Anomalies in an EFT:
The On-Shell Way

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Recap: what is an Anomaly?

- A classical symmetry, violated at the quantum level
- Corresponds to non-invariance of the path integral measure
- Example: the abelian anomaly for an axial current

$$\mathcal{L} = -\frac{1}{4}F^{\alpha}_{\mu\nu}F^{\alpha\mu\nu} - \overline{\psi}\not{D}\psi$$

Axial rotation: $U(x) = e^{i\epsilon(x)q\gamma_5}$

$$\psi(x) \rightarrow \psi'(x) = U(x)\psi(x)$$
$$\overline{\psi}(x) \rightarrow \overline{\psi}'(x) = \overline{\psi}(x)\overline{U}(x)$$

Noether procedure →

$$J_5^{\mu} = iq\,\overline{\psi}\gamma^{\mu}\gamma_5\psi$$

Classically conserved

What happens quantum mechanically? Look at path integral.

Abelian Anomaly

- Path integral measure not invariant under chiral rotation

$$\int \mathcal{D}\psi' \mathcal{D}\overline{\psi}' = \int \mathcal{D}\psi \mathcal{D}\overline{\psi} e^{i \int \mathrm{d}^4 x \epsilon(x) \mathrm{a}(x)} \qquad \mathrm{a}(x) = -\frac{q^3}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}(x) F_{\rho\sigma}(x)$$

Fujikawa method

- Chiral current not conserved quantum mechanically

$$\int \mathcal{D}\psi \mathcal{D}\overline{\psi} e^{-i\bar{\psi} \not{D} \psi} = \int \mathcal{D}\psi \mathcal{D}\overline{\psi} e^{-i\bar{\psi} \not{D} \psi} \left[1 + i \int \mathrm{d}^4 x \epsilon(x) \left(\mathrm{a}(x) + \partial_\mu J_5^\mu(x) \right) + \dots \right]$$

$$\longrightarrow \boxed{\partial_\mu \left\langle J_5^\mu(x) \right\rangle = \frac{q^3}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}(x) F_{\rho\sigma}(x)}$$

Anomalous Ward identity

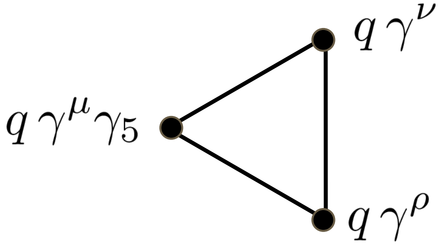
Abelian Anomaly - Connection to Triangle Diagram

- From the effective action for A

$$\frac{\delta}{\delta A_\nu(y)}\frac{\delta}{\delta A_\rho(z)}\partial_\mu \left\langle J_5^\mu(x) \right\rangle \Big|_{A=0} = \partial_\mu \Gamma_5^{\mu\nu\rho}(x,y,z)$$

- Where

$$\Gamma_5^{\mu\nu\rho}(-p-q,p,q) =$$



- The loop has a well known regularization ambiguity. Regularizing such that the vector current is conserved, we get

$$-i(p+q)_\mu \Gamma_5^{\mu\nu\rho}(-p-q,p,q) = -\frac{q^3}{2\pi^2}\,\epsilon^{\nu\rho\lambda\sigma}p_\lambda q_\sigma$$

Consistently with the path integral measure derivation

Gauge Anomalies - the Big No-No

- Gauging a symmetry $\longrightarrow$ coupling to a current

$$S = \int \mathcal{D}A \, e^{i\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + A_\mu \langle J^\mu \rangle\right)}$$

- If $\partial_\mu \langle J^\mu \rangle \neq 0$ gauge invariance is broken

- (If A is massless) we cannot establish equivalence of Lorentz and Unitary gauge

$\longrightarrow$ Cannot quantize Lorentz invariant theory

Solution in SM:

Anomaly cancellation!
Charges of all SM fermions conspire to cancel all triangle diagrams

Anomaly Cancelation : a Toy Example

D'Hoker and Farhi 84'

- $SU(2)_L$ x $U(1)_R$ gauge theory with N=2n generations of Weyl fermions

$$\psi^i_L(\mathbf{2})_{y_i} \qquad \psi^i_R = \left(\psi^i_{R1}(\mathbf{1})_{y_i+\frac{1}{2}},\, \psi^i_{R2}(\mathbf{1})_{y_i-\frac{1}{2}}\right)$$

  - The Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F^2_{1\mu\nu} - \frac{1}{2}\,\mathrm{Tr}\,F^2_{2\mu\nu} + \sum_{i=1}^{N}\left[i\overline{\psi}^i_L \not{D}^i_L \psi^i_L + i\overline{\psi}^i_R \not{D}^i_R \psi^i_R\right]$$

$$D^i_{L;\mu} = \partial_\mu + ig_2 A_\mu + ig_1 y_i B_\mu \qquad D^i_{R;\mu} = \partial_\mu + ig_1\left(y_i + \tau_3\right)B_\mu$$

- Fermions couple chirally $\longrightarrow$ potential $U(1)_R^3$ and $SU(2)_L$x$U(1)_R$ anomalies

Anomaly Cancelation : a Toy Example

D'Hoker and Farhi 84'

- Fermions couple chirally $\longrightarrow$ potential $U(1)_R{}^3$ and $SU(2)_L{}^2 x U(1)_R$ anomalies

U(1)^3 anomaly:

$$\partial_\mu \left\langle J^\mu_{\mathrm{U}(1)} \right\rangle_{\mathrm{U}(1)^2} = -\frac{g_1^2}{48\pi^2} F_{1\mu\nu}\tilde{F}_1^{\mu\nu} \sum \left[2y_i^3 - (y_i + \frac{1}{2})^3 - (y_i - \frac{1}{2})^3\right]$$

Mixed anomaly:

$$\partial_\mu \left\langle J^\mu_{\mathrm{U}(1)} \right\rangle_{\mathrm{SU}(2)^2} = -\frac{g_2^2}{24\pi^2} \varepsilon^{\mu\nu\kappa\lambda}\partial_\mu \,\mathrm{Tr}\left[A_\nu \partial_\kappa A_\lambda + \frac{1}{2} i g_2 A_\nu A_\kappa A_\lambda\right] \sum y_i$$

$$\partial_\mu \left\langle J^{\mu\, a}_{\mathrm{SU}(2)} \right\rangle_{\mathrm{SU}(2)\times\mathrm{U}(1)} = -\frac{g_1\, g_2}{24\pi^2} \varepsilon^{\mu\nu\kappa\lambda}\partial_\mu \,\mathrm{Tr}\left[\tau^a \left(2B_\nu \partial_\kappa A_\lambda + \frac{1}{2} i g_2 B_\nu A_\kappa A_\lambda\right)\right] \sum y_i$$

- Anomalies cancel if $\sum y_i = 0$

Toy Example: Higgsing the Theory

D'Hoker and Farhi 84'

- What happens if we Higgs the theory?
- Introduce Higgs $\phi(\mathbf{2})_{\frac{1}{2}}$ with a VEV that breaks $SU(2)_L \times U(1)_R \rightarrow U(1)_V$
- Choose Higgs quartic, gauge couplings and Yukawas such that

$$m_Z,\, m_W,\, m^f_{i<N},\, \ll \mathrm{E} \ll m^f_N,\, m_h$$

- Integrate the radial mode and the Nth generation of fermions

↓

EFT for A, Z, W with N-1 "massless" fermions and $\sum y_i \neq 0$

Is this EFT consistent? How can we quantize in a Lorentz invariant way?

Anomaly Cancelation in the EFT

D'Hoker and Farhi 84'
Goldstone and Wilczek 81'

- Is the EFT consistent? Missing Nth generation of fermions
- D'Hoker and Farhi:
  - Integrating out radial mode leaves an EFT for the Goldstone matrix U
  - Integrating out the Nth generation generates gauged WZW (or GW) terms
  - The gauged WZW cancel the anomaly from the N-1 light fermions

- Effective action:

$$\mathcal{S}_{\rm EFT} = \underbrace{\mathcal{S}_{\rm nl\sigma m}(A_\mu, B_\mu, \psi^i, U)}_{\text{Fixed by the nonlinear realization of SU(2)xU(1)}} + \Gamma_{\rm WZW}(U) + \int d^4x\, g_1\, y_N\, B_\mu J^\mu_{\rm GW}$$

$U \in \dfrac{\mathrm{SU}(2)_L \times U(1)_R}{\mathrm{U}(1)_V}$ is the Goldstone matrix

Anomaly Cancelation in the EFT

D’Hoker and Farhi 84’

- WZW term:

$$\Gamma_{\mathrm{WZW}}(U) = -\frac{i}{240\pi^2}\int_Q d^5x\,\epsilon^{\alpha\beta\gamma\delta\epsilon}\,\mathrm{Tr}\left[\tilde{U}^\dagger\partial_\alpha\tilde{U}\tilde{U}^\dagger\partial_\beta\tilde{U}\tilde{U}^\dagger\partial_\gamma\tilde{U}\tilde{U}^\dagger\partial_\alpha\tilde{U}\tilde{U}^\dagger\partial_\delta\tilde{U}\tilde{U}^\dagger\partial_\epsilon\tilde{U}\right]$$

  This term is anomalous under the non-perturbative SU(2) anomaly. This exactly cancels the contribution of the light N-1 fermions.

Anomaly Cancelation in the EFT

D'Hoker and Farhi 84'
Goldstone and Wilczek 81'

- Improved Goldstone-Wilczek term $\int d^4x\, g_1\, y_N\, B_\mu \hat{J}^\mu_{\rm GW}$ where

$$J^\mu_{\rm GW} = \frac{1}{24\pi^2}\epsilon^{\mu\alpha\beta\gamma}{\rm Tr}\Big[\underbrace{U^\dagger D_\alpha U U^\dagger D_\beta U U^\dagger D_\gamma U - i\frac{3g_2}{2}F_{2\alpha\beta}D_\gamma U U^\dagger - i\frac{3g_1}{2}F_{1\alpha\beta}\tau_3 U^\dagger D_\gamma U}$$

Cancels the $U(1)_R^3$ and $SU(2)_L^2 \times U(1)_R$ anomalies

$$\underbrace{-g_2^2\left(A_\alpha F_{2\beta\gamma} - \frac{1}{2}ig_2 A_\alpha A_\beta A_\gamma\right)}\Big]$$

Local counterterm that does not cancel the $SU(2)_L^2 \times U(1)_R$ anomaly but only shifts it only to the $U(1)_R$. Analogous to the regulator ambiguity of the chiral anomaly - we can choose to preserve $SU(2)_L$ or $U(1)_R$, but not both.

Anomaly Cancelation in the EFT

D'Hoker and Farhi 84'

Main point: The U(1)^3 and U(1)xSU(2)^2 anomalies of the GW current are exactly equal to those of the integrated out Nth fermion generation, and the anomalies are canceled.

$$\partial_\mu \left\langle J^{\mu\, a}_{\mathrm{SU(2)}} \right\rangle_{\mathrm{SU(2)\times U(1)}} = -\frac{g_1\, g_2}{24\pi^2} \varepsilon^{\mu\nu\kappa\lambda} \partial_\mu \,\mathrm{Tr}\left[\tau^a \left(2B_\nu \partial_\kappa A_\lambda + \frac{1}{2} i g_2 B_\nu A_\kappa A_\lambda\right)\right] \sum_i^{N-1} y_i + \delta S_{GW} = 0$$

From the local counterterm that makes the mixed anomaly violate U(1)$_R$ but not SU(2)$_L$

$$\partial_\mu \left\langle J^{\mu}_{\mathrm{U(1)}} \right\rangle_{\mathrm{U(1)^2}} = \frac{1}{32\pi^2} g_1^2 F_{1\mu\nu} \tilde{F}_1^{\mu\nu} \sum_i^{N-1} y_i + \partial_\mu J^\mu_{\mathrm{GW}}|_{A=0} = 0$$

$$\partial_\mu \left\langle J^{\mu}_{\mathrm{U(1)}} \right\rangle_{\mathrm{SU(2)^2}} = -\frac{g_2^2}{24\pi^2} \varepsilon^{\mu\nu\kappa\lambda} \partial_\mu \,\mathrm{Tr}\left[\left(2B_\nu \partial_\kappa A_\lambda + \frac{1}{2} i g_2 B_\nu A_\kappa A_\lambda\right)\right] \sum_i^{N-1} y_i + \partial_\mu J^\mu_{\mathrm{GW}}|_{B=0} = 0$$

From the Goldstone dependent terms that cancel the anomalies

Quick Summary

- Gauge anomalies are bad (... or were bad in 1984)

- Gauge anomalies in an EFT
    - Start in the UV with an anomaly free theory
    - Higgs the theory and integrate out one heavy generation of fermions
    - The EFT in anomaly free
    - The anomaly from the light fermions is canceled by the GW current

Is this a *gauge independent* statement?

Another Look at EFT Anomaly Cancelation

- Back to the improved GW current:

$$J^{\mu}_{\rm GW} = \frac{1}{24\pi^2}\epsilon^{\mu\alpha\beta\gamma}{\rm Tr}\left[U^\dagger D_\alpha U U^\dagger D_\beta U U^\dagger D_\gamma U - i\frac{3g_2}{2}F_{2\alpha\beta}D_\gamma U U^\dagger - i\frac{3g_1}{2}F_{1\alpha\beta}\tau_3 U^\dagger D_\gamma U \right.$$
$$\left. - g_2^2\left(A_\alpha F_{2\beta\gamma} - \frac{1}{2}ig_2 A_\alpha A_\beta A_\gamma\right)\right]$$

- Go to unitary gauge: U is eaten, left only with the local counterterm

$$g_1 B_\mu J^{\mu}_{\rm GW} = -\frac{g_1 g_2^2}{24\pi^2}\epsilon^{\mu\alpha\beta\gamma}B_\mu {\rm Tr}\left[\left(A_\alpha F_{2\beta\gamma} - \frac{1}{2}ig_2 A_\alpha A_\beta A_\gamma\right)\right]$$

But this counterterm only *shifts* the mixed anomaly to the U(1)...

What cancels the anomalies?

The Plot Thickens

- Well known paper by Preskill 91':

Gauge Anomalies in an Effective Field Theory

  - There is no problem with quantizing an anomalous gauge theory, as long as the gauge bosons are *massive* (the theory is in the Higgs phase or "spontaneously broken").

  - D'Hoker and Farhi's anomaly cancelation is a *gauge artifact*
  - The theory in the Higgs phase has a cutoff Λ related to the GB mass

Preskill's Argument

- Massive U(1) coupled to a single Weyl fermion

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\mu^2}{2}A_\mu A^\mu + i\bar{\psi}_L \not{D}\psi$$

Under $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \omega$ the theory has an anomaly $\frac{e^2 q^2}{48\pi^2}\omega F_{\mu\nu}\tilde{F}^{\mu\nu}$

We can cancel the anomaly by introducing a gauge artifact field b:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\left(\partial_\mu b - \mu A_\mu\right)^2 + i\bar{\psi}_L \not{D}\psi - \frac{e^3 q^3}{48\pi^2 \mu}\, b\, F_{\mu\nu}\tilde{F}^{\mu\nu}$$

The anomaly cancels if $b \to b + \frac{\mu}{e}\omega$ under a gauge transformation

Preskill's Argument

- In the b theory, we can always go to unitary gauge and set b=0, and the Lagrangian reduces to the original, anomalous Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\mu^2}{2}A_\mu A^\mu + i\bar{\psi}_L \not{D}\psi$$

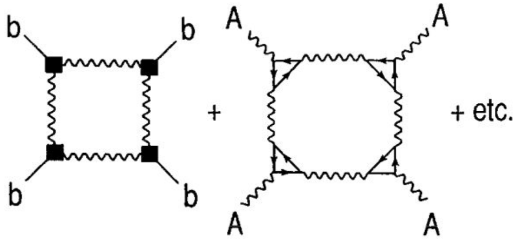
- The two theories, with and without the b field, are identical in unitary gauge, and so all of their physical observables are the same

- Consequently, there is *no physical difference* between an anomalous massive theory, and a massive theory with "WZW" anomaly cancellation

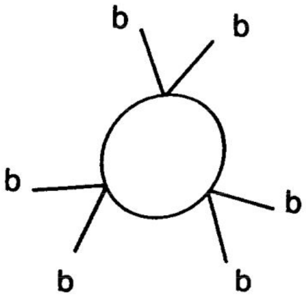
Preskill's Argument

- The massive theory has an inherent cutoff Λ, which is evident in the b≠0 gauge. The theory has a tower of divergent diagrams, generating effective operators:

$$\left(\frac{\Lambda}{4\pi v}\right)^n \frac{1}{(4\pi v)^{m-2}} \frac{1}{v^{m-2}} (\partial b - \mu A)^m$$



where $\frac{1}{v} = \frac{(eQ)^3}{16\pi^2 \mu}$ and n arbitrarily high



Clearly the theory is only calculable if

$$\Lambda \leq 4\pi v$$

A Pause for Confusion

- By Preskill's argument, an *anomalous* EFT is equivalent to an *anomaly canceled* EFT. But what about the Ward identities? Anomalous or not?

- Also...
  - Didn't we say that anomalous gauge theories cannot be quantized?
  - Apparently *massive* anomalous gauge theories *can* be quantized

Our research question:

What's the fundamental difference between the massless and massive theory that only allows to quantize the latter?

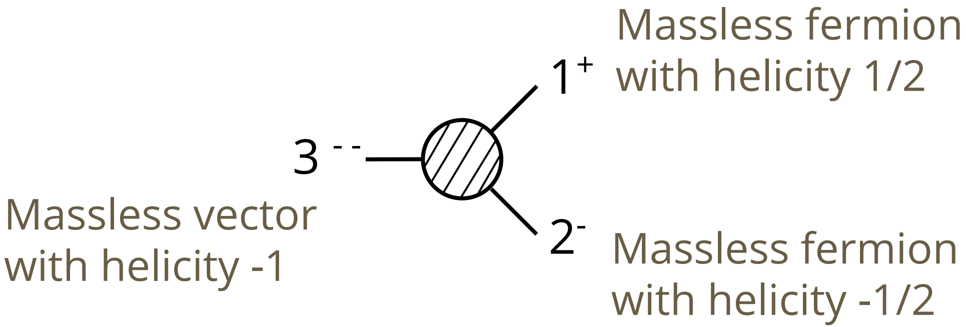
We want a manifestly gauge invariant answer!

On-Shell Methods: a Manifestly Gauge Invariant Formalism

- To understand the consistency of massive anomalous gauge theories, we first focus on the inconsistency of massless ones
- This inconsistency should arise in a *gauge invariant* way, i.e. in scattering amplitudes
- The *on-shell formalism* allows us to compute tree and loop level scattering amplitudes without introducing any action / gauge freedom
- We first review the formalism, and then arrive at the on-shell notion of a gauge anomaly as *tension between locality and unitarity @ 1-loop*

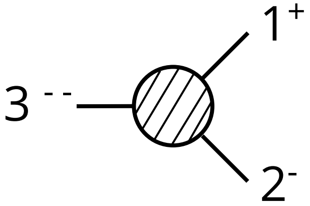
On-Shell Methods: a Manifestly Gauge Invariant Formalism

- In the *on-shell formalism*, a theory is specified not by an action, but by the representations of the scattering particles under the Lorentz group (actually little group)
- The basic building blocks are tree-level three-point amplitudes



Extremely Quick Intro to Spinor-Helicity

- 3-pt amplitudes are uniquely determined by their little group transformation. i.e. saying that a helicity -1 vector scatters with helicity ±1/2 fermion is enough to fix the amplitude (up to color factors)



$$= \frac{\langle 23\rangle^2}{\langle 12\rangle}$$

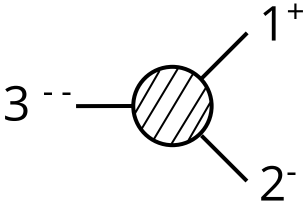
The spinor-helicity variables are defined by $p_i^\mu \, \bar{\sigma}_\mu^{\dot{\alpha}\alpha} = |i\rangle^{\dot{\alpha}} \, [i|^\alpha$ or explicitly

$$|i\rangle^{\dot{\alpha}} = \sqrt{2E_i} \begin{pmatrix} \cos\theta_i/2 \\ e^{i\phi_i} \sin\theta_i/2 \end{pmatrix}$$ with little group weight -1/2

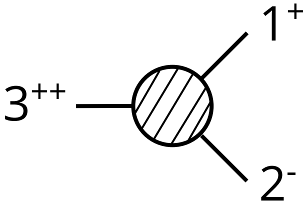
$$[i|^\alpha = \sqrt{2E_i} \begin{pmatrix} \cos\theta_i/2 \\ e^{-i\phi_i} \sin\theta_i/2 \end{pmatrix}$$ with little group weight +1/2

Rules For Forming Helicity Amplitudes

- The little group transformation of the amplitude is known and has to be saturated by stacking up spinor-helicity variables.
- Dotted and undotted indices have to be contracted separately, with $\epsilon^{\dot{\alpha}\dot{\beta}}$, $\epsilon^{\alpha\beta}$
- A spinor in the denominator has the opposite little group weight (helicity)



$$= \frac{\langle 23\rangle^2}{\langle 12\rangle}$$

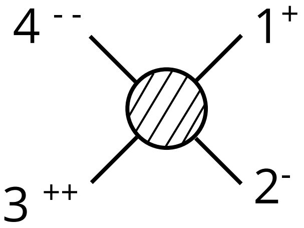


$$= \frac{[13]^2}{[12]}$$

- Locality requires that all amplitudes factorize on their poles. Since 3-pt amplitudes can’t factorize, they cannot have poles. Dimensional analysis further reduces them to these unique expressions.

4-pt Tree Level Amplitudes

- Little group fixes 4-pt amplitudes up to functions of the Mandelstam variables with simple poles, namely

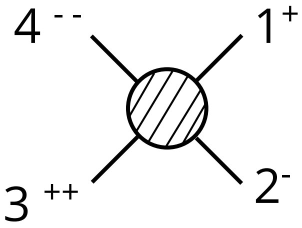


$$= \frac{\langle 24\rangle^3}{\langle 12\rangle\langle 23\rangle\langle 34\rangle}\left[\frac{c_s}{s}+\frac{c_t}{t}\right]$$

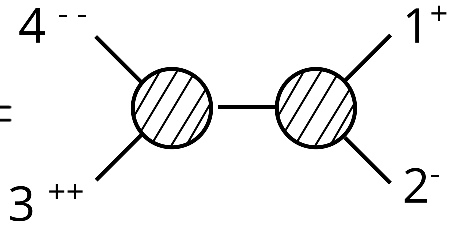
Note that we are considering *color ordered* amplitudes, i.e. the order of external legs is fixed and we cannot have a u-pole.

Once again, c_s and c_t are determined by locality / factorization. We have:

$$\operatorname*{Res}_{s\to 0}$$



$$=$$

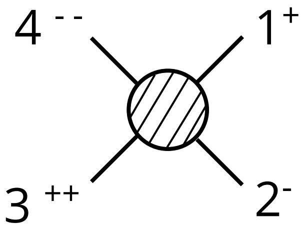


$$= \frac{\langle 4l\rangle^3}{\langle l3\rangle\langle 34\rangle}\times\frac{[1l]^2}{[12]}+\frac{[l3]^3}{[34][4l]}\times\frac{\langle 2l\rangle^2}{\langle 12\rangle}$$

$$l = p_1 + p_2$$

4-pt Tree Level Amplitudes

- From factorization on the s and t poles we get $c_s=2s$, $c_t=2t$ and so



$$= 2\frac{\langle 24\rangle^3}{\langle 12\rangle\langle 23\rangle\langle 34\rangle}$$

- Similarly, we can determine all tree-level amplitude for massless vectors and fermions, up to the specification of the structure constants f^{abc} of the gauge group

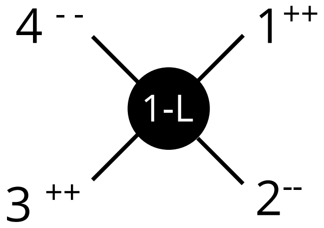
- So far we have used little group, dimensional analysis, and locality / factorization

- To determine the form of loop-level amplitudes we have to use one more ingredient: *unitarity*

Massless 4-Vector 1-loop Amplitude

Chen, Huang and McGady 14'

- A theory with a single Weyl Fermion coupled to massless vectors is inconsistent
- The inconsistency arises in the 1-loop 4-vector amplitudes
  - Constructed in a manifestly *unitary* way, these amplitudes cannot be also *local*
  - This is seen using *generalized unitarity*



$$= \underbrace{\frac{\langle 24\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 41\rangle}} \underbrace{F\left(s_{12}, s_{13}, s_{14}\right)}$$

Little group factor A^{PT} (Parke-Taylor)

Function of Mandelstams determined by unitarity cuts up to rational terms

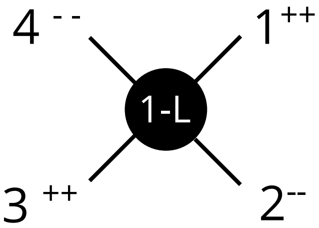
How do we determine F?

Massless 4-Vector 1-loop Amplitude

Bern, Dunbar, Kosower 95’

Forde 07’

Expand amplitude in basis of all possible loop integrals (like Passarino-Veltman)



$$= A^{\mathrm{PT}}\left[c_4 I_4\left(s_{12}, s_{14}\right) + \left(c_3^{(12)} + c_3^{(34)}\right) I_3\left(s_{12}\right)\right.$$

$$\left. + \left(c_3^{(14)} + c_3^{(23)}\right) I_3\left(s_{14}\right) + c_2^{(12)} I_2\left(s_{12}\right) + c_2^{(14)} I_2\left(s_{14}\right) + R(s_{12}, s_{13}, s_{14})\right]$$

The $I_i\left(s\right)$ are known master integrals with unique branch cuts. We can extract the coefficients c_i by performing unitarity cuts on both sides. R is a rational term, unfixed by unitarity alone.

Massless 4-Vector 1-loop Amplitude

Bern, Dunbar, Kosower 95'
Forde 07'

Let's extract $c_3^{(23)}$ by performing three unitarity cuts on both sides of the expansion

On the RH side we have

$$c_3^{(23)}\, I_3(s_{14}) \;=\; i c_3^{(23)} \int d^4\ell\, \frac{1}{\ell^2\,(\ell-p_1)^2\,(\ell+p_4)^2}$$

Cutting the integral means replacing:

$$\frac{i}{\ell^2} \to 2\pi\delta\left[\ell^2\right],\; \frac{i}{(\ell-p_1)^2} \to 2\pi\delta\left[(\ell-p_1)^2\right],\; \frac{i}{(\ell+p_4)^2} \to 2\pi\delta\left[(\ell+p_4)^2\right],$$

$$c_3^{(23)}\, I_3(s_{14}) \;\to\; -\,c_3^{(23)}\,(4\pi)^3 \int d^4\ell\;\delta\left[\ell^2\right]\,\delta\left[(\ell-p_1)^2\right]\,\delta\left[(\ell+p_4)^2\right]$$

Massless 4-Vector 1-loop Amplitude

Bern, Dunbar, Kosower 95'
Forde 07'

Next, we eliminate the delta functions by choosing an appropriate parametrization for l

$$\ell^\mu = t\,\frac{\langle 1|\bar{\sigma}^\mu|4]}{2} \quad \text{such that} \quad \ell^2 = (\ell - p_1)^2 = (\ell + p_4)^2 = 0$$

and

$$c_3^{(23)}\, I_3(s_{14}) \;\rightarrow\; -\, c_3^{(23)}\,(4\pi)^3 \int dt\, J_t$$

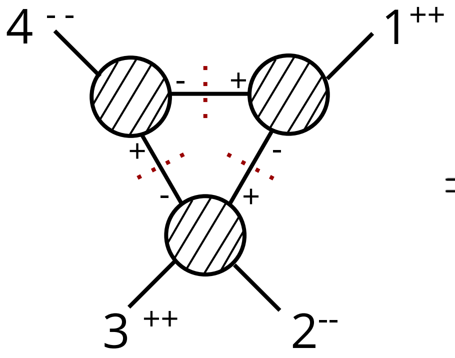
where J_t is the Jacobian for the t parametrization. Other integrals on the RH side have different t dependence in the integral. We will see that the LH side has a unique term that matches to this one, so we can compute $c_3^{(23)}$.

Massless 4-Vector 1-loop Amplitude

Bern, Dunbar, Kosower 95'
Forde 07'

On the RH side, we also cut $\ell^2 = (\ell - p_1)^2 = (\ell + p_4)^2 = 0$

By *cutting rules*, the amplitude has the form



$$= (4\pi)^3 \int dt\, J_t\, A_1(t) A_2(t) A_3(t)$$

where the $A_i(t)$ are tree level 3- and 4-pt amplitudes involving the external vector and the internal Weyl Fermion

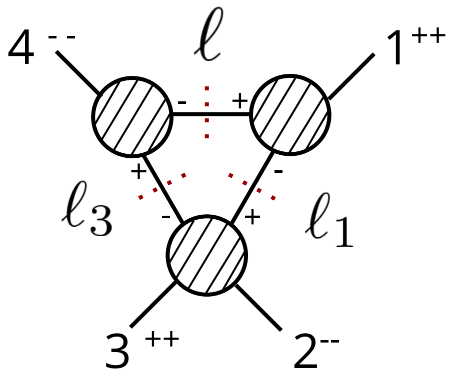
Massless 4-Vector 1-loop Amplitude

Bern, Dunbar, Kosower 95'
Forde 07'

In our case

$$A_1 A_2 A_3 = \frac{[1\ell]^2}{[\ell\ell_1]} \times \frac{\langle 2\ell_3\rangle^3}{\langle \ell_1\ell_3\rangle \langle \ell_3 3\rangle \langle 23\rangle} \times \frac{\langle 4\ell\rangle^2}{\langle \ell_3\ell\rangle}$$

$$|\ell\rangle = t\,|1\rangle\,, \quad |\ell] = |4]$$
$$|\ell_1\rangle = t\,|1\rangle\,, \quad |\ell_1] = -t^{-1}\,|1] + |4]$$
$$|\ell_3\rangle = t\,|1\rangle + |4\rangle\,, \quad |\ell_3] = |4]$$



All satisfy cut conditions $\ell^2 = \ell_1^2 = \ell_3^2 = 0$

Massless 4-Vector 1-loop Amplitude

Bern, Dunbar, Kosower 95'
Forde 07'

In fact, one can show

$$\int dt\, J_t\, A_1 A_2 A_3 \;=\; \mathrm{Inf}_t^0\left[A_1 A_2 A_3\right] \int dt\, J_t +$$ Poles in t corresponding to Box Integral I_4

where $\mathrm{Inf}_t^0\left[A_1 A_2 A_3\right]$ is the constant term in a Laurent expansion in t

Plugging everything in, we have

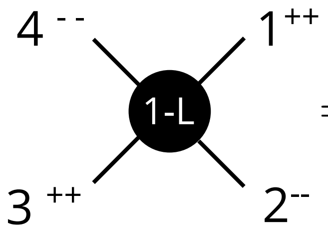
$$c_3^{(23)} \;=\; -\,\frac{\mathrm{Inf}_t^0\left[A_1 A_2 A_3\right]}{A^{\mathrm{PT}}} \;=\; \frac{s_{12} s_{14}^4}{2 s_{13}^4}$$

And so we've successfully calculated the coefficient of the triangle master integral

Massless 4-Vector 1-loop Amplitude

Bern, Dunbar, Kosower 95’
Forde 07’

Similarly, we obtain all other coefficients in



$$= A^{\mathrm{PT}}\left[c_4 I_4(s_{12}, s_{14}) + \left(c_3^{(12)} + c_3^{(34)}\right) I_3(s_{12}) + \left(c_3^{(14)} + c_3^{(23)}\right) I_3(s_{14}) + c_2^{(12)} I_2(s_{12}) + c_2^{(14)} I_2(s_{14}) + R(s_{12}, s_{13}, s_{14})\right]$$

$$c_4^L = \frac{s_{12}^2 s_{14}^4}{2 s_{13}^4} \quad , \quad c_4^R = \frac{s_{12}^4 s_{14}^2}{2 s_{13}^4}$$

$$c_3^{L,(14)} = c_3^{L,(23)} = \frac{s_{12} s_{14}^4}{2 s_{13}^4} \quad , \quad c_3^{R,(14)} = c_3^{R,(23)} = \frac{s_{12}^3 s_{14}^2}{2 s_{13}^4}$$

$$c_3^{L,(34)} = c_3^{L,(12)} = \frac{s_{14}^3 s_{12}^2}{2 s_{13}^4} \quad , \quad c_3^{R,(34)} = c_3^{R,(12)} = \frac{s_{14} s_{12}^4}{2 s_{13}^4}$$

$$c_2^{(12)} = \frac{s_{14}\left(2 s_{14}^2 - 5 s_{12} s_{14} - s_{12}^2\right)}{6 s_{13}^3} \quad , \quad c_2^{R,(12)} = \frac{s_{14}\left(2 s_{14}^2 + 7 s_{12} s_{14} + 11 s_{12}^2\right)}{6 s_{13}^3}$$

$$c_2^{(14)} = \frac{s_{12}\left(2 s_{12}^2 + 7 s_{12} s_{14} + 11 s_{14}^2\right)}{6 s_{13}^3} \quad , \quad c_2^{R,(14)} = \frac{s_{12}\left(2 s_{12}^2 - 5 s_{12} s_{14} - s_{14}^2\right)}{6 s_{13}^3} \,. \quad \text{(C.41)}$$

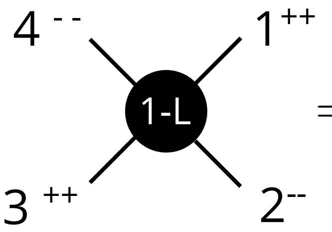
Coefficients for LH/RH fermions running in the loop. Since the vectorlike contribution cannot be anomalous, we set

$$c_i = c_i^L - c_i^R$$

Massless 4-Vector 1-loop Amplitude

Bern, Dunbar, Kosower 95’
Forde 07’

Similarly, we obtain all other coefficients in



$$= A^{\mathrm{PT}} \Big[c_4 I_4 (s_{12}, s_{14}) + \left(c_3^{(12)} + c_3^{(34)} \right) I_3 (s_{12}) + \left(c_3^{(14)} + c_3^{(23)} \right) I_3 (s_{14}) + c_2^{(12)} I_2 (s_{12}) + c_2^{(14)} I_2 (s_{14}) + R(s_{12}, s_{13}, s_{14}) \Big]$$

$$\begin{aligned}
c_4^L &= \frac{s_{12}^2 s_{14}^4}{2 s_{13}^4} \quad , \quad c_4^R = \frac{s_{12}^4 s_{14}^2}{2 s_{13}^4} \\
c_3^{L,(14)} = c_3^{L,(23)} &= \frac{s_{12} s_{14}^4}{2 s_{13}^4} \quad , \quad c_3^{R,(14)} = c_3^{R,(23)} = \frac{s_{12}^3 s_{14}^2}{2 s_{13}^4} \\
c_3^{L,(34)} = c_3^{L,(12)} &= \frac{s_{14}^3 s_{12}^2}{2 s_{13}^4} \quad , \quad c_3^{R,(34)} = c_3^{R,(12)} = \frac{s_{14} s_{12}^4}{2 s_{13}^4} \\
c_2^{(12)} &= \frac{s_{14} \left(2 s_{14}^2 - 5 s_{12} s_{14} - s_{12}^2 \right)}{6 s_{13}^3} \quad , \quad c_2^{R,(12)} = \frac{s_{14} \left(2 s_{14}^2 + 7 s_{12} s_{14} + 11 s_{12}^2 \right)}{6 s_{13}^3} \\
c_2^{(14)} &= \frac{s_{12} \left(2 s_{12}^2 + 7 s_{12} s_{14} + 11 s_{14}^2 \right)}{6 s_{13}^3} \quad , \quad c_2^{R,(14)} = \frac{s_{12} \left(2 s_{12}^2 - 5 s_{12} s_{14} - s_{14}^2 \right)}{6 s_{13}^3} \, . \quad \text{(C.41)}
\end{aligned}$$

What about the rational term?

Can’t determine from cuts!

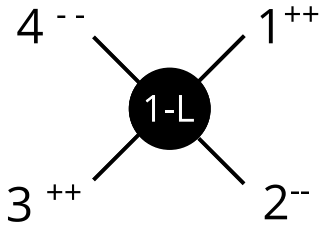
Coefficients for LH/RH fermions running in the loop. Since the vectorlike contribution cannot be anomalous, we set

$$c_i = c_i^L - c_i^R$$

Touching Base

Chen, Huang and McGady 14’

- We look for an inconsistency in the on shell construction of the 1-loop, massless 4-vector amplitude (on shell manifestation of a “gauge anomaly”)
- Using generalized unitarity, we can nail down the functional form of



up to an unknown rational term.

- Further constraint: *locality* implies that the full 1-loop amplitude, including the rational term, factorizes correctly on all of its poles
- For our inconsistent theory, *no rational term* can lead to correct factorization

Constraints on the Rational Term

Chen, Huang and McGady 14’

- First of all, our amplitude is *color ordered* so external legs cannot cross and we cannot have a pole in the u (or s_{13}) channel. Taking the limit $s_{13} \to 0$, we have

$$\lim_{s_{13}\to 0} A^{1-\text{loop}} = \lim_{s_{13}\to 0} A^{\text{PT}}\left[\sum c_i I_i + R\right] = \lim_{s_{13}\to 0} A^{\text{PT}}\left[-\frac{s_{12}}{s_{13}} + R\right]$$

- Demanding a sign flip under the cyclic shift $A(1^{++},2^{--},3^{++},4^{--}) \to A(4^{--},1^{++},2^{--},3^{++})$ we arrive at the only viable rational term:

$$R = \frac{s_{12} - s_{14}}{2 s_{13}}$$

- But $A^{\text{PT}} = \frac{[13]^2 \langle 24\rangle^2}{s_{12}s_{14}}$, and so the rational term modifies the residues of the full amplitude in the s_{12} and s_{14} channels.

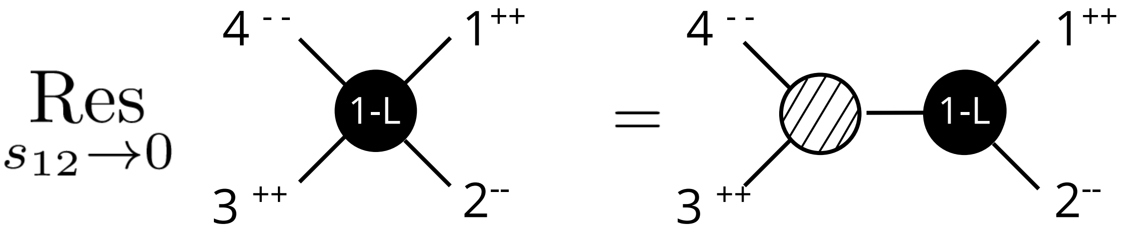
Constraints on the Rational Term

Chen, Huang and McGady 14’

- The extra s_{12} residue from the rational term is:

$$\underset{s_{12}\to 0}{\text{Res}} \, A^{\text{PT}} R = \frac{[13]^2 \langle 24 \rangle^2}{2 s_{14}}$$

- By locality, this residue should be a product of two 3-pt amplitudes:



- However, there are no possible 3-pt amplitudes that can yield this residue

The On Shell Inconsistency of a Massless Anomalous Gauge Theory

- The inconsistency of a massless gauge anomalous theory arises in the 1-loop, 4-vector amplitude
- Constructed in a manifestly unitary way, there is no choice of rational term that could lead to consistent factorization on all channels
- From an on-shell perspective, the tension is between *unitarity* and *locality*
- This is different from our field theory intuition, where the gauge anomaly signals a disconnect between *unitarity* and *Lorentz invariance* (different gauges)

Chen, Huang and McGady 14’

The On Shell Consistency of a Massive Anomalous Gauge Theory

- The demonstrate the consistency of the massive theory, we developed a formalism for generalized unitarity with *massive* external vectors. This is a fusion of the generalized unitarity formalism of Forde 07' and the massive amplitude formalism of Arkani-Hamed, Huang and Huang 17'.

- Massive particles correspond to **bolded** spinors $|\mathbf{i}\rangle^{\dot{\alpha}}{}_{I}$, $[\mathbf{i}|^{J\alpha}$, transforming as □ of their SU(2) little group, and defined so that

$$p_i^{\mu}\bar{\sigma}_{\mu}^{\dot{\alpha}\alpha} = |\mathbf{i}\rangle^{\dot{\alpha}}{}_{I}\,[\mathbf{i}|^{I\alpha}$$

This work, 19'

A 1-Loop Massive Amplitude

In this work we perform the first generalized unitarity calculation in the massive amplitude formalism. Expanding in the master integral basis, we have as usual

$$A^{1-\text{loop}}(\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4}) = \sum_i c_i\, I_i\left(s_{12}, s_{14}\right) + R$$

The SU(2) little group constrains the coefficients to be of the form

$$c_i = \frac{1}{m^4}\left(\prod_{j=1}^{4} \langle \mathbf{j}|_{\dot{\alpha}_j}\, |\mathbf{j}]_{\alpha_j}\right) T_i^{\dot{\alpha}_1\alpha_1\dot{\alpha}_2\alpha_2\dot{\alpha}_3\alpha_3\dot{\alpha}_4\alpha_4}(s_{12}, s_{13}, s_{14}, m)$$

where we suppress SU(2) little group indices. The T_i are made of $p_k^{\dot{\alpha}\alpha}$, $\epsilon^{\dot{\alpha}\alpha}$ and Mandelstams

Example Calculation: c_4

We demonstrate the calculation of c_4 the coefficient of the box integral

$$I_4 = \int d^4\ell \, \frac{1}{\ell^2 \ell_1^2 \ell_2^2 \ell_3^2}$$

where $\ell_1 = \ell - p_1, \, \ell_2 = \ell_1 - p_2, \, \ell_3 = \ell + p_4 \, .$

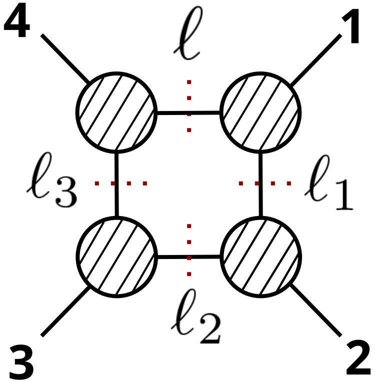
Cutting this integral four times completely localizes the integral:

$$I_4 \to \int d^4\ell \prod_i \delta(\ell_i) = 1$$

(we dropped the $(2\pi)^4$ appearing both on the amplitude and master integral sides)

Calculation of c_4

On the amplitude side we have



$$= \frac{1}{m^4} \langle \ell \mathbf{1} \rangle [\mathbf{1} \ell_1] \times \langle \ell_1 \mathbf{2} \rangle [\mathbf{2} \ell_2] \times$$
$$\times \langle \ell_2 \mathbf{3} \rangle [\mathbf{3} \ell_3] \times \langle \ell_3 \mathbf{4} \rangle [\mathbf{4} \ell]$$

$$= \frac{1}{m^4} \langle \mathbf{4} | \ell | \mathbf{1}] \langle \mathbf{1} | \ell_1 | \mathbf{2}] \langle \mathbf{2} | \ell_2 | \mathbf{3}] \langle \mathbf{3} | \ell_3 | \mathbf{4}]$$

And we just need to solve for $\ell^2 = \ell_1^2 = \ell_2^2 = \ell_3^2 = 0$

Calculation of c_4

To solve the cut conditions, we need to express ℓ in a basis of massless vectors constructed from external momenta. Following Forde 07' we define

$$\gamma = -p_1 \cdot p_4 - \sqrt{\Delta}, \quad \Delta = (p_1 \cdot p_4)^2 - m^4$$

$$p_1^{b,\mu} = \frac{\gamma}{\gamma^2 - m^4} \left(\gamma p_1^\mu + m^2 p_4^\mu\right) \quad , \quad p_4^{b,\mu} = -\frac{\gamma}{\gamma^2 - m^4} \left(m^2 p_1^\mu + \gamma p_4^\mu\right)$$

so that $(p_1^b)^2 = (p_4^b)^2 = 0$. Expressing ℓ in this basis, we get

$$\ell^\mu = x\,(p_1^{\flat\mu} + p_1^{\flat\mu}) + t\,\frac{\langle 1^\flat|\bar{\sigma}^\mu|4^\flat]}{2} + \frac{x^2}{t}\frac{\langle 4^\flat|\bar{\sigma}^\mu|1^\flat]}{2}, \qquad x \equiv \frac{m^2}{m^2+\gamma}$$

which immediately gives $\ell^2 = \ell_1^2 = \ell_3^2 = 0$.

Calculation of c_4

We have one more variable to fix - t , and one more cut condition $\ell_2^2 = 0$.

Solving the cut algebraically, we get the full solution of the 4 cut conditions:

$$t_\pm = \langle 4^b|2|1^b]\,\hat{t}_\pm \qquad \hat{t}_\pm = \frac{1}{\gamma s_{13}}\left(p_1 \cdot p_4 \pm \sqrt{\Delta}\sqrt{1 - \frac{4m^4}{s_{12}s_{14}}}\right)$$

By Fierzing the spinors we can get

$$\langle \mathbf{a}|l|\mathbf{b}] = x\,\langle \mathbf{a}\,|p_1^b + p_4^b|\,\mathbf{b}] + \hat{t}\,\langle \mathbf{a}\,|1^b 2 4^b|\,\mathbf{b}] + \frac{x^2}{A_t\hat{t}}\langle \mathbf{a}\,|4^b 2 1^b|\,\mathbf{b}]$$

For any bolded spinors **a** and **b**, and for $A_t = 4(p_2 \cdot p_1^\flat)(p_2 \cdot p_4^\flat) - \gamma m^2$

Calculation of c_4

Now we can easily compute c_4

$$c_4 = \frac{1}{2}\sum_{t_\pm}\frac{1}{m^4}\langle \mathbf{4}|\ell|\mathbf{1}]\,\langle \mathbf{1}|\ell_1|\mathbf{2}]\,\langle \mathbf{2}|\ell_2|\mathbf{3}]\,\langle \mathbf{3}|\ell_3|\mathbf{4}]$$

By substituting

$$\langle \mathbf{a}|l|\mathbf{b}] = x\left\langle \mathbf{a}\left|p_1^b + p_4^b\right|\mathbf{b}\right] + \hat{t}\left\langle \mathbf{a}\left|1^b 2 4^b\right|\mathbf{b}\right] + \frac{x^2}{A_t\hat{t}}\left\langle \mathbf{a}\left|4^b 2 1^b\right|\mathbf{b}\right]$$

$$\begin{aligned}
\langle \mathbf{1}|\ell_1|\mathbf{2}] &= \langle \mathbf{1}|\ell|\mathbf{2}] - \langle \mathbf{1}|1|\mathbf{2}] \\
\langle \mathbf{2}|\ell_2|\mathbf{3}] &= \langle \mathbf{2}|\ell|\mathbf{3}] - \langle \mathbf{2}|1|\mathbf{3}] - \langle \mathbf{2}|2|\mathbf{3}] \\
\langle \mathbf{3}|\ell_3|\mathbf{4}] &= \langle \mathbf{3}|\ell|\mathbf{4}] + \langle \mathbf{3}|4|\mathbf{4}]
\end{aligned}$$

Checking the Massless Limit for c_4

$$c_4 = \frac{1}{2}\sum_{t_\pm}\frac{1}{m^4}\langle\mathbf{4}|\ell|\mathbf{1}]\,\langle\mathbf{1}|\ell_1|\mathbf{2}]\,\langle\mathbf{2}|\ell_2|\mathbf{3}]\,\langle\mathbf{3}|\ell_3|\mathbf{4}]$$

In the massless limit, for A(1^{++},2^{--},3^{++},4^{--}), we have $c_4 = A^{\mathrm{PT}}\frac{s_{12}^2 s_{14}^4}{2s_{13}^4}$

To reach this limit from the massive result, we must first single out the (1^{++},2^{--},3^{++},4^{--}) helicity assignment, as described in Arkani-Hamed et. al. To do this we unbold the spinors in the following way:

$$\langle\mathbf{1}| \to m\frac{\langle n_1|}{\langle n_1 1\rangle}\,, \quad |\mathbf{1}] \to |1] \qquad \langle\mathbf{2}| \to \langle 2|\,, \quad |\mathbf{2}] \to m\frac{|n_2]}{[2n_2]}$$

$$\langle\mathbf{3}| \to m\frac{\langle n_3|}{\langle n_3 3\rangle}\,, \quad |\mathbf{3}] \to |3] \qquad \langle\mathbf{4}| \to \langle 4|\,, \quad |\mathbf{4}] \to m\frac{|n_4]}{[4n_4]}$$

where n_i are reference spinors whose arbitrariness reflects the emergence of unbroken gauge symmetry at $m \to 0$

Checking the Massless Limit for c_4

Unbolding the spinors, and noting that for m=0 we have

$$\gamma = -s_{12}, \; x = 0, \; \hat{t}_+ = -s_{13}^{-1}, \; \hat{t}_- = 0$$

$$p_1^\flat = p_1, \quad p_4^\flat = -p_4$$

and so

$$\ell^\mu = -\frac{s_{12}}{s_{13}} \frac{\langle 24 \rangle}{\langle 12 \rangle} \frac{\langle 1|\bar{\sigma}^\mu|4]}{2}$$

After a lot of algebra, we get that indeed $c_4 \to A^{\mathrm{PT}} \frac{s_{12}^2 s_{14}^4}{2 s_{13}^4}$ as expected,

with all of the reference spinors dropping out

Status: Generalized Unitarity for 4-Massive Vectors

- We've calculated all of the coefficients

$$c_4,\ c_3^{(12)},\ c_3^{(34)},\ c_3^{(14)},\ c_3^{(23)},\ c_2^{(12)},\ c_2^{(14)}$$

  using our generalized unitarity formalism for massive external particles

- All of the coefficients match their correct massless limits
- We are still working on extracting the s_{13} pole and finding the rational term which cancels it (lots of algebra!)

Expectations from Calculation

- We know from the field theory side that the massive anomalous theory *is* consistent, so we expect to find a rational term that can factorize on all the poles

- It would be interesting to understand if there's a spurious s_{13} pole that can be resolved by a rational term, without modifying all of the other residues, or if there's no s_{13} pole to begin with.

- Next, we plan to study how the spurious poles emerge in the $m \to 0$, or alternatively the high energy limit. This will provide us with a natural cutoff for the massive EFT, analogous to the one explored by Preskill

- Most of all, we want to gain *physical intuition* why massive theory is consistent, while the massless one isn't. Is there a way to know in advance that the massive theory resolved the tension between *unitarity* and *locality*, without all the gory detail?

Summary

- We presented a counterintuitive field theory argument (due to Preskill) that massive anomalous gauge theories are equivalent to massive “anomaly canceled” theories
- We asked the gauge invariant question why massless theories are inconsistent while massive ones are
- We are close to a technical resolution of the question: the spurious poles arising in the 1-loop 4-vector amplitude should disappear in the massive theory. The tension between *Unitarity* and *locality* is resolved.
- We are still wondering about the gauge invariant physics that makes one theory consistent, while the other one isn’t. Perhaps a close look at the poles will provide a more fundamental resolution of this issue.

Thank You!