A three-dimensional bridge between physics and mathematics

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- Mina Aganagic
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I’d like to discuss an example of this symbiosis, in a correspondence that I helped develop, which has been one of the major themes in my work.
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“3d-3d correspondence”
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\[ M \ , \ g \ \sim \rightarrow \ T_g[M] \]

3-manifold A,D,E 3d (N=2) SUSY field theory

first hints: [Dimofte-Gukov-Hollands ’10]
“3d-3d correspondence”

\[ M, \, g \overset{\sim}{\mapsto} T_g[M] \]

3-manifold \( A, D, E \) \quad 3d (N=2) SUSY field theory

depending only on topology of \( M \)!

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\[ M, g \xrightarrow{\sim} T_g[M] \]

3-manifold \( A,D,E \) \hspace{1cm} 3d (N=2) SUSY field theory

depending only on topology of \( M \)!

\[ T_g[M] \] is a top-level top’l inv’t of \( M \)

- its observables (quantities one can compute) all correspond to classical, quantum, or categorical topological invariants, some old, but many new.

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3-manifold \( A, D, E \) \hspace{2cm} 3d (N=2) SUSY field theory depending only on topology of \( M \)!

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e.g.: hyperbolic volume \( \text{Vol}(M) \) \hspace{2cm} [Mostow ’73]

space of flat \( G_\mathbb{C} \) conn’s \( \mathcal{M}_{\text{flat}}(M, G_\mathbb{C}) \)
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complex Chern-Simons part’n f’n \( \mathcal{Z}^{G_\mathbb{C}}_{CS}(M) \)

cf. Jones poly, WRT \[\text{[Witten '89, Reshetikhin-Turaev '91]}\]
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More than just theory!

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More than just theory!

“Most” \( M \ , \ g = \mathfrak{sl}_2 \) : explicit construction of \( T_g[M] \)

[Dimofte-Gaiotto-Gukov ’11]
[ Cecotti-Cordova-Vafa ’11]
[ Dimofte-Gaiotto-v.d.Veen ’13]

\( g = \mathfrak{sl}_n \) :

[Dimofte-Gabella-Goncharov ’13]
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Main tool: (topological) ideal triangulations
+ a generalization of Thurston-Neumann-Zagier gluing methods from hyperbolic geometry (’80’s)

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Results?
Math: New “quantum” topological invariants, a comb’r definition of the \( G_C \) Chern-Simons part’n function  
\[ Z_{CS}^{G_C}(M) \]
for all CS levels \( k \in \mathbb{Z} \)  

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for all CS levels \( k \in \mathbb{Z} \)

- analyzing asymptotics of \( Z_{CS}^{G_\mathbb{C}}(M) \) (easy!)

\( \sim \) simple, conjectured (tested) formula for
\( G_\mathbb{C} \)-twisted Reidemeister-Ray-Singer torsion of \( M \)

[Dimofte-Gaiotto-Gukov ’11]
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[Dimofte-Garoufalidis ’12]
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<predictions for asymptotics of colored Jones poly’s (hard!; play a role in Volume Conjecture)
  [Kashaev ’97, Murakami-Murakami ’99, Gukov ’03]
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Results?

Math: a comb’r definition of the \( G_C \) Chern-Simons part’n function \( Z_{CS}^G(M) \)

Hopefully: a combinatorial definition for \( G_C \) 3-manifold homology!

in progress w/ Gaiotto-Moore

(Analogous to Khovanov homology for \( G \))

- analyzing asymptotics of \( Z_{CS}^G(M) \) (easy!)

\[ \sim\Rightarrow \text{simple, conjectured (tested) formula for} \]

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Physics: Intuition!

Properties of 3d N=2 theories are governed by the geometry of 3-manifolds!
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Properties of 3d N=2 theories are governed by the geometry of 3-manifolds!

- geometric description of (IR) dualities within a large class of 3d N=2 SUSY gauge theories
  (from different ways to cut/glue the same \( M \))
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- geometric description of (IR) dualities within a large class of 3d N=2 SUSY gauge theories (from different ways to cut/glue the same \( M \))

- systematic construction of superconformal interfaces

\[ T_g[M] \]

[Dimofte-Gaiotto-v.d.Veen ‘13]

4d N=2 T  \quad 4d N=2 T’
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[S-duality: \( g_{YM} \) \( g'_{YM} \sim 1/g_{YM} \)]

[Dimofte-Gaiotto-v.d.Veen ’13]
Remainder of the talk:

\[ T_g[M] \]

S-duality:

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- a few more details on the correspondence, and observables of $T_g[M]$
- tetrahedra, formulas, and examples
- first look at homological/categorical invariants
The correspondence

Starting point: 6d \( (2,0) \) SCFT

"theory \( \mathcal{X} \)"

[Strominger, Witten '90's]

super-conformal-field-theory
The correspondence

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Starting point: 6d (2,0) SCFT

“theory $\mathcal{X}$”

(world-volume theory of M5 branes)

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The correspondence

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labelled by an ADE symmetry algebra \( \mathfrak{g} \)

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“theory \(\mathcal{X}\)”

(world-volume theory of M5 branes)

labelled by an ADE symmetry algebra \(\mathfrak{g}\)

\[\mathcal{X}_g\] on \(M \times \mathbb{R}^3\) (topological twist on M)

\(\leadsto\) effective theory \(T_g[M]\) on \(\mathbb{R}^3\)
The correspondence

Starting point: 6d (2,0) SCFT
“theory $\mathcal{X}$”
(world-volume theory of M5 branes)
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$\mathcal{X}_g$ on $M \times \mathbb{R}^3$ (topological twist on M)
$\sim \Rightarrow$ effective theory $T_g[M]$ on $\mathbb{R}^3$

How to describe $T_g[M]$?
The correspondence

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How to describe $T_g[M]$?

- direct, first-principles is hard: $\mathcal{X}$ has no Lagrangian
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super-conformal-field-theory

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How to describe $T_g[M]$?

- direct, first-principles is hard: $\mathcal{X}$ has no Lagrangian

- nevertheless, can infer many properties of $T_g[M]$ + its compact’ns
The correspondence

Basic property:

\[ \{ \text{vacua of } T_g[M] \text{ on } \mathbb{R}^2 \times S^1 \} = \{ \text{flat } G_{\mathbb{C}} \text{ connections on } M \} \]

\[ \mathcal{M}_{\text{flat}}(M, G_{\mathbb{C}}) \]

To see this:

\[ \begin{align*}
6d & \quad \mathcal{X}_g & \quad M \times \mathbb{R}^2 \times S^1 \\
3d & \quad T_g[M] & \quad \mathbb{R}^2 \times S^1 \\
2d & & \quad \mathbb{R}^2
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5d \text{ super-YM} & & M \times \mathbb{R}^2
\end{align*}
\]

look at the vacua of 5d SYM, topologically twisted on M

[has a Lagrangian!]
The correspondence

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look at the vacua of 5d SYM, topologically twisted on M

fields \quad A_a, \phi_a \in g \\
(a = 1, 2, 3)
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\[ \sim \rightarrow \quad A_a = A_a + i \phi_a \]

\( G_C \) connection on \( M \)
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look at the vacua of 5d SYM, topologically twisted on M

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\[ \rightsquetext{A_a = A_a + i\phi_a} \]

minimize potential:

\[ dA + A \wedge A = 0 \]

\[ \text{flat} \]
The correspondence

$$T_g[M]$$

{vacua on $\mathbb{R}^2 \times S^1$} = {flat $G_C$ connections} $\mathcal{M}_{\text{flat}}(M, G_C)$

[Dimofte-Gukov-Hollands '10]

look at the vacua of 5d SYM, topologically twisted on $M$

fields $A_a, \phi_a \in g$ ($a = 1, 2, 3$)

$\leadsto$ $A_a = A_a + i\phi_a$

$G_C$ connection on $M$

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[Dimofte-Gukov-Hollands '10]

quantize!

look at the vacua of 5d SYM, topologically twisted on \( M \)

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[Dimofte-Gukov-Hollands '10]

quantize!

\( G_\mathbb{C} \) Chern-Simons theory on \( M \)

\( \mathcal{A} \): \( g_\mathbb{C} \)-valued 1-form

\[ Z_{CS}[M] = \int \mathcal{D}A \mathcal{D}\overline{A} e^{\frac{k+i\sigma}{8\pi i} I_{CS}(A) + \frac{k-i\sigma}{8\pi i} I_{CS}(\overline{A})} \]

[Witten '91]

\[ I_{CS}(\mathcal{A}) := \int_M \text{Tr} \left( \mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right) \]
The correspondence

\[ T_g[M] \quad \{ \text{vacua on } \mathbb{R}^2 \times S^1 \} = \quad \{ \text{flat } G_\mathbb{C} \text{ connections} \} \quad \mathcal{M}_{\text{flat}}(M, G_\mathbb{C}) \]

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[I_{CS}(A) := \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \]

\[ k \in \mathbb{Z} \quad \sigma \in \mathbb{R} \text{ (or } \mathbb{C} \text{)} \]

\(G_\mathbb{C}\) Chern-Simons theory on \(M\)

\(A\) : \(g_\mathbb{C}\)-valued 1-form
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[Dimofte-Gukov-Hollands '10]

quantize!

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- classical sol’ns are flat \( G_\mathbb{C} \) connections
The correspondence

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[Dimofte-Gukov-Hollands '10]

quantize!

\( \mathbf{G}_\mathbb{C} \) Chern-Simons theory on \( M \)
\( \mathcal{A} : \ \mathfrak{g}_\mathbb{C} \)-valued 1-form

\[ Z_{CS}[M] = \int \mathcal{D}\mathcal{A}\mathcal{D}\overline{\mathcal{A}} e^{\frac{k+i\sigma}{8\pi i} I_{CS}(\mathcal{A}) + \frac{k-i\sigma}{8\pi i} I_{CS}(\overline{\mathcal{A}})} \]

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- classical sol’ns are flat \( \mathbf{G}_\mathbb{C} \) connections

- cf. compact \( \mathbf{G} \) CS thy: on knot complements, get Jones polys

(combinatorial definition)

[Witten '89]

[Reshetikhin-Turaev '90, etc.]
The correspondence

\[ T_g[M] \]

\{vacua on \( \mathbb{R}^2 \times S^1 \}\) \hspace{1cm} = \hspace{1cm} \{\text{flat } G_\mathbb{C} \text{ connections} \} \quad M_{\text{flat}}(M, G_\mathbb{C})

[Dimofte-Gukov-Hollands ’10]

quantize!

\( G_\mathbb{C} \) Chern-Simons theory on \( M \)

\( A : g_\mathbb{C} \)-valued 1-form

\[ Z_{CS}[M] = \int \mathcal{D}A \mathcal{D}\overline{A} e^{\frac{k+i\sigma}{8\pi i} I_{CS}(A) + \frac{k-i\sigma}{8\pi i} I_{CS}(\overline{A})} \]

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- classical sol’ns are flat \( G_\mathbb{C} \) connections

- cf. compact \( G \) CS thy: on knot complements, get Jones polys (combinatorial definition)

- combinatorial def’n missing for \( G_\mathbb{C} \) until recently!

[Reshetikhin-Turaev ’90, etc.]
The correspondence

\[ T_g[M] \]

\{vacua on \( \mathbb{R}^2 \times S^1 \}\)

\[ \overset{\text{}}{= \quad \{ \text{flat } G_\mathbb{C} \text{ connections} \}} \quad \overset{\text{}}{\mathcal{M}_{\text{flat}}(M, G_\mathbb{C})} \]

[Dimofte-Gukov-Hollands '10]

\[ Z^{(k, \sigma)}_{CS}[M] \]

\[ Z_{CS}[M] = \int \mathcal{D}A \overline{\mathcal{D}A} e^{\frac{k+i\sigma}{8\pi i} I_{CS}(A) + \frac{k-i\sigma}{8\pi i} I_{CS}(\overline{A})} \]

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- classical sol’ns are flat \( G_\mathbb{C} \) connections

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(combinatorial definition)

[Reshetikhin-Turaev '90, etc.]

- combinatorial def’n missing for \( G_\mathbb{C} \) until recently!
The correspondence

\[ T_g[M] \]

\{vacua on \( \mathbb{R}^2 \times S^1 \}\) = \{flat \( G_{\mathbb{C}} \) connections\} \( \mathcal{M}_{\text{flat}}(M, G_{\mathbb{C}}) \)

[Dimofte-Gukov-Hollands '10]

\[ Z_{T_g}[M][L(k, 1)\sigma] = Z_{CS}^{(k,\sigma)}[M] \]

part’n function on ellipsoidally-deformed lens space

\[ Z_{CS}[M] = \int \mathcal{D}A \mathcal{D}\overline{A} e^{\frac{k+i\sigma}{8\pi i} I_{CS}(A) + \frac{k-i\sigma}{8\pi i} I_{CS}(\overline{A})} \]  

[Witten '91]

\[ k \in \mathbb{Z} \quad \sigma \in \mathbb{R} \ (\text{or } \mathbb{C}) \quad I_{CS}(A) := \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) \]

- classical sol’ns are flat \( G_{\mathbb{C}} \) connections

- cf. compact \( G \) CS thy: on knot complements, get Jones polys

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\[ L(k, 1)_{\sigma} = S^3_{\sigma}/\mathbb{Z}_k \]

\[ \simeq \{ b^2|z|^2 + b^{-2}|w|^2 = 1 \} \in \mathbb{C}^2 \bigg/ (z, w) \sim (e^{\frac{2\pi i}{k}}z, e^{\frac{2\pi i}{k}}w) \]

\[ b^2 = \frac{k - i\sigma}{k + i\sigma} \]
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k=1:

[Terashima-Yamazaki '11]
[Dimofte-Gaiotto-Gukov '11]
[Cordova-Jafferis '13] — physical proof
The correspondence

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[Dimofte-Gukov-Hollands '10]

\[ Z_{T_g[M]}[L(k, 1)_\sigma] = \] \[ Z_{(k, \sigma)}[M] \]

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to tie this all together:

\[ Z^{k,\sigma}_{CS}[M] = \sum_{\text{flat } \alpha} B^{k+i\sigma}[M] \overline{B}^{k+i\sigma}[M] \]
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\[ L(k, 1) \simeq (D^2 \times S^1) \cup_{\varphi \in SL(2, \mathbb{Z})} (D^2 \times S^1) \]

[Beem-Dimofte-Pasquetti '12]
The correspondence

$$\frac{T_g[M]}{\{\text{vacua on } \mathbb{R}^2 \times S^1\}} = \frac{M}{\{\text{flat } G_\mathbb{C} \text{ connections}\} \mathcal{M}_{\text{flat}}(M, G_\mathbb{C})}$$

$$Z_{T_g[M]}[L(k, 1)_\sigma] = Z_{CS}^{(k, \sigma)}[M]$$

So: quantum invariants of 3-manifolds can be understood via 3d SUSY theories on lens spaces!

to tie this all together:

$$Z_{T_g[M]}[L(k, 1)_\sigma] = \sum_{\text{vacua } \alpha} B^{k+i\sigma}_\alpha[M] \overline{B^{k+i\sigma}_\alpha[M]}$$

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So: quantum invariants of 3-manifolds can be understood via 3d SUSY theories on lens spaces!

One more step: categorify
The correspondence

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One more step: categorify

\[
k=0: \quad Z_{T[M]}(S^2 \times S^1) = \text{Tr}_{\mathcal{H}(S^2)}(-1)^F q^J + \frac{F}{2}
\]

is a an index
The correspondence

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So: by studying more refined observables of \( T_g[M] \), like Hilbert spaces, one obtains homological lifts of quantum inv’ts!

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\[
\begin{array}{ccc}
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This was the “pedestrian” version!

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Full picture: study $M$ with boundary

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The right way to compactify $\mathcal{X}_g$ on a space w/ bdy is to stretch to bdy to asymptotic regions

$\mathcal{X}_g$ on $\mathbb{R}^3 \times \partial M_1 \times \mathbb{R}_+ \times \partial M_2 \times \mathbb{R}_+$
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The right way to compactify $\mathcal{X}_g$ on a space w/ bdy is to stretch to bdy to asymptotic regions

$\mathcal{X}_g$ on $\mathbb{R}^3 \times M \cong \partial M_1 \times \mathbb{R}_+ \cup \partial M_2 \times \mathbb{R}_+$

3d interface between 4d (N=2) SUSY theories

[Dimofte-Gaiotto-v.d.Veen ’13]
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“2d-4d correspondence”

[Dimofte-Gaiotto-v.d.Veen ‘13]

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The correspondence is functorial:

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The correspondence is functorial:

\[ T_g : \text{Cobordism category of 2-manifolds} \rightarrow \text{Cat. of 4d N=2 SUSY thy’s} \]

objects: 2-manifolds
morphisms: 3-cobordisms

\[ \mathcal{X}_g \ 	ext{on} \ R^3 \times M \]
\[ \partial M_1 \times R_+ \quad \downarrow \quad \partial M_2 \times R_+ \]

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Can extend further,

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“4d-2d correspondence”

cf. [Gadde-Gukov-Putrov ’13]
The correspondence is effective
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For a large class of 3-manifolds, can explicitly compute $T_g[M]$
give an explicit 3d Lagrangian density
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  = admit a metric of constant neg. curvature)
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- method of computation: cut $M$ into (topological) ideal tetrahedra

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$$M = \bigcup_{i=1}^{N} \Delta_i$$

$$T_g[M] = \left( \bigotimes_{i=1}^{N} T_g[\Delta_i] \right) / \sim$$

truncated vertices
The correspondence is effective

Remainder of the talk: \[ g = sl_2 \quad G_\mathbb{C} = SL(2, \mathbb{C}) \quad \text{(or} \quad PSL(2, \mathbb{C}) \quad = \quad SL(2, \mathbb{C})/\{\pm 1\}) \]

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- for simplicity, and some added intuition \( g = sl_n \)

[Dimofte-Gabella-Goncharov '13]

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[Dimofte-Gabella-Goncharov ’13]

- \( PSL(2, \mathbb{C}) \) flat connections are (roughly) hyperbolic metrics

So: \( T_g[M] \) quantizes, categorifies, etc. classical hyperbolic geometry!

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Single (ideal, hyperbolic) tetrahedron:

$$\partial \mathbb{H}^3 \cong \mathbb{C} \cup \{\infty\}$$
The correspondence is effective

Single (ideal, hyperbolic) tetrahedron:

- vertices at on the bdy of $\mathbb{H}^3$
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Single (ideal, hyperbolic) tetrahedron:

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- the hyperbolic structure is encoded in 6 complexified dihedral angles

$$z = e^{(\text{torsion})+i(\text{angle})}$$
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equal on opposite edges, and satisfy

$$zz'z'' = -1$$

$$z'' + z^{-1} - 1 = 0$$

[W. Thurston, late '70's]
The correspondence is effective

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\]

[Dimofte '10]

\[
\bigcup \mathcal{M}_{\text{flat}}(\Delta, G_{\mathbb{C}}) = \{z'' + z^{-1} - 1 = 0\}
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Tetrahedron theory: \( T[\Delta] = \) single free chiral superfield

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[Dimofte-Gaiotto-Gukov '11]
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\[
\Phi \quad \text{or} \quad \phi, \psi
\]
complex scalar, complex fermion

(function on \( \mathbb{R}^3 \)) (section of spinor bundle on \( \mathbb{R}^3 \))

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Lagrangian: \( \mathcal{L} = |\partial_\mu \phi|^2 + \overline{\psi}(\sigma \cdot \partial)\psi \)

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Tetrahedron theory: \( T[\Delta] = \) single free chiral superfield

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complex scalar, complex fermion

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[Dimofte-Gaiotto-Gukov ’11]

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E.g. \( Z_{T[Δ]}[S^{2} × S^{1}] = \prod_{r=0}^{∞} \frac{1 - q^{1 - \frac{m}{2}} \zeta^{-1}}{1 - q^{\frac{m}{2}} \zeta} \) \[ q = e^{\frac{2π}{σ}} \]
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Categorical/homological invariant:

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That categorifies the volume of a hyperbolic tetrahedron.
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In general, glue

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3 chiral multiplets \( \Phi_1, \Phi_2, \Phi_3 \)
cubic superpotential \( W = \Phi_1 \Phi_2 \Phi_3 \)
i.e. \( \mathcal{L} = ... + \psi_1 \psi_2 \phi_3 + |\phi_1 \phi_2|^2 + ... \)
The correspondence is good for physics

$T[M] = 3d \text{ SQED}$

2 chiral multiplets $\Phi_1, \Phi_2$

U(1) gauge sym. $+1$ $-1$

$T[M] = \text{“XYZ model”}$

3 chiral multiplets $\Phi_1, \Phi_2, \Phi_3$

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cf. classical \( \text{Li}_2(x) + \text{Li}_2(y) = \text{Li}_2\left(\frac{x}{1-y}\right) + \text{Li}_2\left(\frac{y}{1-x}\right) + \text{Li}_2\left(\frac{(1-x)(1-y)}{xy}\right) + \text{logs} \)
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Two examples:

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\[ g^2 \quad T[M] \]

\[ g'^2 \sim 1/g^2 \]

\[ \quad \rightarrow \quad 4 \text{ or } 5 \Delta's \]

\[ \quad \rightarrow \quad T[M] \]
Moral

There is interesting structure to be discovered and developed,

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- moduli spaces
- partition functions
- (SUSY) Hilbert spaces
- topological invariants
- categorification
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Relations like the 3d-3d correspondence allow both kinds of structure to be developed in tandem, with double the power and intuition.
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Relations like the 3d-3d correspondence allow both kinds of structure to be developed in tandem, with double the power and intuition.

I hope this type of work will find a place here at Davis.