sigma}\frac{1}{4\pi^2}\sum_j \partial_\nu(\alpha_j + \beta_j) f_{\rho\sigma}.$$

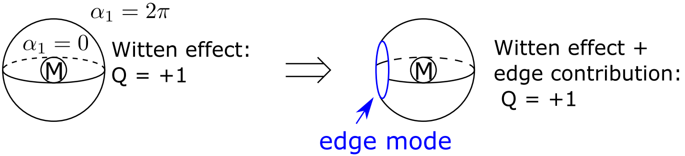
Monopole bag and pancake

- Let us consider the monopole surrounded by the wall of α_1.
- Witten effect: Because there is a monopole, $f = \sin\theta\, d\theta d\varphi/2$, the charges are

$$J^\mu = \varepsilon^{\mu\nu\rho\sigma} \frac{1}{4\pi^2} \sum_j \partial_\nu(\alpha_j + \beta_j) f_{\rho\sigma},$$

$$\Rightarrow \quad Q = \int d^3x\, J^0 = \frac{1}{2\pi} \sum_j \int_0^\infty dr\, \partial_r(\alpha_j + \beta_j) = 1.$$

- This object corresponds to the kink in the s-wave theory.
- The wall with a boundary is the pancake. There are edge modes that contribute to the charge so that the total charge is an integer.



The edge state

- By substituting the 2π jump of α_j into the effective Lagrangian $\sum_j(\alpha_j+\beta_j)f\wedge f/(8\pi^2)$, we obtain the Chern-Simons theory as the theory on the wall:

$$\frac{1}{4\pi}\int a\wedge f$$

- When the wall has the boundary, the CS theory is not gauge invariant.
  $\Rightarrow$ There has to be a chiral edge mode:

$$\frac{1}{4\pi}\int_{\mathbb{R}\times D^2} a\wedge f+\frac{1}{4\pi}\int_{\partial D^2}\left(D_x\phi(D_t\phi+vD_x\phi)\,dxdt-\phi f\right),$$
$$D\phi := d\phi - a,\quad a\to a+d\lambda,\quad \phi\to\phi+\lambda.$$

- ϕ is a 2π-periodic scalar, thus we can define the winding number of ϕ,

$$\frac{1}{2\pi}\int_{\partial D^2} d\phi.$$

- The pancake with the exited edge state with this winding number ± 1 can be considered as a fermion.

The charge of the edge state

- The $U(1)$ charge is

$$Q = \frac{1}{2\pi}\int_{\partial D^2}(d\phi - a) + \frac{1}{2\pi}\int_{D^2} f.$$

  In the gauge where the Dirac string does not penetrate the pancake, the charge is given as a winding number of ϕ around the edge.
- The quantum eigenstate of $\int_{\partial D^2} d\phi/2\pi$ with the eigenvalue $+1$ (-1) in the edge theory has the charge ± 1.
- Classically, the state corresponds to the solution of the eq. of motion of the edge theory. If we neglect the gauge fields, it is

$$\phi = \pm 2\pi(x - vt)/L.$$

- By introducing the background fields of the maximal torus of $SU(4)$, we can confirm that the edge state also have the flavor charge corresponding to the fundamental representation.

The spin of the pancake

- The spin of the object is given as the generator of the translation along the edge:

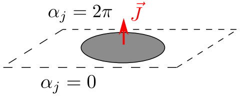
$$J^z = \frac{L}{2\pi} P_x = \frac{L}{8\pi^2} \int_0^L dx\, (\partial_x \phi)^2,$$

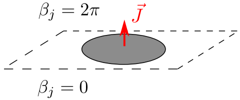
where we neglect the gauge fields.

- By substituting the solution $\phi = \pm 2\pi(x - vt)/L$, we obtain

$$J^z = \frac{1}{2}.$$

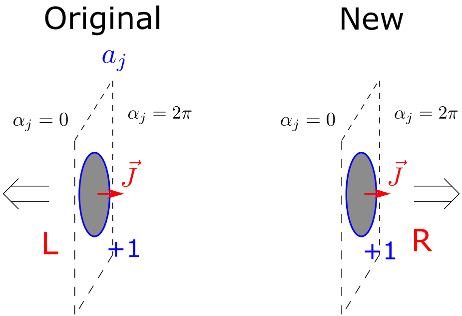
- The direction of the spin depends only on the orientation of the wall, and does not depend on the charge.





Original and new fermions

- The pancakes can be regarded as fermions.
- Some of them have opposite helicity to the original fermions.
  $\Rightarrow$ New fermions!
- The final state of the monopole-fermion scattering is identified with the new fermions.



Massive case

- We introduce the mass term

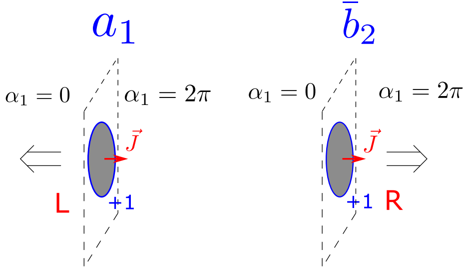
$$m_1(a_1 b_2 + a_2 b_1 + \text{h.c.}) + m_2(a_3 b_4 + a_4 b_3 + \text{h.c.}).$$

- The global symmetry reduces to $U(1) \times U(1)$, whose generators are

$$H_1 = \text{diag}(1, -1, 0, 0), \quad H_2 = \text{diag}(0, 0, 1, -1).$$

- There are no quantum number corresponding to $H_3 = \text{diag}(1, 1, -1, -1)$.

a_1: $(L, +1, Q_{H_1} = 1, Q_{H_2} = 0)$
$\bar{b}_2$: $(R, +1, Q_{H_1} = 1, Q_{H_2} = 0)$



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Boundary condition at the core of the monopole

- In the process of the scattering, the boundary condition at the core of the monopole plays an important role.
- For the action to be finite, it has to be satisfied that

$$\sum_j (\alpha_j + \beta_j) = 0,$$

- For the background gauge field of $SU(4)$ to be coupled to the theory, it has to be satisfied that

$$\alpha_j - \beta_j - \alpha_k + \beta_k = 0, \quad \forall j, k.$$

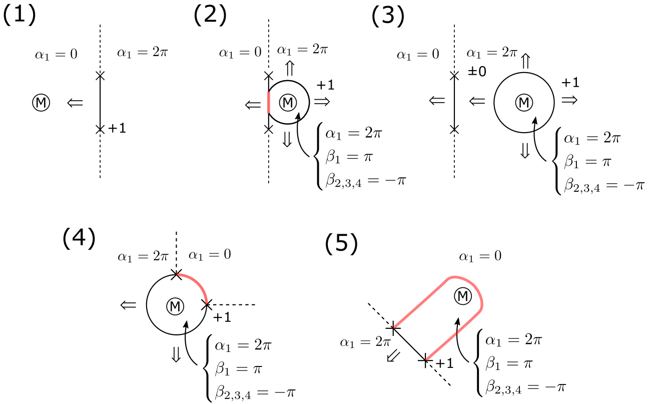
- When α_1 changes from 0 to 2π at the core, the β_j change as

$$\beta_1 = \alpha_1/2, \quad \beta_2 = \beta_3 = \beta_4 = -\alpha_1/2.$$

to maintain the boundary condition.

Pancake-monopole scattering

- The wall of the red line corresponds to an $SU(4)$ rotation (associated with a $U(1)$ gauge transformation) of the phases, and thus it is transparent, i.e., there is no object actually.



Massive case

- In the massive case, there appears additional condensates:

$$\langle a_j b_{j+1} \rangle \propto \frac{m^3}{r^6}, \quad \langle a_{j+1} b_j \rangle \propto \frac{m^3}{r^6}, \quad \text{for odd } j.$$

- In the massive case, the region

$$\alpha_1 = 2\pi, \quad \beta_1 = \pi, \quad \beta_2 = \beta_3 = \beta_4 = -\pi$$

  is no longer the vacuum, because the additional condensates change.

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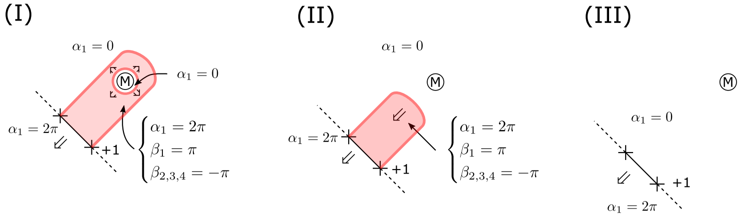


Summary

- When a charged fermion collides with a monopole, the helicity of the s-wave component of the fermion has to flip irrespective of what the UV theory is.
- The only possible source of this flip is the chiral anomaly. This means that when there is a monopole, the effect of the anomaly is not suppressed in QED.
- As a consequence, there is a fermion condensate violating the fermion number conservation.
- Puzzle: If there are two flavors of massless Dirac fermions, any fermion cannot be the final state of the monopole-fermion scattering due to the flavor charge conservation.
- We solve this puzzle by identifying the final state as the new fermion, which is the soliton of the fermion condensates in the monopole background.

Backup

The $U(1)$ charge and the $SU(4)$ charges in s-wave approximation

- The $U(1)$ charge is

$$Q = \frac{1}{2\pi} \sum_j \int_0^\infty dr(\partial_r \phi_j) = \frac{1}{2\pi} \sum_j (\phi_j(\infty) - \phi_j(0)).$$

- The $SU(4)$ charge corresponding to $\mathrm{diag}(1, -1, 0, 0)$ is

$$Q_{(1,-1,0,0)} = \frac{1}{2\pi}(\phi_1(\infty) - \phi_2(\infty) - \phi_1(0) + \phi_2(0))$$

- The $SU(4)$ charge corresponding to $\mathrm{diag}(0, 0, 1, -1)$ is

$$Q_{(0,0,1,-1)} = \frac{1}{2\pi}(\phi_3(\infty) - \phi_4(\infty) - \phi_3(0) + \phi_4(0))$$

- The expression of the $SU(4)$ charge corresponding to $\mathrm{diag}(1, 1, -1, -1)$ depends on the particles are incoming or outgoing. For incoming particles $\partial_t \phi_j^{\mathrm{in}} = \partial_r \phi_j^{\mathrm{in}}$, it is

$$4\pi r^2 J^t_{(1,1,-1,-1)} = \frac{1}{2\pi}\partial_t(\phi_1^{\mathrm{in}} + \phi_2^{\mathrm{in}} - \phi_3^{\mathrm{in}} - \phi_4^{\mathrm{in}}) = \frac{1}{2\pi}\partial_r(\phi_1^{\mathrm{in}} + \phi_2^{\mathrm{in}} - \phi_3^{\mathrm{in}} - \phi_4^{\mathrm{in}})$$
$$\Rightarrow Q_{(1,1,-1,-1)} = \frac{1}{2\pi}((\phi_1^{\mathrm{in}} + \phi_2^{\mathrm{in}} - \phi_3^{\mathrm{in}} - \phi_4^{\mathrm{in}})(\infty) - (\phi_1^{\mathrm{in}} + \phi_2^{\mathrm{in}} - \phi_3^{\mathrm{in}} - \phi_4^{\mathrm{in}})(0))$$

The currents and the boundary condition

- We couple the background gauge fields for the maximal torus of $SU(4)$.
- The covariant derivative is given as

$$\left(d - ia - i\sum_{l=1}^{3} A_l [H_l]_{jj}\right) a_j, \quad H_l: \text{ Cartan generators of } SU(4).$$

- The anomaly implies the terms in the effective theory of α_j, β_j:

$$\frac{1}{8\pi^2}\sum_j \left(\alpha_j (f + \sum_l F_l [H_l]_{jj})^2 + \beta_j (-f + \sum_l F_l [H_l]_{jj})^2\right).$$

- The currents are, for $F_l = 0$,

$$\star J = \frac{1}{4\pi^2}\sum_j (d\alpha_j + d\beta_j) f, \quad \star J_l = \frac{1}{4\pi^2}\sum_j [H_l]_{jj} (d\alpha_j - d\beta_j) f.$$

- For the action to be finite, the term containing f has to be zero at the core of the monopole, and thus

$$\sum_j (\alpha_j + \beta_j) = 0,$$

$$\alpha_j - \beta_j - \alpha_k + \beta_k = 0, \quad \forall j, k, \quad \text{at } r = 0.$$

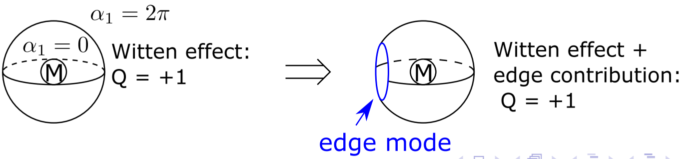
Monopole bag and pancake

- Let us consider the monopole surrounded by the wall of α_1.
- Because there is a monopole, $f = \sin\theta\, d\theta d\varphi/2$, the charges are

$$Q = \int d^3x J^0 = \frac{1}{2\pi}\sum_j \int_0^\infty dr\,\partial_r(\alpha_j + \beta_j) = 1,$$

$$Q_l = \int d^3x J_l^0 = \frac{1}{2\pi}\sum_j [H_l]_{jj} \int_0^\infty dr\,\partial_r(\alpha_j - \beta_j) = [H_l]_{11}.$$

  This means the object has the same charges of a_1.
- This object corresponds to the kink in the s-wave theory.
- The wall with a boundary is the pancake. There are edge modes that contribute to the charge so that the total charge is an integer.



Pancake soliton

- When the wall has the boundary, there has to be a chiral edge mode to maintain the gauge invariance:

$$\frac{1}{4\pi}\int_{\mathbb{R}\times D^2}(a+\sum_l A_l[H_l]_{jj})(f+\sum_{l'} F_{l'}[H_{l'}]_{jj})$$
$$+\frac{1}{4\pi}\int_{\partial D^2}(D_x\phi(D_t\phi+vD_x\phi)\,dxdt-\phi(f+\sum_l F_l[H_l]_{jj})),$$
$$D\phi:=d\phi-a-A_l[H_l]_{jj}, a\to a+d\lambda,\ A_l\to A_l+d\lambda_l,\ \phi\to\phi+\lambda+\lambda_l[H_l]_{jj}.$$

- ϕ is a 2π-periodic scalar.
- The $U(1)$ and $SU(4)$ charges are

$$Q=\frac{1}{2\pi}\int_{\partial D^2}D\phi+\frac{1}{2\pi}\int_{D^2}(f+\sum_l F_l[H_l]_{jj}),$$
$$Q_l=\frac{1}{2\pi}\int_{\partial D^2}D\phi[H_l]_{jj}+\frac{1}{2\pi}\int_{D^2}(f+\sum_m F_m[H_m]_{jj})[H_l]_{jj}.$$

In the gauge where the Dirac string does not penetrate the pancake, the charges are given as a winding number of ϕ around the edge.

- Additional condensates are

$$\langle a_j b_{j+1} \rangle \propto \frac{m^3}{r^6}, \quad \langle a_{j+1} b_j \rangle \propto \frac{m^3}{r^6}, \quad \text{for odd } j.$$

- The genuine variables are

$$\varphi_{12} = \alpha_1 + \beta_2, \quad \varphi_{21} = \alpha_2 + \beta_1, \quad \varphi_{34} = \alpha_3 + \beta_4, \quad \varphi_{43} = \alpha_4 + \beta_3,$$
$$\theta_{13} = \alpha_1 - \beta_1 - \alpha_3 + \beta_3$$

- The pancake of $\theta_{13} : 0 \to 2\pi$, that is, the string of $(\alpha_1 : 0 \to 2\pi, \beta_2 : 0 \to -2\pi)$, or $(\alpha_2 : 0 \to 2\pi, \beta_1 : 0 \to -2\pi)$, etc., does not have any charge because we cannot couple the background gauge field corresponding to $H_3 = \mathrm{diag}(1, 1, -1, -1)$.
$\Rightarrow$ The pancake of α_1 and that of β_2 cannot be distinguished.