The missing final state puzzle in the monopole-fermion scattering

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When the proton collides with a GUT monopole, it decays into a positron and mesons.

The effect has been used to put constraint on the monopole flux in the Universe.
Helicity flip: One-flavor case

\[ f^+_R \quad \rightarrow \quad \text{Monopole} \quad \left\langle \bar{f} f \right\rangle \propto \frac{1}{r^3} \quad f^+_L \]

- The helicity flips even in the massless QED.
- The only possible source of the fermion number violation is the chiral anomaly, which is a non-perturbative effect of QED.
- As the effect of the anomaly, the fermion condensate is nonzero.
Puzzle: Two-flavor massless case

Any fermion cannot be the final state due to the flavor charge conservation.

\[ \langle \bar{f}^1 f^1 \rangle = 0 \]
\[ \langle (\bar{f}^1 f^1)(\bar{f}^2 f^2) \rangle \propto \frac{1}{r^6} \]

? = Pancake (Wall with boundary)

Soliton of the phase of the fermion condensate
Outline

1. Missing final state puzzle
2. Review: S-wave approximation
3. Pancake soliton
4. Soliton picture of the scattering
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Helicity flip

- If there is a monopole and a charge, the electromagnetic field carries an angular momentum, \( \hat{r}_0 \): The unit vector pointing from the monopole to the charge

\[
\vec{J}_{EM} = \frac{1}{4\pi} \int d^3x \vec{r} \times (\vec{E} \times \vec{B}) = -\frac{1}{2} \hat{r}_0.
\]

- If the incoming particle has a helicity \(-1/2\) (left-handed), the total angular momentum is zero.

- After the scattering, the angular momentum from the electromagnetic field has the opposite direction to the particle momentum.

- To conserve the total angular momentum, the helicity of the particle has to flip.

\[
\vec{J}_{EM} = -\frac{1}{2} \hat{r}_0 \quad \text{L} \quad \vec{J}_{h} = \frac{1}{2} \hat{r}_0 \quad \text{M}
\]

\[
\vec{J}_{EM} = -\frac{1}{2} \hat{r}_0 \quad \text{R} \quad \vec{J}_{h} = \frac{1}{2} \hat{r}_0 \quad \text{M}
\]
Set up

- We consider an $SU(2)$ gauge theory with an adjoint Higgs and 4 flavors of Weyl fermions, where $SU(2)$ is spontaneously broken down to $U(1)$.
- The global symmetry is $SU(4)$.

$SU(4)$ quadruplets (fund. rep.)

$SU(2)$ doublets $\begin{cases} 
(a_1^+, a_2^+, a_3^+, a_4^+) \\
(b_1, b_2, b_3, b_4) 
\end{cases}$

- The theory can be regarded as an approximation of $SU(5)$ GUT:

\[
\begin{align*}
(a_1) &= (e_L^+), \\
(b_1) &= (d_L^3), \\
(a_2) &= (\bar{d}_L^3), \\
(b_2) &= (\bar{e}_L), \\
(a_3) &= (\bar{u}_L^1), \\
(b_3) &= (\bar{\bar{u}}_L^2), \\
(a_4) &= (\bar{\bar{u}}_L^1), \\
(b_4) &= (\bar{u}_L^1).
\end{align*}
\]
The low energy effective theory

- We approximate the theory as the gauge theory of the unbroken $U(1)$.
  
  $a :$ The $U(1)$ gauge field,  
  $a_j :$ The Weyl fermions with charge +1,  
  $b_j :$ The Weyl fermions with charge $-1$.  

- The 't Hooft-Polyakov monopole is approximated by the (background) Dirac monopole.  

- $X, Y$ bosons, GUT Higgs bosons are considered to be infinitely heavy.
The missing final state puzzle

- The helicity, the \( U(1) \) charge and the representation of \( SU(4) \) are
  
  \[ a_j : (L, +1, \Box), \quad b_j : (L, -1, \Box), \quad \bar{a}_j : (R, -1, \bar{\Box}), \quad \bar{b}_j : (R, +1, \bar{\Box}) \]

- If the initial state is \( a_j \), the final state should be a particle with \( (R, +1, \Box) \). However, there are no particle with this quantum number.

- The \( s \)-wave approximation implies that, when the initial state is \( a_1 \), the final state is something like

  \[ \frac{b_1}{2} + \frac{\bar{b}_2}{2} + \frac{\bar{b}_3}{2} + \frac{\bar{b}_4}{2}. \]

  \( b_j/2 \): “Semiton”, the state with halves of the \( U(1) \) charge, the flavor charge and the spin of \( b_j \).

It is hard to interpret what this “semiton state” actually is.
Interpretation of the final state


- In the $SU(5)$ GUT,

$$a_1 = e^+_L, \quad a_2 = d^3_L, \quad a_3 = u^1_L, \quad a_4 = u^2_L,$$
$$b_1 = d^3_L, \quad b_2 = e^-_L, \quad b_3 = u^2_L, \quad b_4 = u^1_L.$$

- “The semiton state” is

$$\frac{1}{2} e^+_R + \frac{1}{2} u^1_R + \frac{1}{2} u^2_R + \frac{1}{2} d^3_L$$

- The state is interpreted as

$$\frac{1}{\sqrt{2}} |e^+_R, M\rangle + \frac{1}{\sqrt{2}} |u^1_R u^2_R d^3_L, M\rangle$$

This is problematic because $e^+_R = \bar{b}_2$ is in the antifundamental representation of $SU(4)$, i.e., the flavor charge is not conserved.

- If massless QED is unitary, there must be a final state.
- The non-perturbative effect of QED can make an appropriate final state.
Outline

1. Missing final state puzzle
2. Review: S-wave approximation
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4. Soliton picture of the scattering
S-wave approximation and bosonization

The final state is obtained as the soliton in the bosonized theory in the s-wave approximation, where we only consider the spherically symmetric fields.

\[
S_{4d} = \int d^4x \left[ -\frac{1}{4g^2} f \star f + \sum_{j=1}^{4} (i \bar{a}_j \bar{\sigma}^\mu D_\mu a_j + i \bar{b}_j \bar{\sigma}^\mu D_\mu b_j) \right],
\]

\[
S_{2d} = \int_0^\infty dr \int_{-\infty}^{\infty} dt \left[ \frac{1}{2\pi} \sum_{i=1}^{4} ((\partial_t \phi_i)^2 - (\partial_r \phi_i)^2) - \frac{g^2}{32\pi^3} \frac{1}{r^2} \left( \sum_{i=1}^{4} \phi_i \right)^2 \right],
\]
Fermions are kinks

- The fermions correspond to the kink solitons.
  
  Kinks = Fermions with the charge $+1$
  
  Anti-kinks = Fermions with the charge $-1$
  
  Incoming (anti-)kinks = $a_j$ ($\bar{a}_j$)
  
  Outgoing (anti-)kinks = $\bar{b}_j$ ($b_j$)

- The kink corresponds to the s-wave state of the fermion in 4d.

\[ \phi_1 : 0 \rightarrow 2\pi \]
Currents and boundary conditions

- The currents of $U(1)$ and the maximal torus of $SU(4)$ is $(\alpha, \beta = t, r)$

\[ 4\pi r^2 J^\alpha = \frac{1}{2\pi} \sum_j \varepsilon_{\alpha\beta} \partial_\beta \phi_j, \]

\[ 4\pi r^2 J^\alpha_{(1,-1,0,0)} = \frac{1}{2\pi} \varepsilon_{\alpha\beta} \partial_\beta (\phi_1 - \phi_2), \]

\[ 4\pi r^2 J^\alpha_{(0,0,1,-1)} = \frac{1}{2\pi} \varepsilon_{\alpha\beta} \partial_\beta (\phi_3 - \phi_4), \]

\[ 4\pi r^2 J^\alpha_{(1,1,-1,-1)} = \frac{1}{2\pi} \partial_\alpha (\phi_1 + \phi_2 - \phi_3 - \phi_4). \]

$J^\alpha_{(...)}$ is the current of $U(1)$ generated by diag(...).

- The boundary condition is determined so that any charge does not flow into the infinitesimal region around the monopole to prohibit the existence of the dyon:

\[ 4\pi r^2 J^r = 0, \quad 4\pi r^2 J^r_{(...)} = 0 \text{ at } r = 0, \]

which implies

\[ \phi_1 + \phi_2 + \phi_3 + \phi_4 = 0, \quad \phi_1 = \phi_2, \quad \phi_3 = \phi_4, \quad \partial_r (\phi_1 + \phi_2 - \phi_3 - \phi_4) = 0. \]
Scattering process

- When the initial state is \( a_1 \), the final state corresponds to the soliton:

\[
\phi_1(0) = \phi_2(0) = \pi, \quad \phi_1(\infty) = 2\pi, \quad \phi_2(\infty) = 0, \quad \phi_3(0) = \phi_4(0) = -\pi, \quad \phi_3(\infty) = \phi_4(\infty) = 0.
\]

- The final state seems to be \( b_1/2 + \bar{b}_2/2 + \bar{b}_3/2 + \bar{b}_4/2 \), which is hardly interpreted.

\[
\begin{align*}
\phi_1 : \pi & \rightarrow 2\pi \\
\phi_2 : \pi & \rightarrow 0 \\
\phi_3 : -\pi & \rightarrow 0 \\
\phi_4 : -\pi & \rightarrow 0
\end{align*}
\]
Massive case: No puzzle

In the massive case, this puzzle disappears.

- When we introduce the mass term,
  \[ m_1(a_1 b_2 + a_2 b_1 + \text{h.c.}) + m_2(a_3 b_4 + a_4 b_3 + \text{h.c.}), \]

- The global symmetry reduces to \( U(1) \times U(1) \), which is generated by
  \[ H_1 = \text{diag}(1, -1, 0, 0), \quad H_2 = \text{diag}(0, 0, 1, -1), \]
  and the charge correspond to
  \[ H_3 = \text{diag}(1, 1, -1, -1) \]
  is not conserved.

- There is a candidate of the final state
  \[ a_1 : (L, +1, Q_{(1,-1,0,0)} = 1, Q_{(0,0,1,-1)} = 0) \]
  \[ \rightarrow \bar{b}_2 : (R, +1, Q_{(1,-1,0,0)} = 1, Q_{(0,0,1,-1)} = 0). \]
Numerical result in massive case

According to the numerical simulation [S. Dawson & A. N. Schellekens (1983)], the scattering process is as follows:

"Semiton state"
What happens when we take the massless limit?

- In the scattering process, when the semitons reach $r \sim 1/m$, the values of $\phi_i$ at the core start to change.
- This means that, in the region where $r \ll 1/m$, the theory can be regarded as the massless theory.
- In the massless limit, every point is near the monopole.
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Fermion condensates

- In the s-wave approximation, the final state is described as a soliton in the bosonized theory.
  \[ \Rightarrow \text{To interpret this in the 4d theory, a "bosonized" picture in 4d is needed.} \]
- The effective theory of the phases of the fermion condensates can be regarded as such a theory.
- **Fact:** In the monopole background, operators of the fermion fields have nonzero expectation values:

  \[
  \langle (\bar{b}^i \sigma_{\mu} a_j)(\bar{a}^k \sigma^{\mu} b_l) \rangle = \frac{1}{r^6} (c_1 \delta_j^i \delta_l^k + c_2 \delta_l^i \delta_j^k),
  \]

  \[
  \langle (a_{i_1} b_{i_2})(a_{i_3} b_{i_4}) \rangle = \frac{1}{r^6} c_3 \epsilon_{i_1 i_2 i_3 i_4}.
  \]

- The condensation is the seed of the helicity flip.
- By "integrating in" the phases of these condensates, and integrating out the fermions fields, we obtain the effective theory of the phases.
The variables of the effective theory is the following four:

\[ \theta_A : \text{the phase of } (a_1 b_2)(a_3 b_4), \quad \theta_{1j} : \text{the phase of } (\bar{b}^1 \sigma_\mu a_1)(\bar{a}^j \sigma^\mu b_j) \]

It is convenient to express them using the phases of the fermion fields. Let \( \alpha_j \) be the phase of \( a_j \) and \( \beta_j \) be that of \( b_j \), which means

\[ a_1 \rightarrow e^{i\theta} a_1 \quad \text{corresponds to } \alpha_1 \rightarrow \alpha_1 + \theta. \]

The genuine variables are expressed as

\[ \theta_A = \sum_j (\alpha_j + \beta_j), \quad \theta_{1j} = \alpha_1 - \beta_1 - \alpha_j + \beta_j \quad \text{for } j = 2, 3, 4. \]

There are configurations of \( \alpha_j \) and \( \beta_j \) that represent an identical configuration of the phases of the condensates.
Pancakes

- We find that pancake configurations of the phases can be fermions by determining the quantum numbers of the object.

\[ \alpha_j : 0 \to 2\pi \sim 0 \]

- The pancake is a \( 2 + 1 \) dimensional object.
- On the bulk of the pancake, the value of \( \alpha_j (\beta_j) \) gradually changes from 0 to \( 2\pi \).
- \( \alpha_j (\beta_j) \) winds around the boundary of the pancake.
  \[ \Rightarrow \] The boundary is a string of the phase.

Slice of pancake

\[ \alpha_j = 2\pi \sim 0 \]

\[ \alpha_j = 0 \]
The effective theory of the phases of the fermion condensates

- Due to the chiral anomaly, the phase shift of the fermion causes the shift of the Lagrangian:

\[ a_j \rightarrow e^{i\theta} a_j \implies \mathcal{L} \rightarrow \mathcal{L} + \frac{1}{8\pi^2} \theta f \wedge f \]

\[ b_j \rightarrow e^{i\theta} b_j \implies \mathcal{L} \rightarrow \mathcal{L} + \frac{1}{8\pi^2} \theta f \wedge f \]

- To reproduce this, the effective theory of \( \alpha_j, \beta_j \) has to contain the term,

\[ \frac{1}{8\pi^2} \sum_j ((\alpha_j + \beta_j)(f \wedge f)) . \]

- The \( U(1) \) current is read off as

\[ J^\mu = \varepsilon^{\mu\nu\rho\sigma} \frac{1}{4\pi^2} \sum_j \partial_\nu (\alpha_j + \beta_j) f_{\rho\sigma} . \]
Monopole bag and pancake

- Let us consider the monopole surrounded by the wall of $\alpha_1$.
- **Witten effect:** Because there is a monopole, $f = \sin \theta d\theta d\varphi/2$, the charges are

  $$J^\mu = \varepsilon^{\mu\nu\rho\sigma} \frac{1}{4\pi^2} \sum_j \partial_\nu (\alpha_j + \beta_j) f_{\rho\sigma},$$

  $$\Rightarrow \quad Q = \int d^3x \, J^0 = \frac{1}{2\pi} \sum_j \int_0^\infty dr \, \partial_r (\alpha_j + \beta_j) = 1.$$  

  This object corresponds to the kink in the $s$-wave theory.

- The wall with a boundary is the pancake. There are edge modes that contribute to the charge so that the total charge is an integer.

![Diagram showing Witten effect and Witten effect + edge contribution]

- Witten effect: $\alpha_1 = 0$, $Q = +1$
- Witten effect + edge contribution: $Q = +1$
The edge state

- By substituting the $2\pi$ jump of $\alpha_j$ into the effective Lagrangian
  \[
  \sum_j (\alpha_j + \beta_j) f \wedge f / (8\pi^2),
  \]
  we obtain the Chern-Simons theory as the theory on the wall:

  \[
  \frac{1}{4\pi} \int a \wedge f
  \]

- When the wall has the boundary, the CS theory is not gauge invariant.
  $\Rightarrow$ There has to be a chiral edge mode:

  \[
  \frac{1}{4\pi} \int_{\mathbb{R} \times D^2} a \wedge f + \frac{1}{4\pi} \int_{\partial D^2} (D_x \phi (D_t \phi + v D_x \phi) \, dx \, dt - \phi f),
  \]

  \[D\phi := d\phi - a, \quad a \rightarrow a + d\lambda, \quad \phi \rightarrow \phi + \lambda.\]

- $\phi$ is a $2\pi$-periodic scalar, thus we can define the winding number of $\phi$,

  \[
  \frac{1}{2\pi} \int_{\partial D^2} d\phi.
  \]

- The pancake with the exited edge state with this winding number $\pm 1$ can be considered as a fermion.
The charge of the edge state

- The $U(1)$ charge is

$$Q = \frac{1}{2\pi} \int_{\partial D^2} (d\phi - a) + \frac{1}{2\pi} \int_{D^2} f.$$ 

In the gauge where the Dirac string does not penetrate the pancake, the charge is given as a winding number of $\phi$ around the edge.

- The quantum eigenstate of $\int_{\partial D^2} d\phi/2\pi$ with the eigenvalue $+1$ ($-1$) in the edge theory has the charge $\pm 1$.

- Classically, the state corresponds to the solution of the eq. of motion of the edge theory. If we neglect the gauge fields, it is

$$\phi = \pm 2\pi(x - vt)/L.$$ 

- By introducing the background fields of the maximal torus of $SU(4)$, we can confirm that the edge state also have the flavor charge corresponding to the fundamental representation.
The spin of the object is given as the generator of the translation along the edge:

\[ J^z = \frac{L}{2\pi} P_x = \frac{L}{8\pi^2} \int_0^L dx (\partial_x \phi)^2, \]

where we neglect the gauge fields. 

By substituting the solution \( \phi = \pm 2\pi(x - vt)/L \), we obtain

\[ J^z = \frac{1}{2}. \]

The direction of the spin depends only on the orientation of the wall, and does not depend on the charge.
Original and new fermions

- The pancakes can be regarded as fermions.
- Some of them have opposite helicity to the original fermions.
  \[ \implies \text{New fermions!} \]
- The final state of the monopole-fermion scattering is identified with the new fermions.

![Diagram of original and new fermions](attachment:image.png)
Massive case

- We introduce the mass term

\[ m_1(a_1 b_2 + a_2 b_1 + \text{h.c.}) + m_2(a_3 b_4 + a_4 b_3 + \text{h.c.}). \]

- The global symmetry reduces to \( U(1) \times U(1) \), whose generators are

\[ H_1 = \text{diag}(1, -1, 0, 0), \quad H_2 = \text{diag}(0, 0, 1, -1). \]

- There are no quantum number corresponding to \( H_3 = \text{diag}(1, 1, -1, -1) \).

\[
\begin{align*}
  a_1: \quad & (L, +1, Q_{H_1} = 1, Q_{H_2} = 0) \\
  \bar{b}_2: \quad & (R, +1, Q_{H_1} = 1, Q_{H_2} = 0)
\end{align*}
\]
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Boundary condition at the core of the monopole

- In the process of the scattering, the boundary condition at the core of the monopole plays an important role.
- For the action to be finite, it has to be satisfied that
  \[ \sum_j (\alpha_j + \beta_j) = 0, \]
- For the background gauge field of $SU(4)$ to be coupled to the theory, it has to be satisfied that
  \[ \alpha_j - \beta_j - \alpha_k + \beta_k = 0, \quad \forall j, k. \]
- When $\alpha_1$ changes from 0 to $2\pi$ at the core, the $\beta_j$ change as
  \[ \beta_1 = \alpha_1/2, \quad \beta_2 = \beta_3 = \beta_4 = -\alpha_1/2. \]
  to maintain the boundary condition.
The wall of the red line corresponds to an $SU(4)$ rotation (associated with a $U(1)$ gauge transformation) of the phases, and thus it is transparent, i.e., there is no object actually.
Massive case

- In the massive case, there appears additional condensates:

\[ \langle a_j b_{j+1} \rangle \propto \frac{m^3}{r^6}, \quad \langle a_{j+1} b_j \rangle \propto \frac{m^3}{r^6}, \quad \text{for odd } j. \]

- In the massive case, the region

\[ \alpha_1 = 2\pi, \quad \beta_1 = \pi, \quad \beta_2 = \beta_3 = \beta_4 = -\pi \]

is no longer the vacuum, because the additional condensates change.
When a charged fermion collides with a monopole, the helicity of the s-wave component of the fermion has to flip irrespective of what the UV theory is. The only possible source of this flip is the chiral anomaly. This means that when there is a monopole, the effect of the anomaly is not suppressed in QED.

As a consequence, there is a fermion condensate violating the fermion number conservation.

**Puzzle:** If there are two flavors of massless Dirac fermions, any fermion cannot be the final state of the monopole-fermion scattering due to the flavor charge conservation.

We solve this puzzle by identifying the final state as the new fermion, which is the soliton of the fermion condensates in the monopole background.
Backup
The \( U(1) \) charge and the \( SU(4) \) charges in s-wave approximation

- The \( U(1) \) charge is
  \[
  Q = \frac{1}{2\pi} \sum_j \int_0^\infty dr (\partial_r \phi_j) = \frac{1}{2\pi} \sum_j (\phi_j(\infty) - \phi_j(0)).
  \]

- The \( SU(4) \) charge corresponding to \( \text{diag}(1, -1, 0, 0) \) is
  \[
  Q_{(1,-1,0,0)} = \frac{1}{2\pi} \left( \phi_1(\infty) - \phi_2(\infty) - \phi_1(0) + \phi_2(0) \right)
  \]

- The \( SU(4) \) charge corresponding to \( \text{diag}(0, 0, 1, -1) \) is
  \[
  Q_{(0,0,1,-1)} = \frac{1}{2\pi} \left( \phi_3(\infty) - \phi_4(\infty) - \phi_3(0) + \phi_4(0) \right)
  \]

- The expression of the \( SU(4) \) charge corresponding to \( \text{diag}(1, 1, -1, -1) \) depends on the particles are incoming or outgoing. For incoming particles \( \partial_t \phi_j^{\text{in}} = \partial_r \phi_j^{\text{in}} \), it is
  \[
  4\pi r^2 J_{(1,1,-1,-1)}^t = \frac{1}{2\pi} \partial_t (\phi_1^{\text{in}} + \phi_2^{\text{in}} - \phi_3^{\text{in}} - \phi_4^{\text{in}}) = \frac{1}{2\pi} \partial_r (\phi_1^{\text{in}} + \phi_2^{\text{in}} - \phi_3^{\text{in}} - \phi_4^{\text{in}})
  \Rightarrow Q_{(1,1,-1,-1)} = \frac{1}{2\pi} \left( (\phi_1^{\text{in}} + \phi_2^{\text{in}} - \phi_3^{\text{in}} - \phi_4^{\text{in}})(\infty) - (\phi_1^{\text{in}} + \phi_2^{\text{in}} - \phi_3^{\text{in}} - \phi_4^{\text{in}})(0) \right)
  \]
The currents and the boundary condition

- We couple the background gauge fields for the maximal torus of $SU(4)$.
- The covariant derivative is given as
  \[
  \left( d - ia - i \sum_{l=1}^{3} A_l[H_l]_{jj} \right) a_j, \quad H_l : \text{ Cartan generators of } SU(4).
  \]
- The anomaly implies the terms in the effective theory of $\alpha_j, \beta_j$:
  \[
  \frac{1}{8\pi^2} \sum_j \left( \alpha_j (f + \sum_l F_l[H_l]_{jj})^2 + \beta_j (-f + \sum_l F_l[H_l]_{jj})^2 \right).
  \]
- The currents are, for $F_l = 0$,
  \[
  \star J = \frac{1}{4\pi^2} \sum_j (d\alpha_j + d\beta_j) f, \quad \star J_l = \frac{1}{4\pi^2} \sum_j [H_l]_{jj} (d\alpha_j - d\beta_j) f.
  \]
- For the action to be finite, the term containing $f$ has to be zero at the core of the monopole, and thus
  \[
  \sum_j (\alpha_j + \beta_j) = 0,
  \alpha_j - \beta_j - \alpha_k + \beta_k = 0, \quad \forall j, k, \quad \text{at } r = 0.
  \]
Monopole bag and pancake

- Let us consider the monopole surrounded by the wall of $\alpha_1$.
- Because there is a monopole, $f = \sin \theta \, d\theta \, d\varphi / 2$, the charges are

$$Q = \int d^3x J^0 = \frac{1}{2\pi} \sum_j \int_0^\infty dr \, \partial_r (\alpha_j + \beta_j) = 1,$$

$$Q_l = \int d^3x J^0_l = \frac{1}{2\pi} \sum_j [H_l]_{jj} \int_0^\infty dr \, \partial_r (\alpha_j - \beta_j) = [H_l]_{11}.$$  

This means the object has the same charges of $a_1$.

- This object corresponds to the kink in the $s$-wave theory.
- The wall with a boundary is the pancake. There are edge modes that contribute to the charge so that the total charge is an integer.

[Diagram of monopole bag and pancake with $\alpha_1 = 2\pi$ and $\alpha_1 = 0$.]
Pancake soliton

- When the wall has the boundary, there has to be a chiral edge mode to maintain the gauge invariance:

\[
\frac{1}{4\pi} \int_{\mathbb{R} \times D^2} \left( a + \sum_l A_l[H_l]_{jj} \right) (f + \sum_{l'} F_{l'}[H_{l'}]_{jj})
\]

\[
+ \frac{1}{4\pi} \int_{\partial D^2} \left( D_x \phi (D_t \phi + v D_x \phi) \right) dx dt - \phi (f + \sum_l F_l[H_l]_{jj}),
\]

\[D\phi := d\phi - a - A_l[H_l]_{jj}, a \to a + d\lambda, \quad A_l \to A_l + d\lambda, \quad \phi \to \phi + \lambda + \lambda_l[H_l]_{jj}.\]

- \(\phi\) is a \(2\pi\)-periodic scalar.

- The \(U(1)\) and \(SU(4)\) charges are

\[
Q = \frac{1}{2\pi} \int_{\partial D^2} D\phi + \frac{1}{2\pi} \int_{D^2} (f + \sum_l F_l[H_l]_{jj}),
\]

\[
Q_l = \frac{1}{2\pi} \int_{\partial D^2} D\phi[H_l]_{jj} + \frac{1}{2\pi} \int_{D^2} (f + \sum_m F_m[H_m]_{jj})[H_l]_{jj}.
\]

In the gauge where the Dirac string does not penetrate the pancake, the charges are given as a winding number of \(\phi\) around the edge.
Additional condensates are

\[ \langle a_j b_{j+1} \rangle \propto \frac{m^3}{r^6}, \quad \langle a_{j+1} b_j \rangle \propto \frac{m^3}{r^6}, \quad \text{for odd } j. \]

The genuine variables are

\begin{align*}
\varphi_{12} &= \alpha_1 + \beta_2, \quad \varphi_{21} = \alpha_2 + \beta_1, \quad \varphi_{34} = \alpha_3 + \beta_4, \quad \varphi_{43} = \alpha_4 + \beta_3, \\
\theta_{13} &= \alpha_1 - \beta_1 - \alpha_3 + \beta_3
\end{align*}

The pancake of \( \theta_{13} : 0 \rightarrow 2\pi \), that is, the string of \((\alpha_1 : 0 \rightarrow 2\pi, \beta_2 : 0 \rightarrow -2\pi)\), or \((\alpha_2 : 0 \rightarrow 2\pi, \beta_1 : 0 \rightarrow -2\pi)\), etc., does not have any charge because we cannot couple the background gauge field corresponding to \( H_3 = \text{diag}(1, 1, -1, -1) \).

\( \Rightarrow \) The pancake of \( \alpha_1 \) and that of \( \beta_2 \) cannot be distinguished.