Universal Dynamics from the Conformal Bootstrap

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in collaboration with
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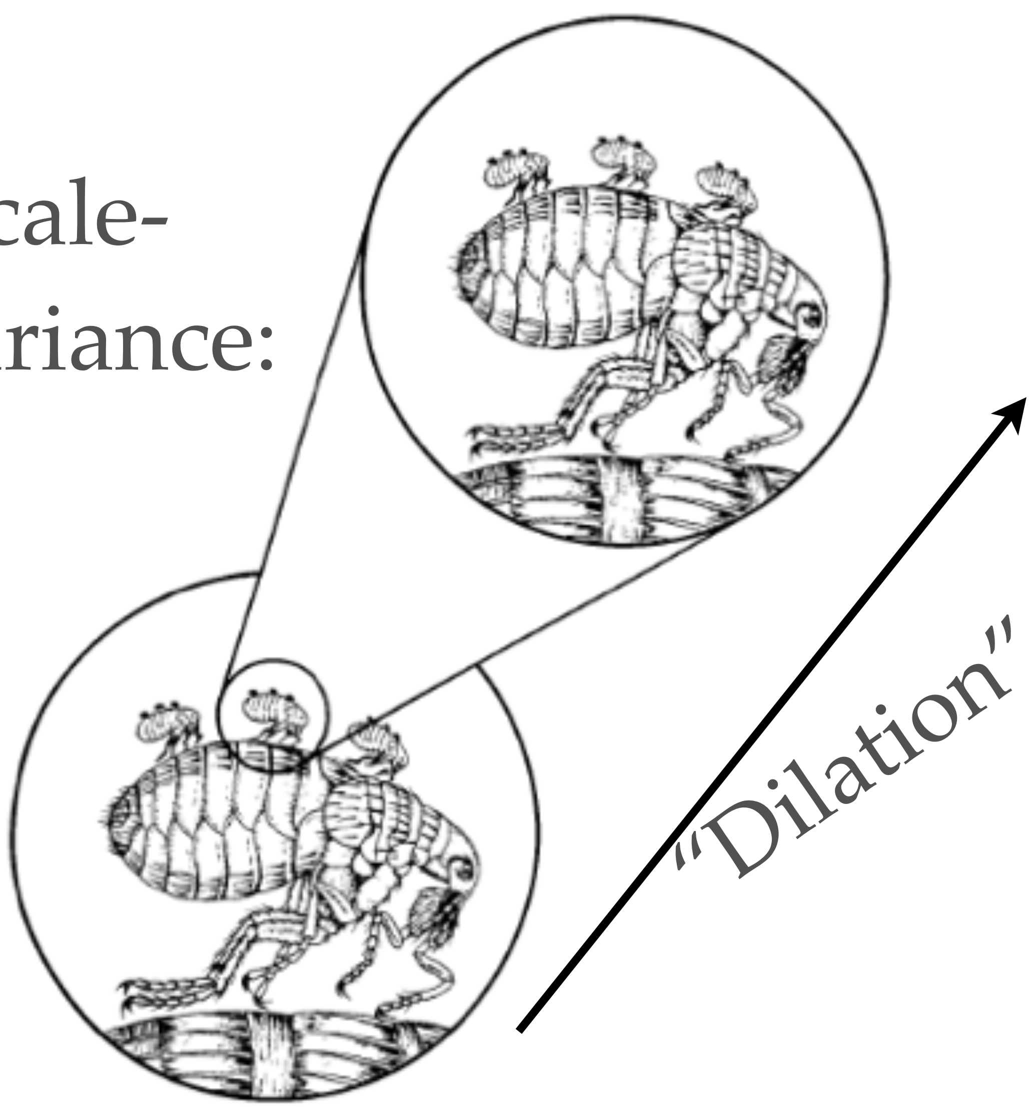
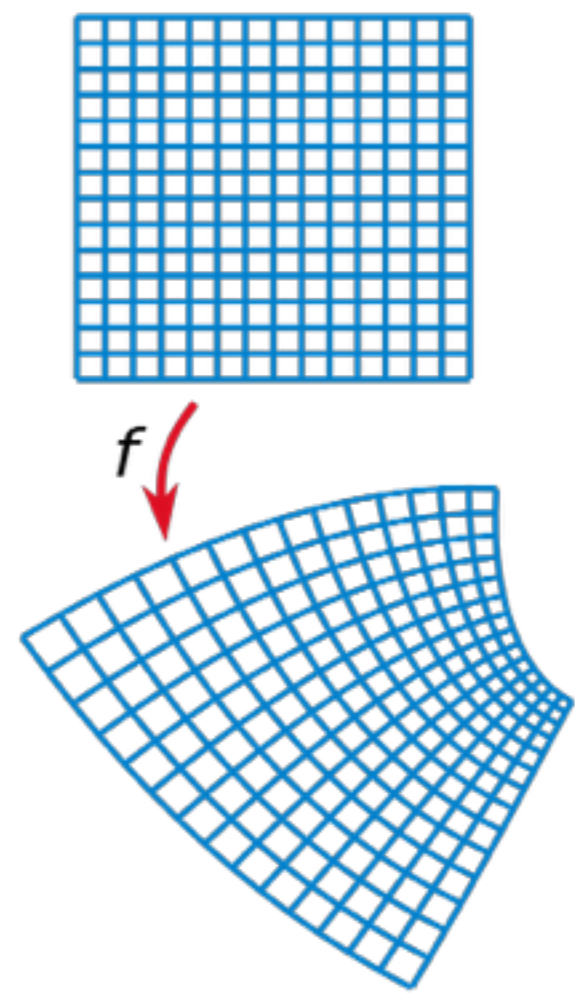
Conformal Symmetry

Conformal = coordinate transformations that preserve angles in special relativity

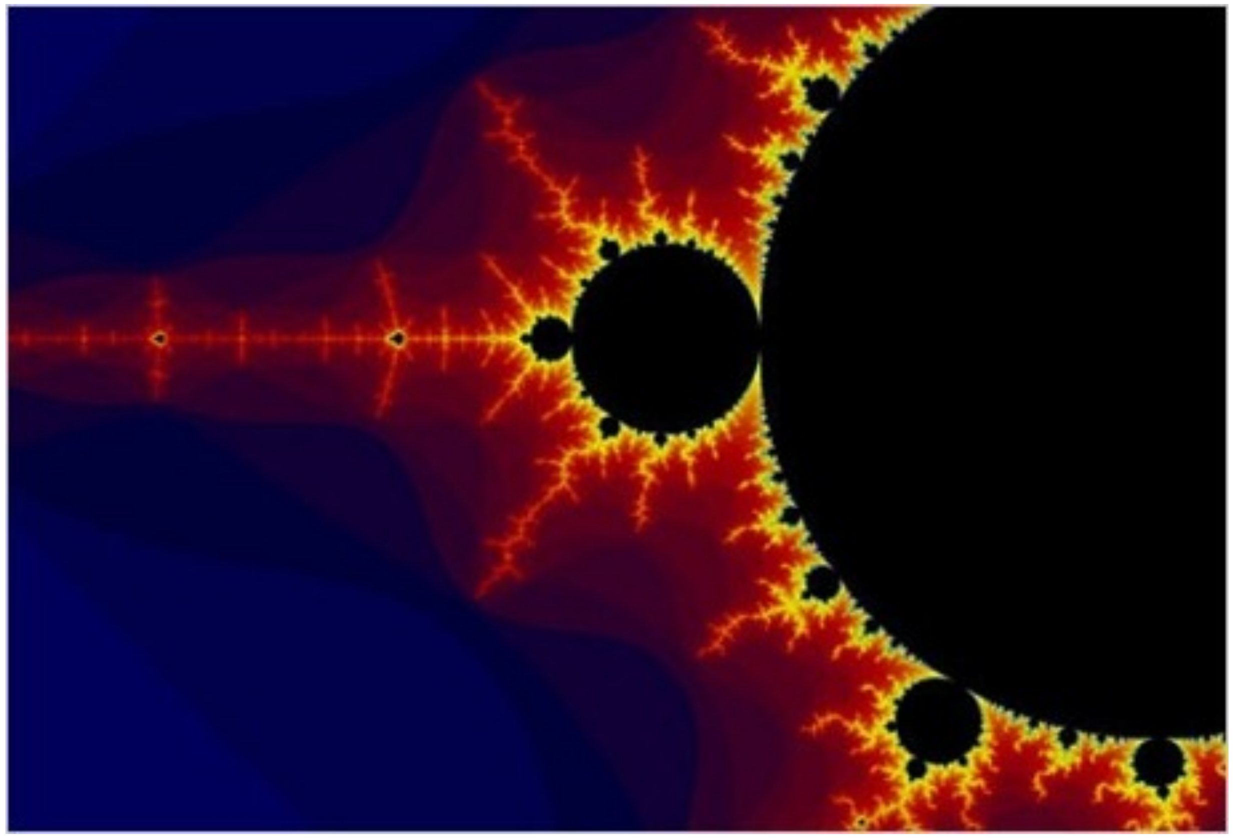
Includes scale transformations, rotations, etc.

Scale-invariance:





Why Study CFTs?



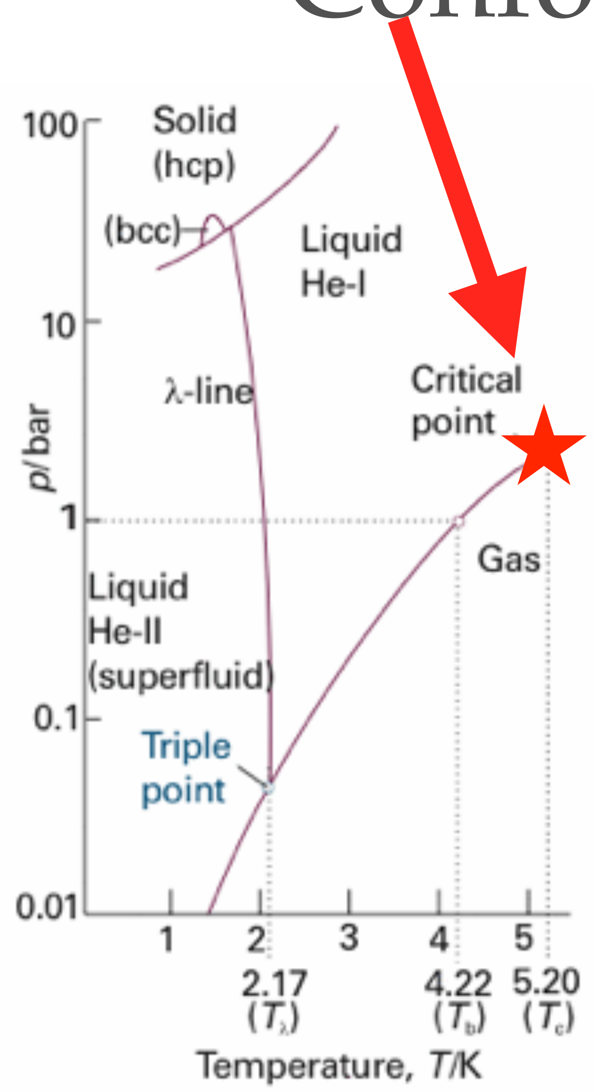
Why Study CFTs?

Conformal Field Theories (CFTs)
are central objects in physics

- Phase Transitions and Critical Phenomena
- Classical Gauge Theories in 4d
- Endpoints and Intermediate Regimes of RG Flow
- *and much more…*

CFTs and Critical Phenomena

Conformal Field Theories describe phase transitions at critical points



Helium-4

$$C_V \sim |T - T_c|^{-\alpha}$$

	CFT	Experiment	
α	0.1099(9)	0.110(7)	liquid-gas
		0.110(6)	binary liquid
	-0.011(1)	-0.012(6)	^{4}He
	-0.115(1)	-0.10(4)	Ferromagnet: EuO

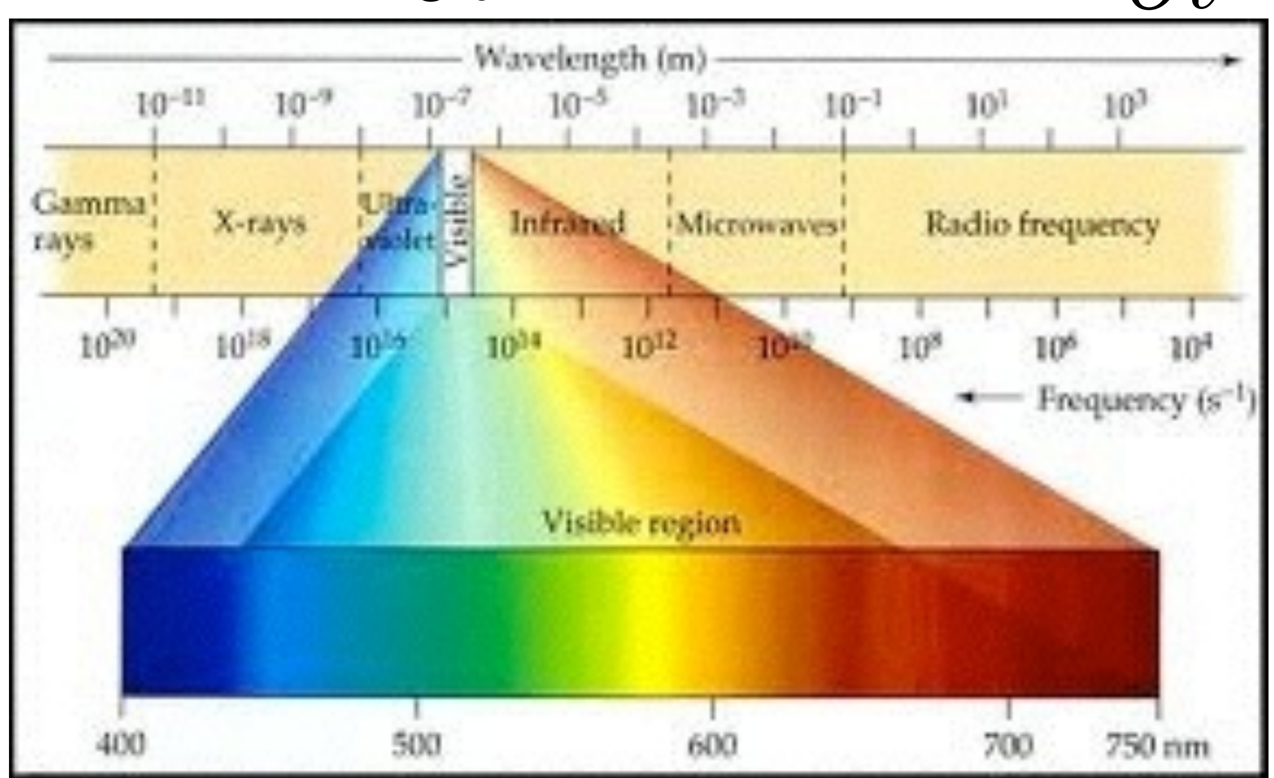
CFTs and Gauge Theory

Classical Gauge Theories

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t}$$

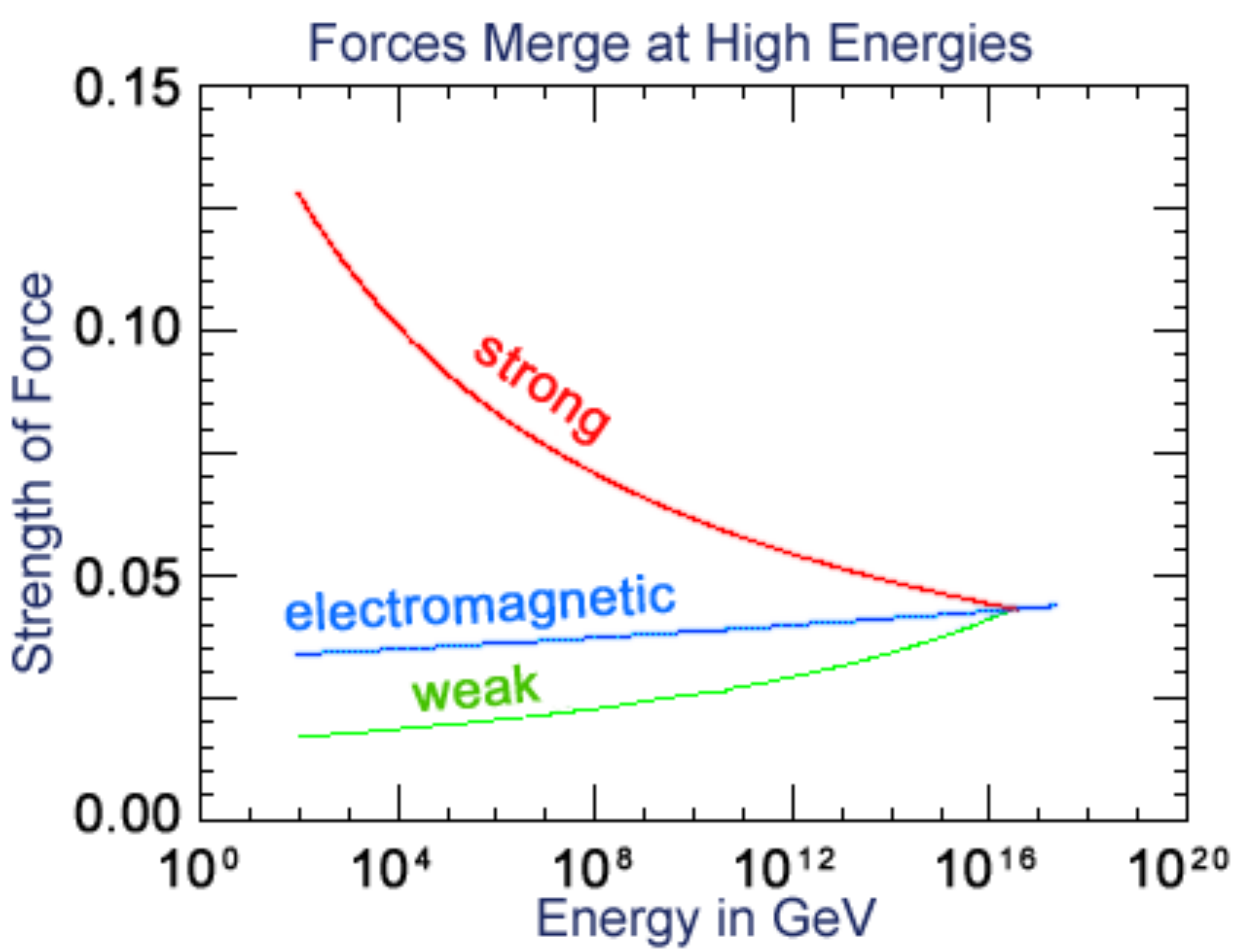
Scale-invariance



RG Flow

When we look at different energies, coupling constants change

Most famous example is grand unification



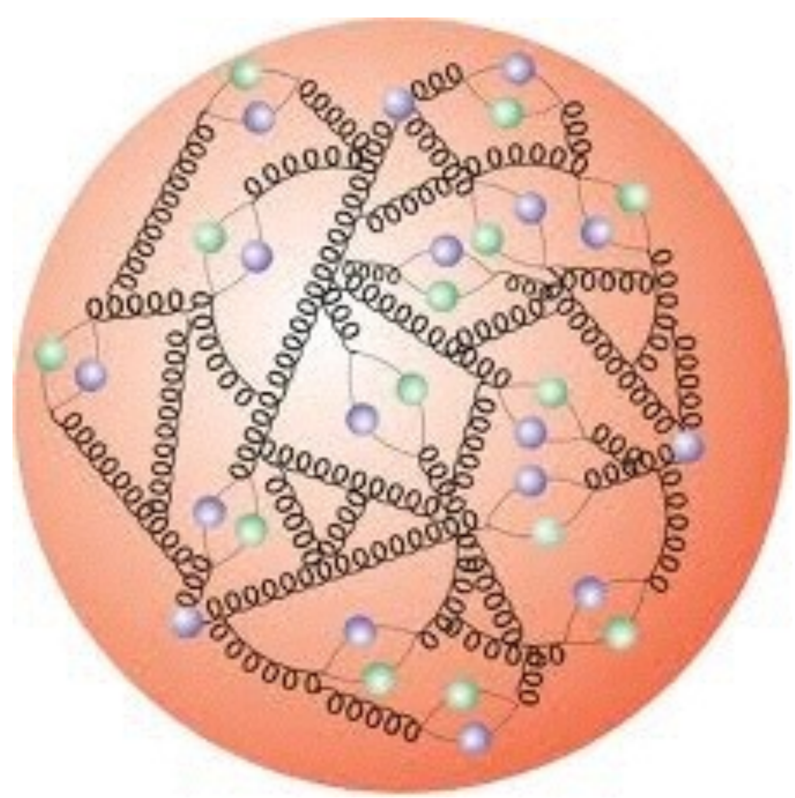
Quantum field theories are approximately scale-invariant over wide regimes separated by a few sudden phase transitions

CFTs and Strong Coupling

Strongly coupled fixed points

Strongly coupled theories are difficult to study.

Conformal invariance can give us a powerful tool to study their behavior.



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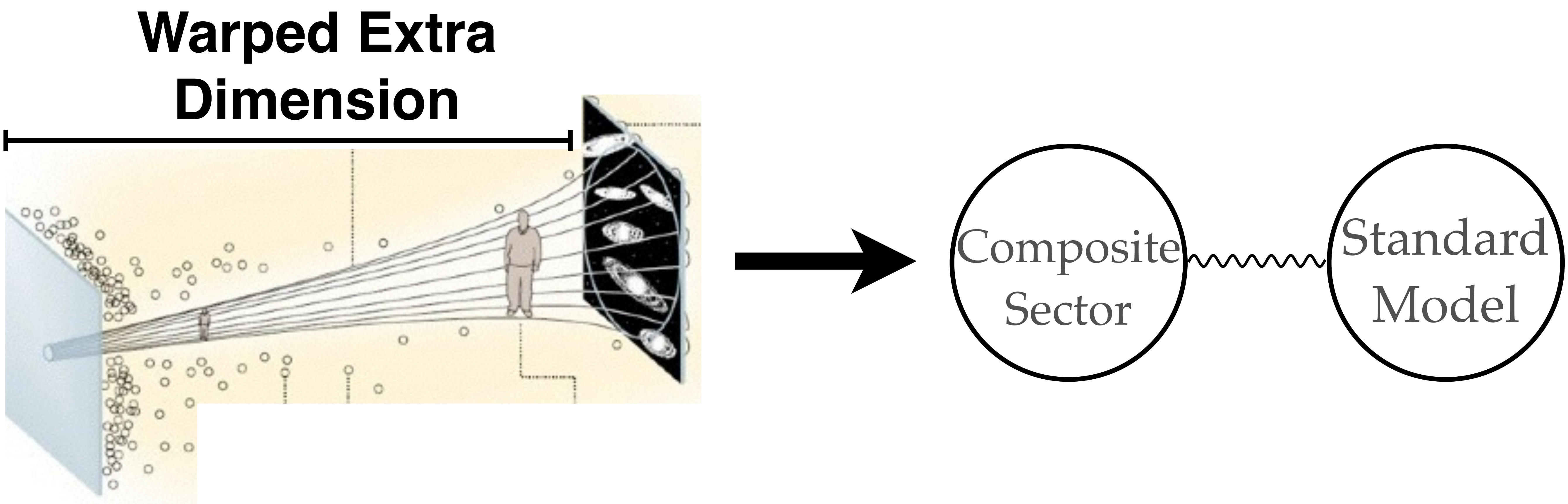
CFTs and Warped Extra Dimensions

5d Randall-Sundrum models have a purely 4d interpretation

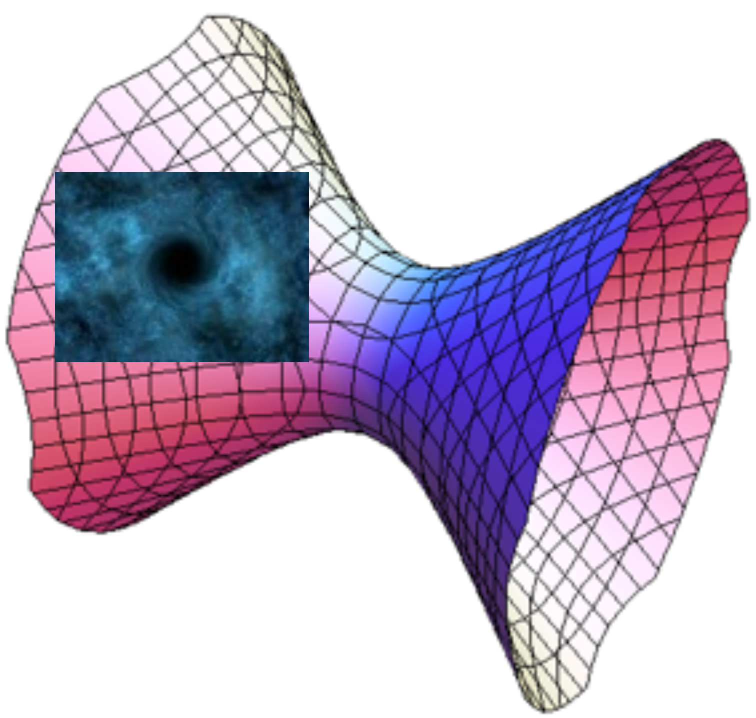


They describe ***some classes*** of strongly-coupled theories.

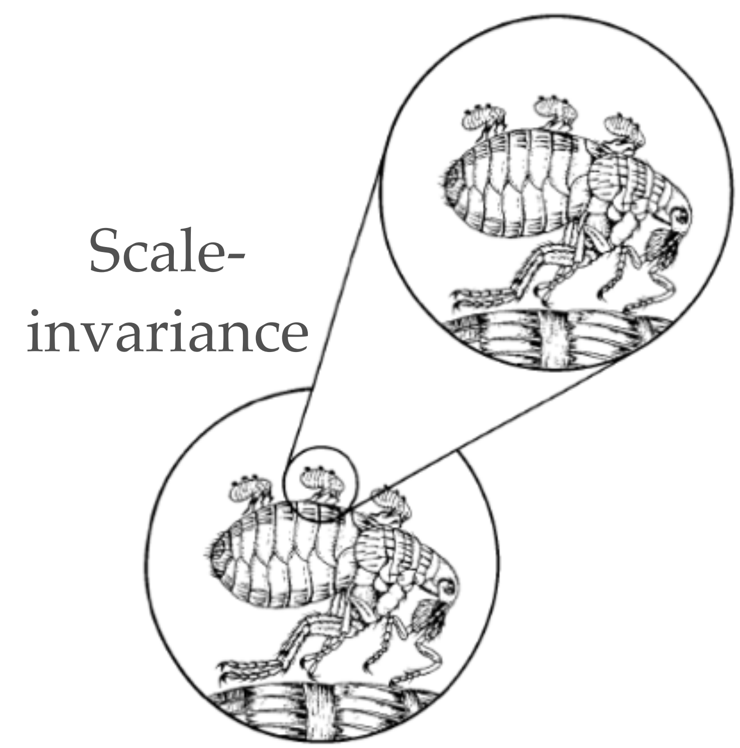
CFTs and Quantum Gravity

Gravity in Anti de Sitter in d+1 dimensions

Conformal Field Theory in d dimensions



equivalent!



So studying CFTs teaches us about gravity, and vice versa!

CFTs and Quantum Gravity

Why does an extra dimension emerge from a CFT?

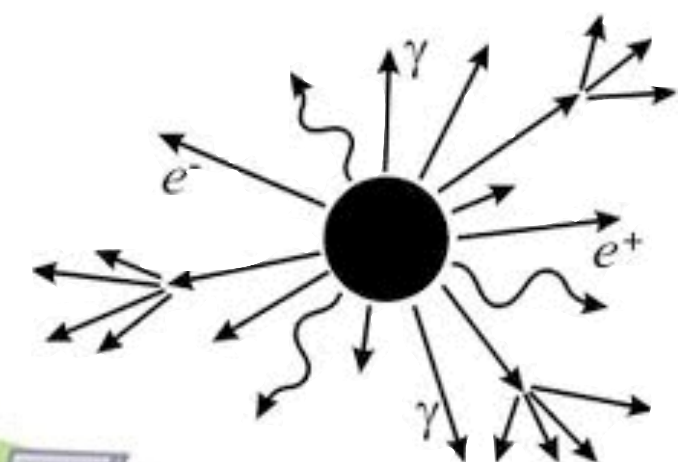
When is physics *local* in this extra dimension?

Why do gravitational interactions emerge?

Do they look like Newtonian gravity/General Relativity VS modified theories of gravity?

CFTs and Quantum Gravity

What can we learn about black hole dynamics?



Hawking radiation: Semi-classical limit says black holes have a temperature.

But if this is exactly true, then information is lost! Not consistent with Quantum Mechanics.

Can we understand “pure” states mimicking a thermal states?

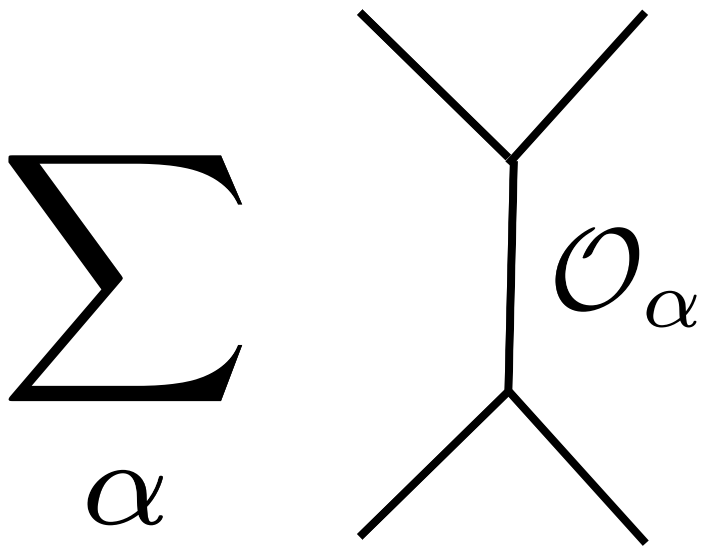
CFTs and Quantum Gravity

Goals:

1) Use the conformal bootstrap to learn about universal properties of CFTs
2) Interpret our results as statements about universal properties of the dual theories with extra dimensions. Specifically, want to derive *locality* and *gravitation* at long-distances.

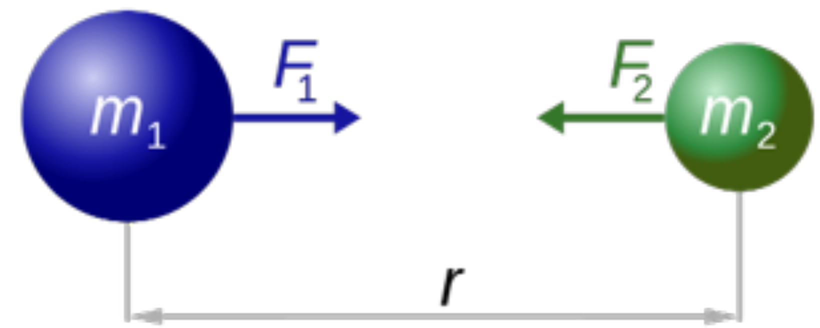
I however *do not* assume anything a priori about the dual gravitational theory.

CFT Bootstrap

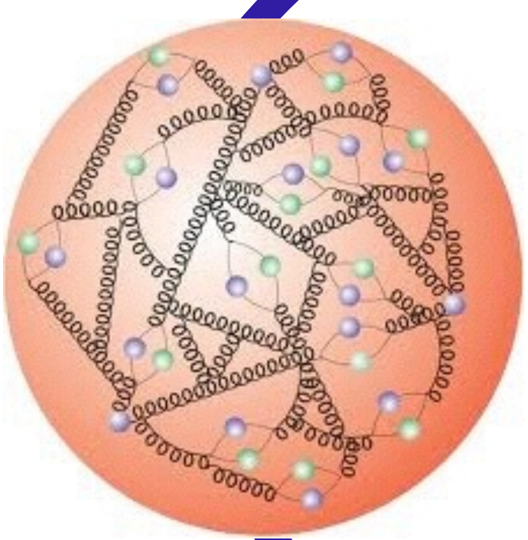


1)

2)



$$F_1 = F_2 = G\frac{m_1 \times m_2}{r^2}$$



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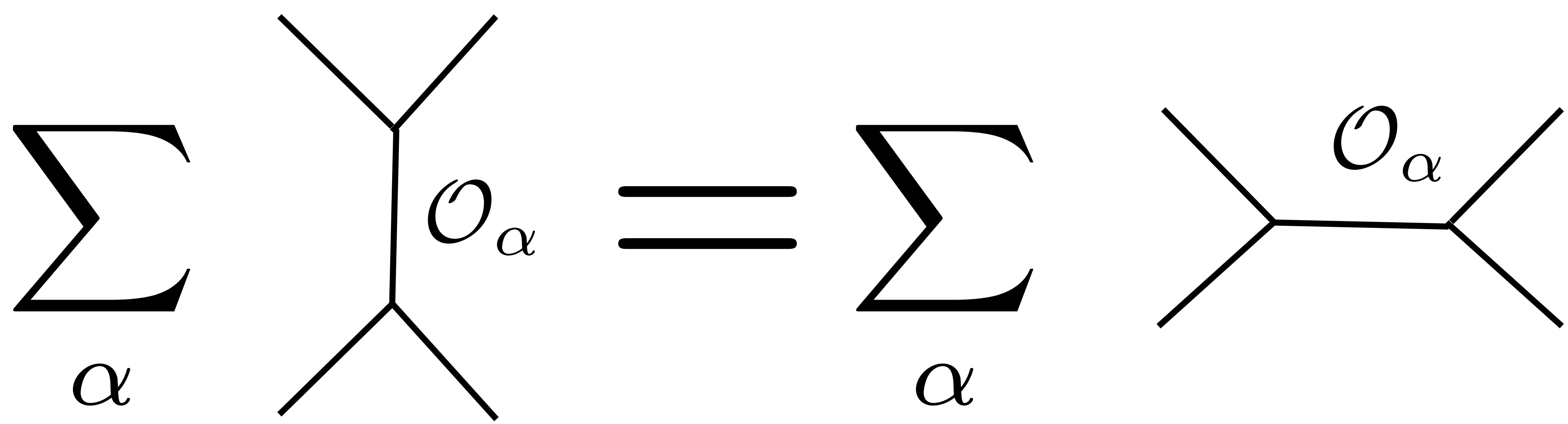
Extra Dimensional Duals

3) 2d CFTs:

Heavy states effectively deform CFT geometry

The Conformal Bootstrap



Operators

In conformal theories, a key role is played by “operators”, which can be any local observable

Simple Example: density operator $\rho(x)$

We study correlation functions among operators

$$\langle \rho(x) \rho(y) \rho(z) \rangle$$

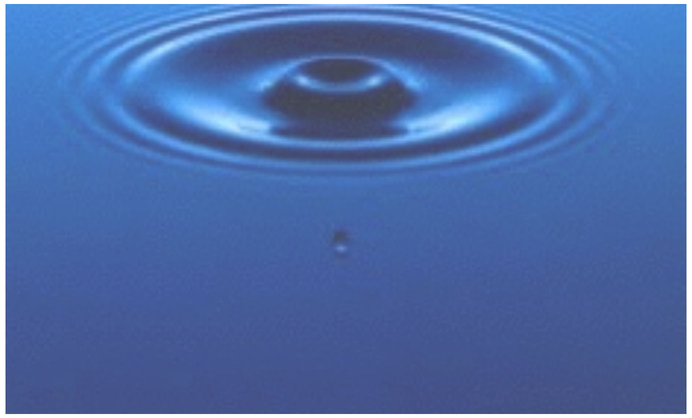
Operators and States

Every operator creates a unique state, and vice versa:

$$\rho(x)|0\rangle \leftrightarrow |\rho\rangle$$

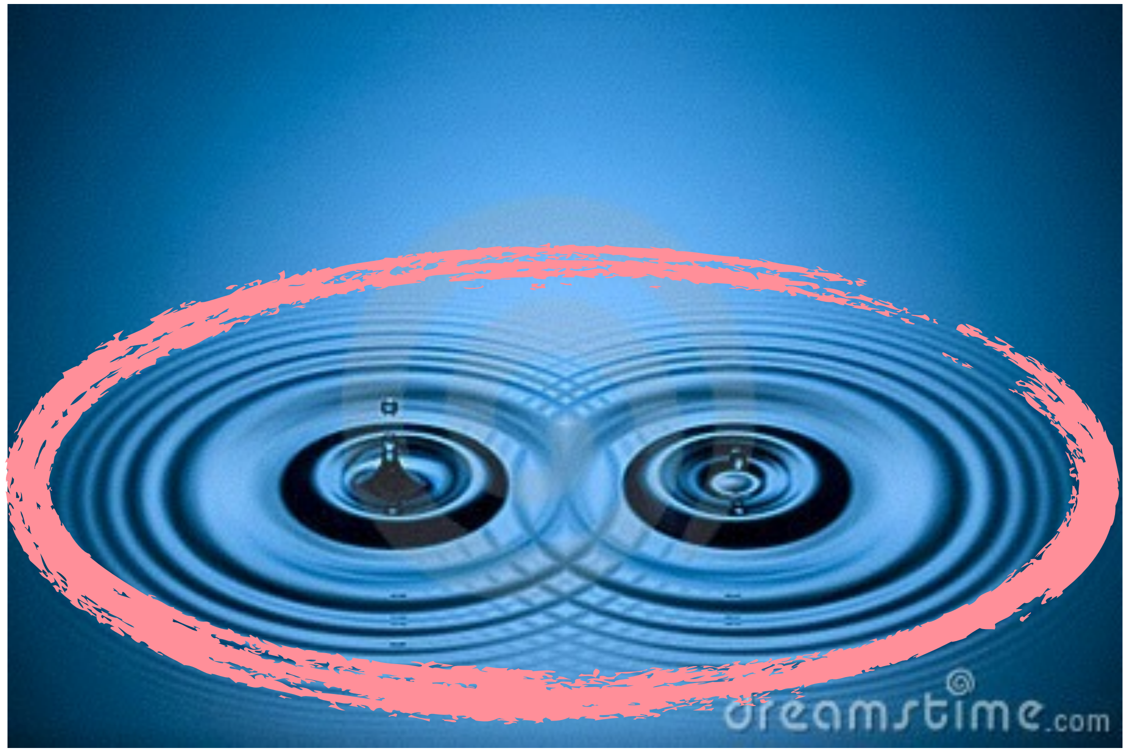
By "measuring" ρ, we perturb the vacuum and put it in a new state.

$$\rho(x)$$



Operator Products

Start with insertion of two operators



$Y_{\ell m}$

Decompose into a convenient basis at a fixed radius.

E.g. Spherical harmonics

Quantum: Decompose wavefunction

$$\psi(\theta, \phi) = \sum c_{\ell,m} Y_{\ell,m}(\theta, \phi)$$

Operator Product Expansion

Products of operators can be expanded as sums of operators

$$\rho(x)\rho(y) = \sum_i c_i(x-y)\mathcal{O}_i(y)$$

Wilson '69
Zimmerman '70

This is very powerful when used inside correlation functions

$$\langle \rho(x_1)\rho(x_2)\rho(x_3)\rho(x_4)\rangle = \sum_\alpha c_\alpha^2 g_{\Delta_\alpha,\ell_\alpha}(x_i)$$

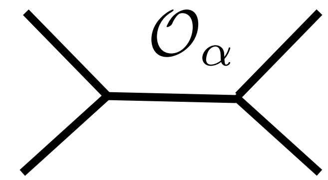
"conformal block"

So all correlators are determined by the two- and three-point functions. Conformal symmetry reduces the terms in the sum to *unknown numbers* times ***known functions***.

Conformal Bootstrap

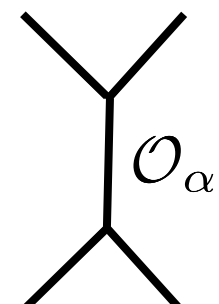
We can contract in the "s-channel"

$$\langle \rho(x_1)\rho(x_2)\rho(x_3)\rho(x_4)\rangle = \sum_\alpha$$



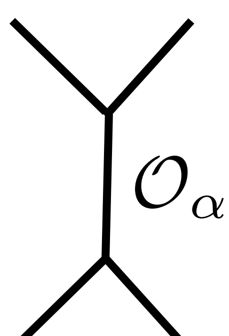
But we can also clearly contract in the "t-channel"

$$\langle \rho(x_1)\rho(x_2)\rho(x_3)\rho(x_4)\rangle = \sum_\alpha$$

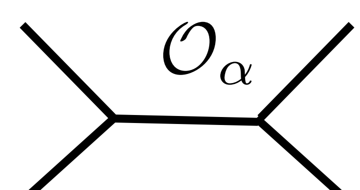


The conformal bootstrap is the constraint that these give the same answer. This simple statement contains an enormous amount of physics.

$$\sum_\alpha$$

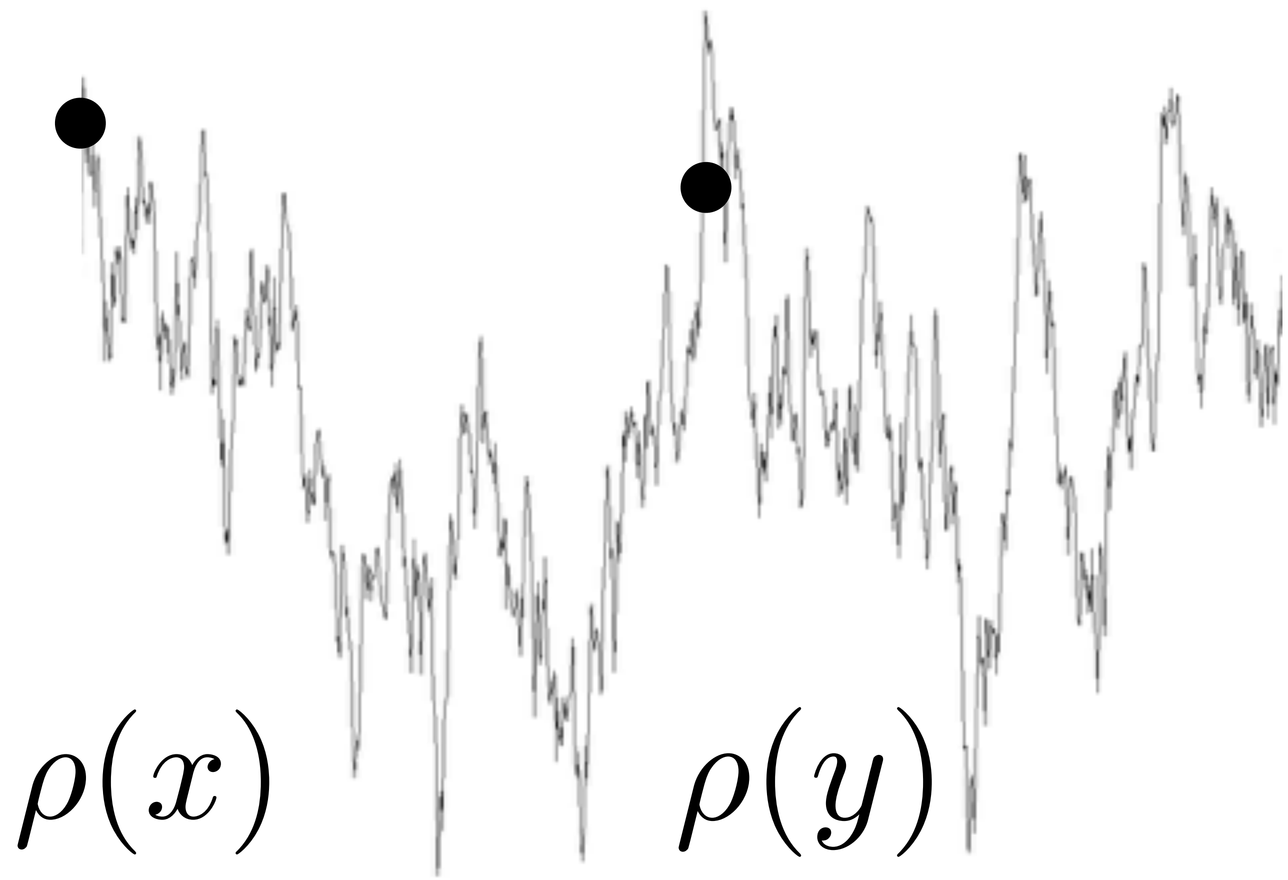


$$= \sum_\alpha$$



Conformal Bootstrap

Variance of fluctuations diverges on short distances



Rate of divergence is fixed by scaling dimensions

$$\langle \rho(x)\rho(y) \rangle \sim \frac{1}{|x-y|^{2\Delta_\rho}}$$

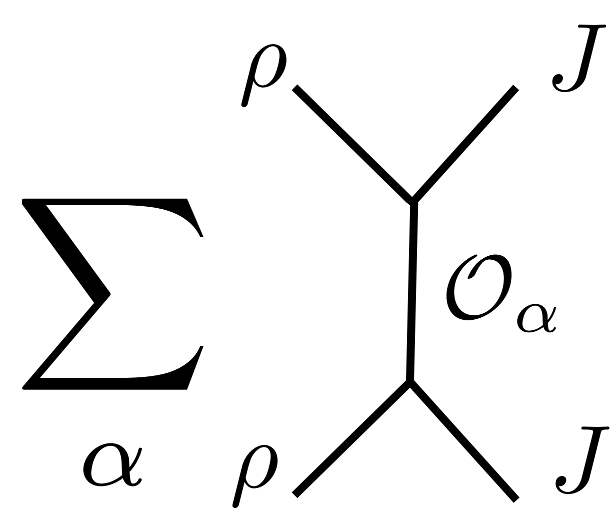
Conformal Bootstrap

This divergence contributes to the correlation function of four operators - e.g. two ρ's and two J's.

$$\langle \rho(x)\rho(y)J(z)J(w)\rangle \supset \langle \rho(x)\rho(y)\rangle\langle J(z)J(w)\rangle$$

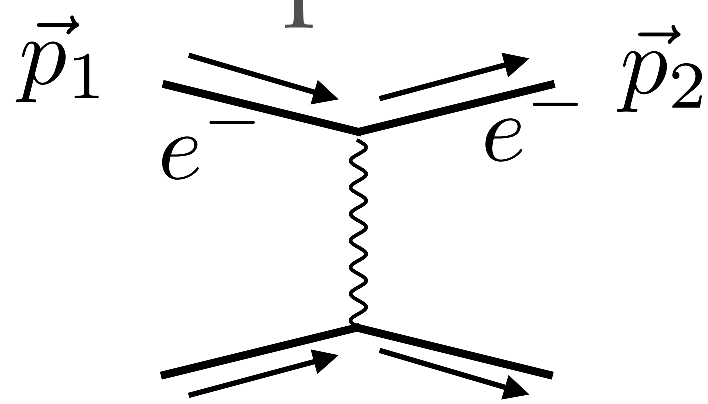
$$\boldsymbol{\rho}(x) \qquad \boldsymbol{\rho}(y) \qquad\qquad \boldsymbol{J}(z) \qquad \boldsymbol{J}(w)$$

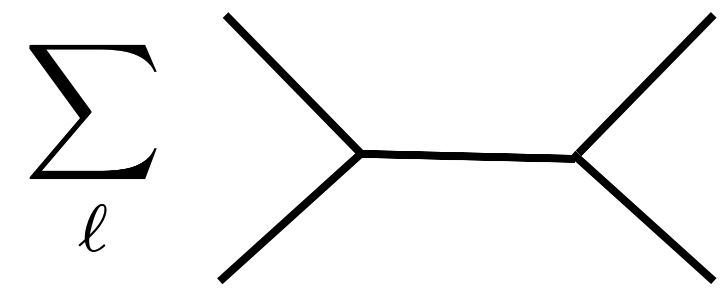
These singularities ***must be reproduced*** by the sums in the bootstrap equation.



Idea of the Method

Simple physical example: spherical harmonic decomposition of Coulomb scattering





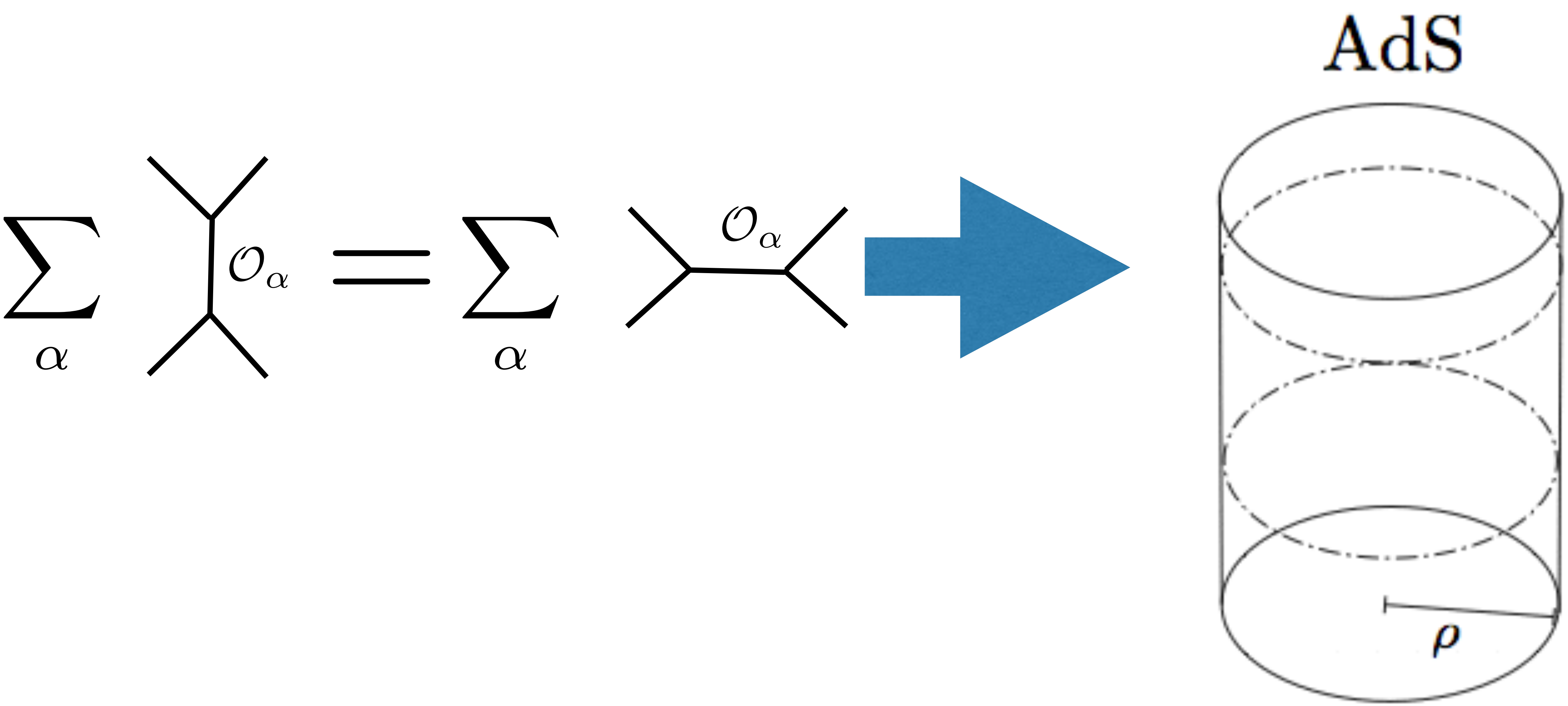
$$\frac{1}{(\vec{p}_1-\vec{p}_2)^2} = \frac{1}{2p^2(1-\cos\theta)} = \sum_\ell a_{\ell m} Y_{\ell m}(\theta,\phi)$$

Singular at cos θ =1

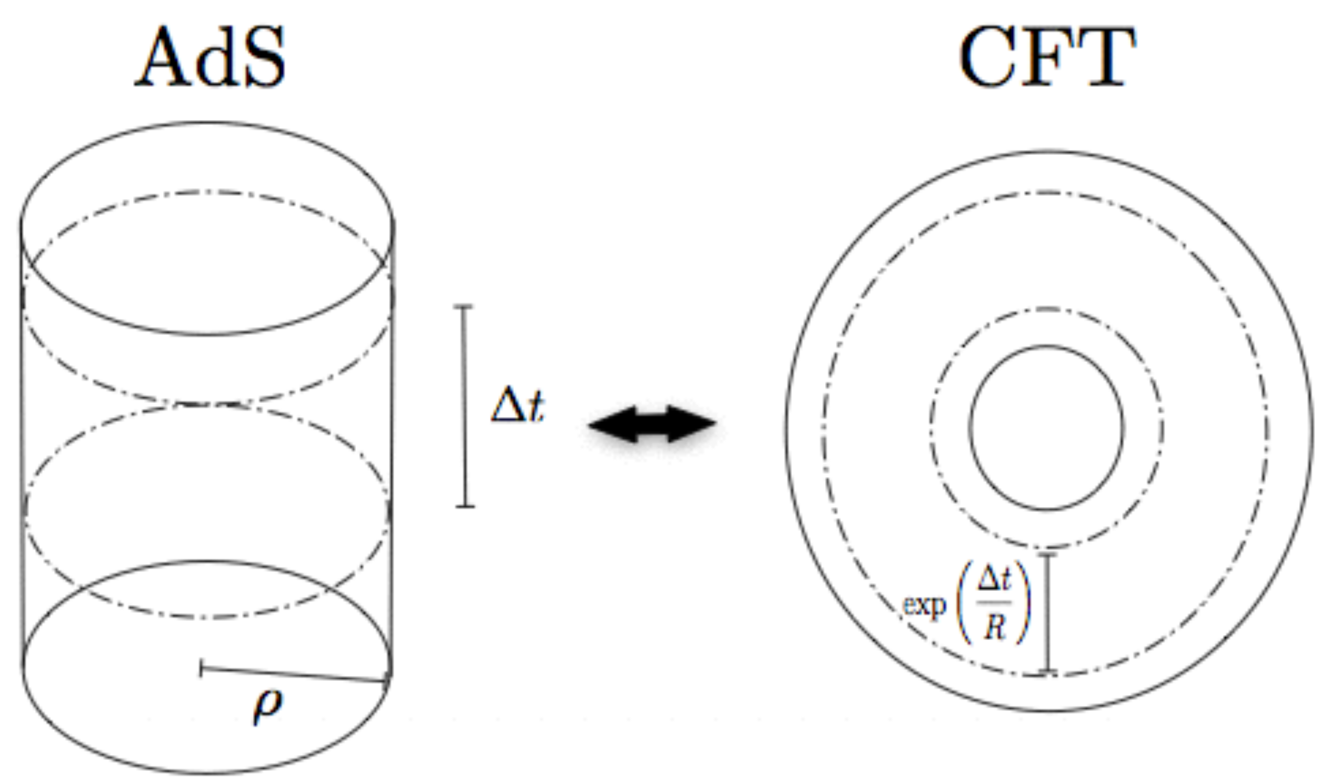
regular at cos θ =1

So we must have an infinite tower of spherical harmonics at large spin

We use this type of analysis of the bootstrap to prove the existence of certain large spin operator in CFTs, as well as obtaining their dimensions and OPE coefficients.
This will demonstrate important universal properties of the long-distance physics of their AdS duals.



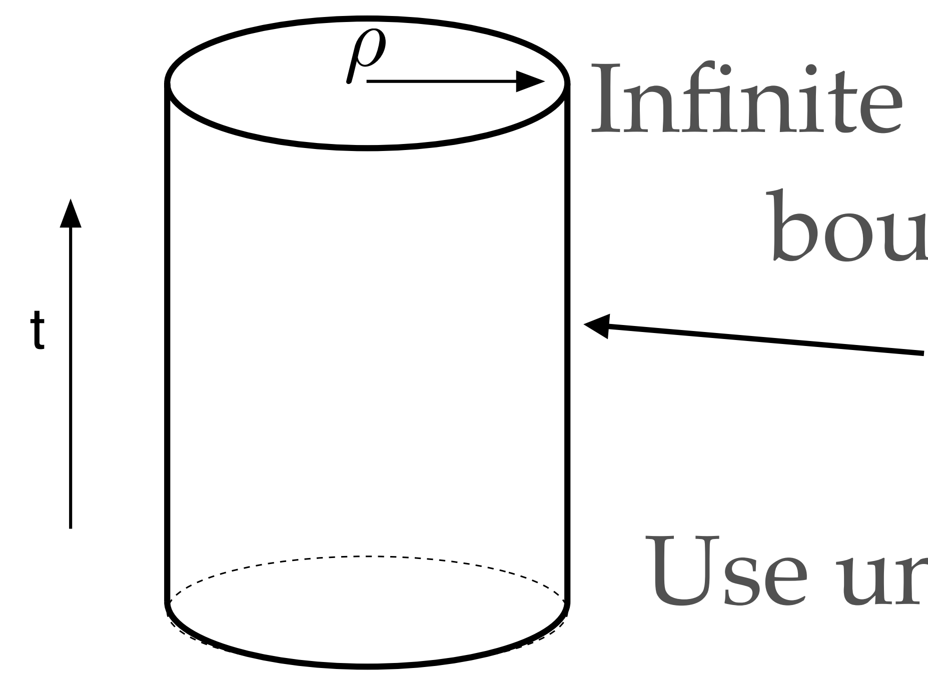
AdS Kinematics and CFT Kinematics



Anti-de Sitter

AdS is a very special box.

$$ds^2 = \frac{R_{\rm AdS}^2}{\cos^2\rho}\left(-dt^2 + d\rho^2 + \sin^2\rho \, d\Omega^2\right)$$



Infinite volume near boundary at $\rho = \frac{\pi}{2}$

Use units of $R_{\rm AdS} = 1$

The isometries of AdS are in one-to-one correspondence with the generators of the conformal group

AdS Energy = CFT Scaling Dimension

$$H_{\mathrm{AdS}} = D_{\mathrm{CFT}}$$

AdS Hamiltonian

Generates time evolution

CFT "Dilatation"

Generates scaling

$\xrightarrow{H_{\mathrm{AdS}}}$

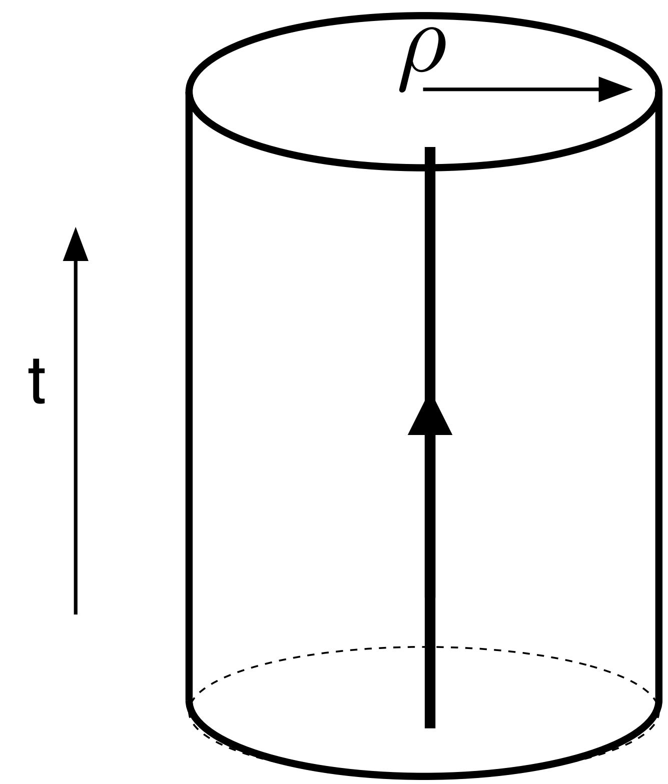
$\xrightarrow{D_{\mathrm{CFT}}}$

States have (center-of-mass) wavefunctions in AdS that are completely determined by symmetry

E.g. Primary states = States at rest in the center of AdS

Primary (roughly, not total derivatives):

$$|\mathcal{O}\rangle \Longleftrightarrow$$



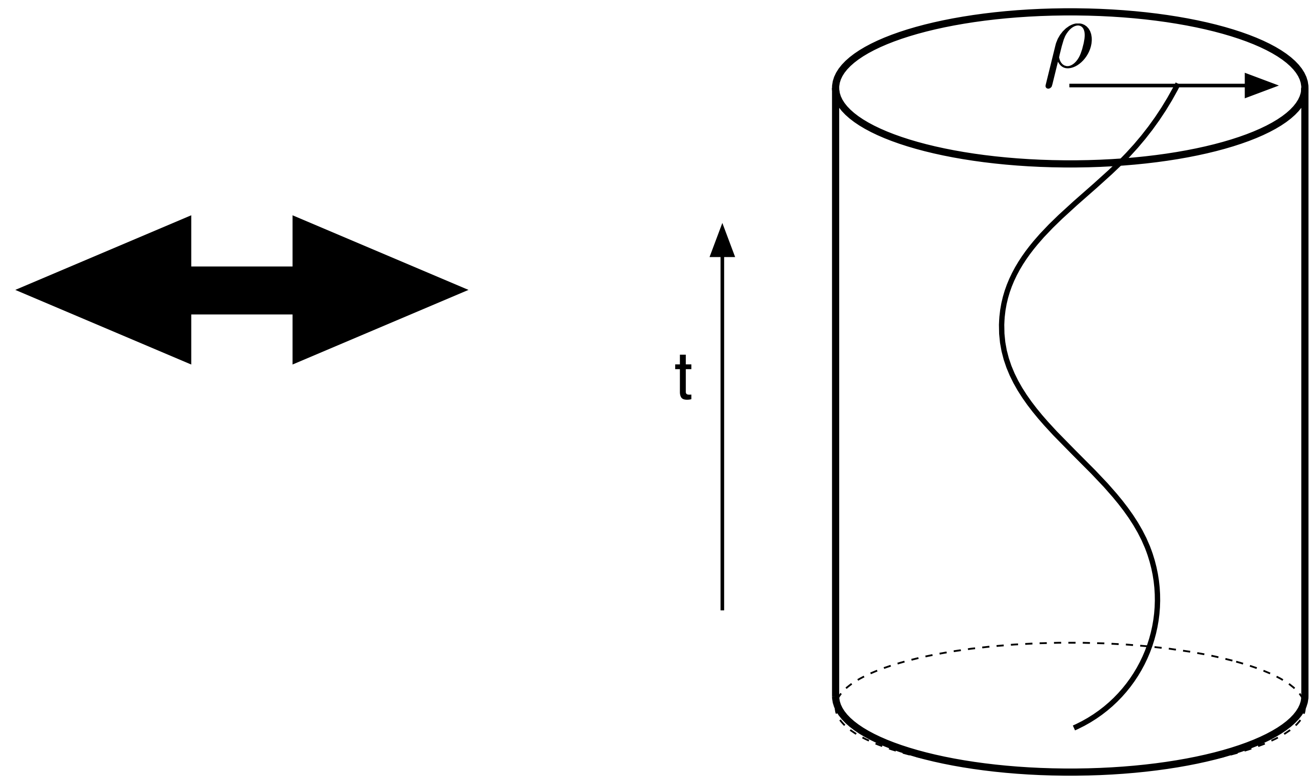
$$\psi(x) = e^{i\Delta t} \cos^{\Delta} \rho$$

AdS Wavefunctions

Descendant states $=$ states oscillating in AdS

$$|\partial \mathcal{O}\rangle$$

$$|\partial^{2n} \mathcal{O}\rangle$$

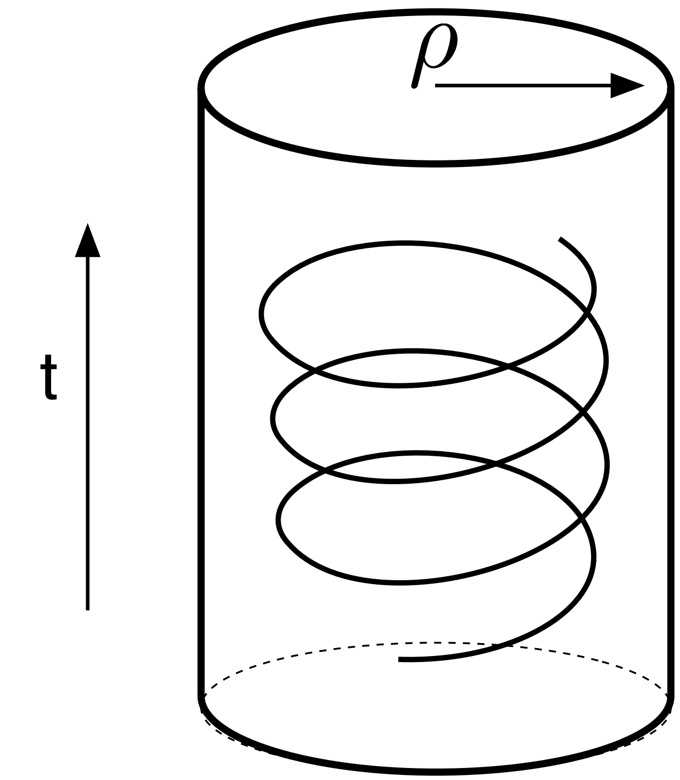


$$e^{i(\Delta+2n)t}\psi_n(\rho)$$

AdS Wavefunctions

Descendant states with large spin $=$ states orbiting center of AdS

$$|\partial_{\mu_1} \cdots \partial_{\mu_\ell} \mathcal{O}\rangle \Longleftrightarrow$$



$$e^{i(\Delta+\ell)t}\psi_\ell(\rho)Y_\ell(\Omega)$$

They orbit center of AdS at a radius given by

$$r \sim L$$

Locality

So far, coordinates in these wavefunctions are just labels. To really count as an extra dimension, there should be a sense of "locality" or "cluster decomposition":

Say we have a "rock" state:

and a "scissors" state:

Can they co-exist peacefully? I.e. do there exist "rock+scissors" states where they are "far apart" and ignore each other?

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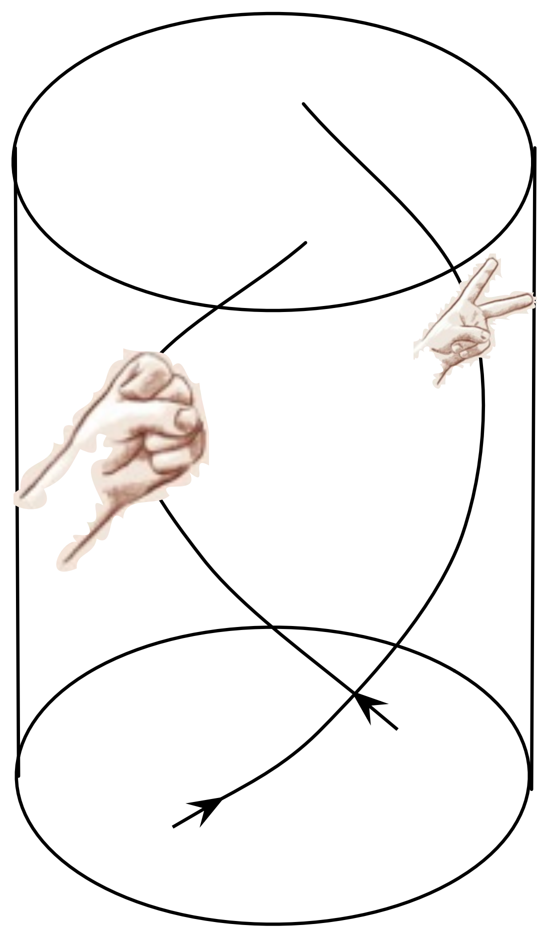
Cluster Decomposition

How do can make these states "far apart in AdS"?

Solution: give each of them large angular momentum so they orbit the center at large distance



So we want to look for states in the CFT with large angular momentum.

ALF, Kaplan, Poland
Simmons-Duffin

Cluster Decomposition

So by "cluster decomposition", I will mean that given any two primary operators ϕ_{rock} and ϕ_{sciss} in the CFT,

Theorem: **all CFTs in d≥3**

There exist an infinite number of operators at large spin, with dimensions

$$\underbrace{\Delta_0(L)}_{\text{“free” part}} \quad \underbrace{+\gamma(L)}_{\text{“interaction part”}} \quad \text{where } \lim_{L\to\infty} \gamma(L) = 0$$

Proof follows from matching singularities in bootstrap equation

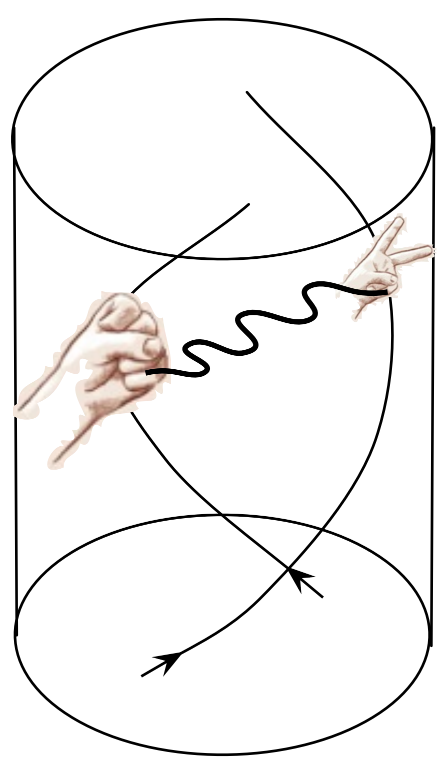
ALF, Kaplan, Poland
Simmons-Duffin

Binding Energy

Furthermore, binding energies for weak AdS interactions show up as a correction to this dimension i.e. an anomalous dimension

$$\Delta(L) = \Delta_0(L) + \gamma(L)$$

This gives us a simple CFT handle on the strength of interactions at large distances



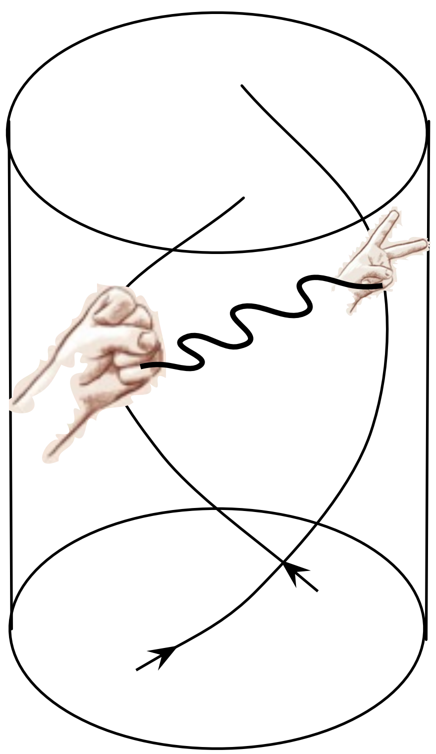
$$\gamma(L) \sim \gamma(r)$$

ALF, Kaplan, Poland
Simmons-Duffin

"Newtonian" Gravity in d≥3

The bootstrap also has a universal singularity from the stress tensor in the CFT.

At long distances, this produces a weak "Newtonian" binding energy



$$\gamma(r) \sim G_N \frac{m_{\text{rock}} m_{\text{sciss}}}{r^{d-2}}$$

AdS		CFT
m_{rock}	$\leftrightarrow$	Δ_{rock}
r	$\leftrightarrow$	L
G_N	$\leftrightarrow$	$\frac{1}{\text{Ndof}}$

All interacting CFTs in d≥3 have this "Newtonian gravity" correction to their binding energy at large spin.

ALF, Kaplan, Poland
Simmons-Duffin

2d CFTs

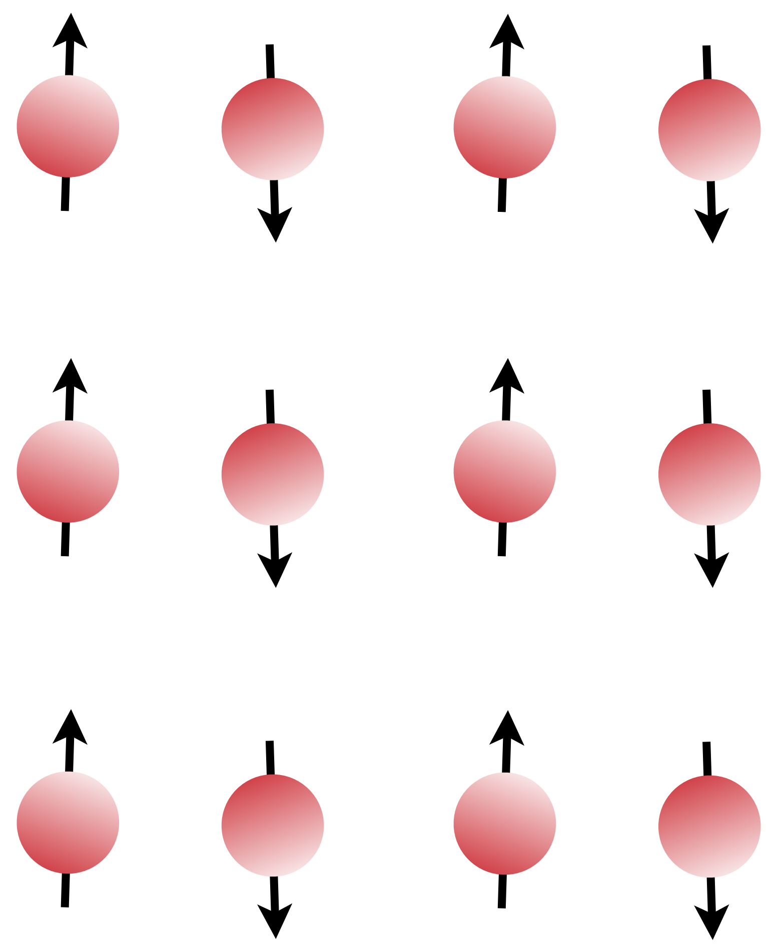
Why Focus on 2d?

Useful toy model: conformal symmetry is much bigger!

Dual to 3d gravity in AdS: Gravitons have no degrees of freedom, but there are still black holes.

Some other toy models:

2d Ising Model



Why Focus on 2d?

Useful toy model: conformal symmetry is much bigger!

Dual to 3d gravity in AdS: Gravitons have no degrees of freedom, but there are still black holes.

Some other toy models:

2d QCD at large N:
the gluon has no DOFs, and the theory is solvable.

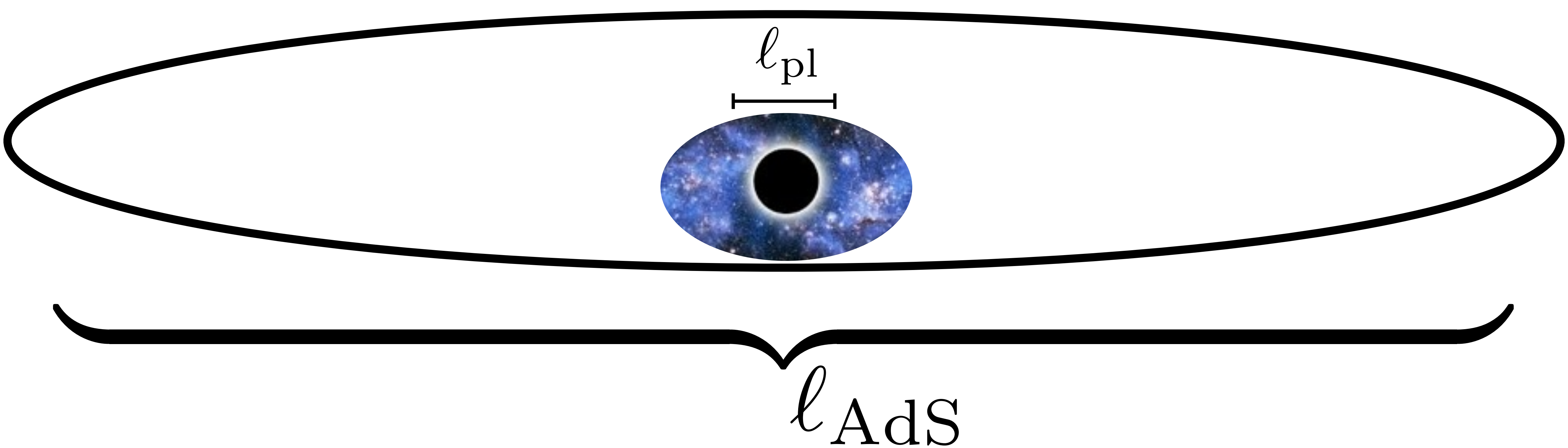
Large C

ALF, Kaplan, Walters

Consider large CFT central charge : essentially, large number of degrees of freedom. Like a classical limit.

The central charge is related to G_N in AdS by $c = \frac{3}{2G_N}$ so this is a "semi-classical" gravity limit

Brown, Henneaux, '86



Large C

How do we get interesting effects in gravity at $G_N \to 0$? Keep $G_N M \sim R$ fixed

What does this limit do in the CFT?

$$G_N \sim \frac{1}{c} \qquad \Delta \sim M \quad \longrightarrow \quad G_N M \sim \frac{\Delta}{c}$$

and in fact, by conformal symmetry

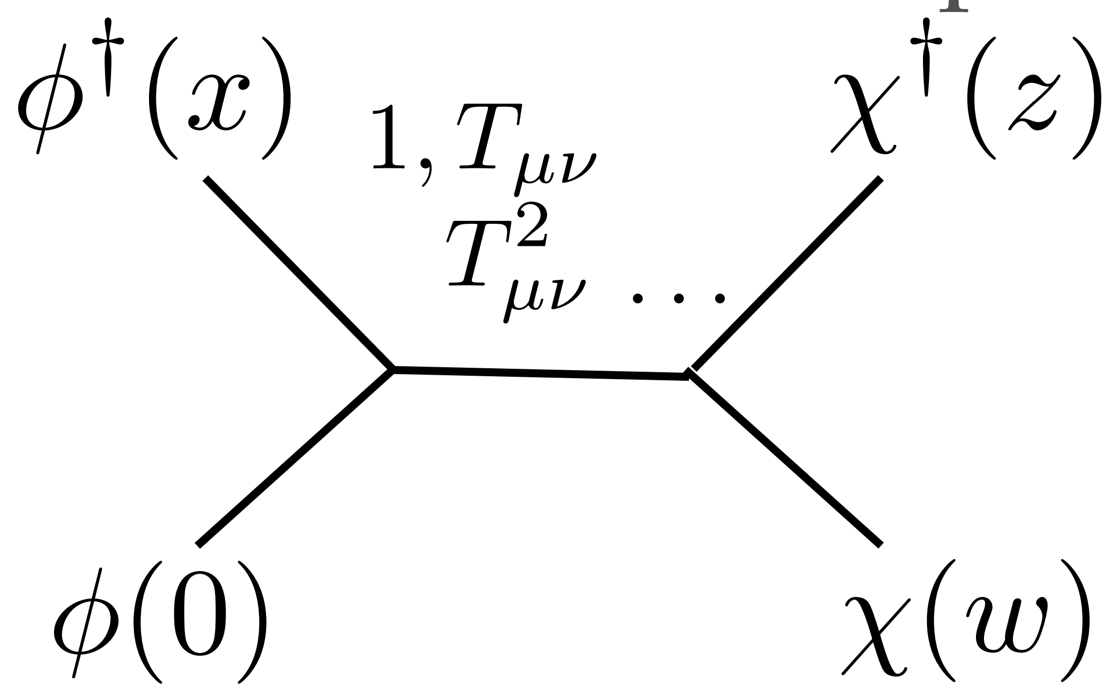
$$\phi(x)\phi(y) \supset \frac{\Delta}{c} T_{\mu\nu} \qquad \phi(x)\phi(y) \supset \left(\frac{\Delta}{c} T_{\mu\nu}\right)^n$$

so at $\frac{\Delta}{c} \sim \mathcal{O}(1)$, we must include all powers of the stress tensor.

This corresponds to resumming all multi-graviton contributions in AdS.

Large C and Classical Backgrounds

Again, focus on four operators:



We want to calculate the contribution from all powers of $T_{\mu\nu}$'s/$h_{\mu\nu}$'s

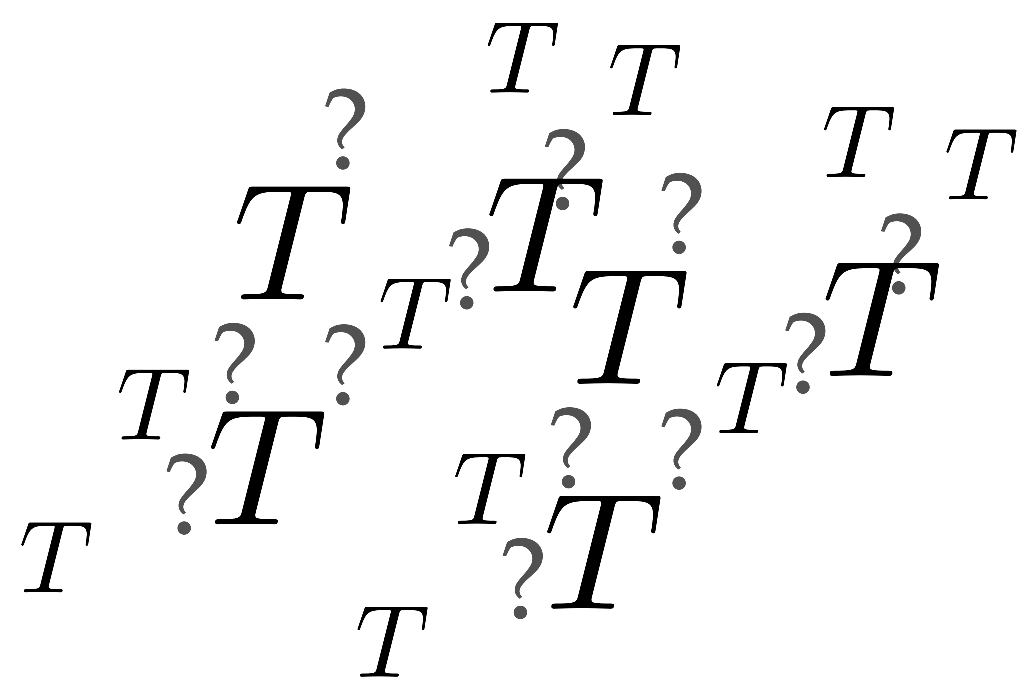
This "dressed" contribution is a **2d conformal block**

2D Conformal

Conformal group in 2d relates all of these contributions to each other!

$1, T, T^2, T^3, \ldots$ All related by symmetry.

So in principle, their contributions are known and computable! But not known in practice.



Large C and Classical Backgrounds

ALF, Kaplan, Walters

Consider the following "probe" limit:

$$c \to \infty$$
$$\Delta_{\rm rock} \to \infty$$

$$\frac{\Delta_{\rm rock}}{c} \text{ fixed} \qquad \frac{\Delta_{\rm sciss}}{c} \ll 1$$

"probe" of the background created by "rock"

Total contribution of *all* T^n can be computed and has a simple effect in this limit.

Large C and Classical Backgrounds

Idea: do a conformal transformation to put the CFT in a new metric.

Due to quantum effects, $T_{\mu\nu}$ gets a vacuum expectation value:

$$\langle \phi_{\rm rock}\phi_{\rm rock} T\rangle \rightarrow \langle \phi_{\rm rock}\phi_{\rm rock} T\rangle - \langle \phi_{\rm rock}\phi_{\rm rock}\rangle\langle T\rangle$$

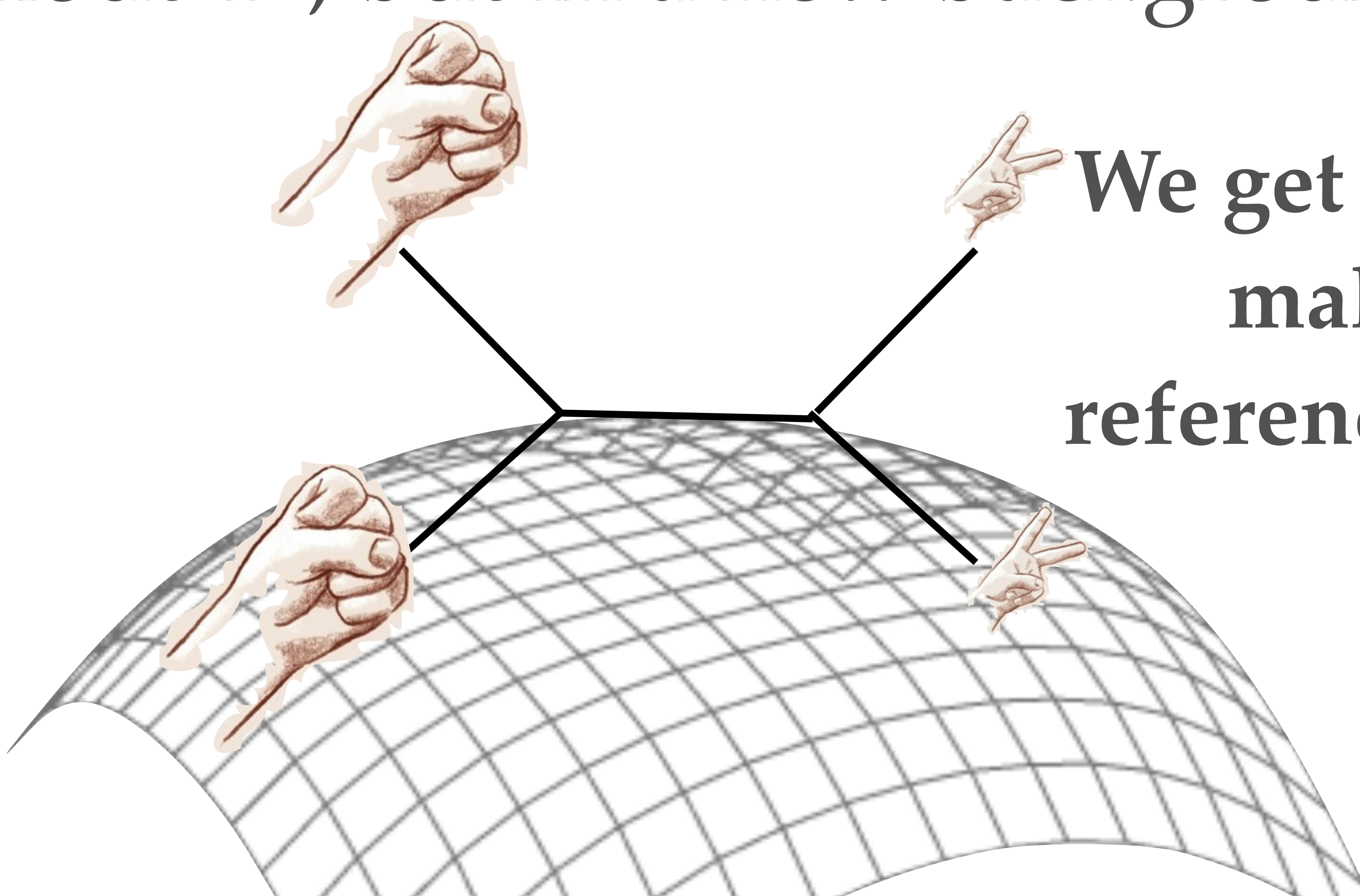
Want this to cancel

It turns out that this can always be done. At large C, all T^n contributions are completely canceled!

ALF, Kaplan, Walters

Large C and Classical Backgrounds

So the "conformal blocks" can be calculated in this limit, and they are just the contribution from the state without T^n, but in a new background metric.



We get this without making any reference to gravity!

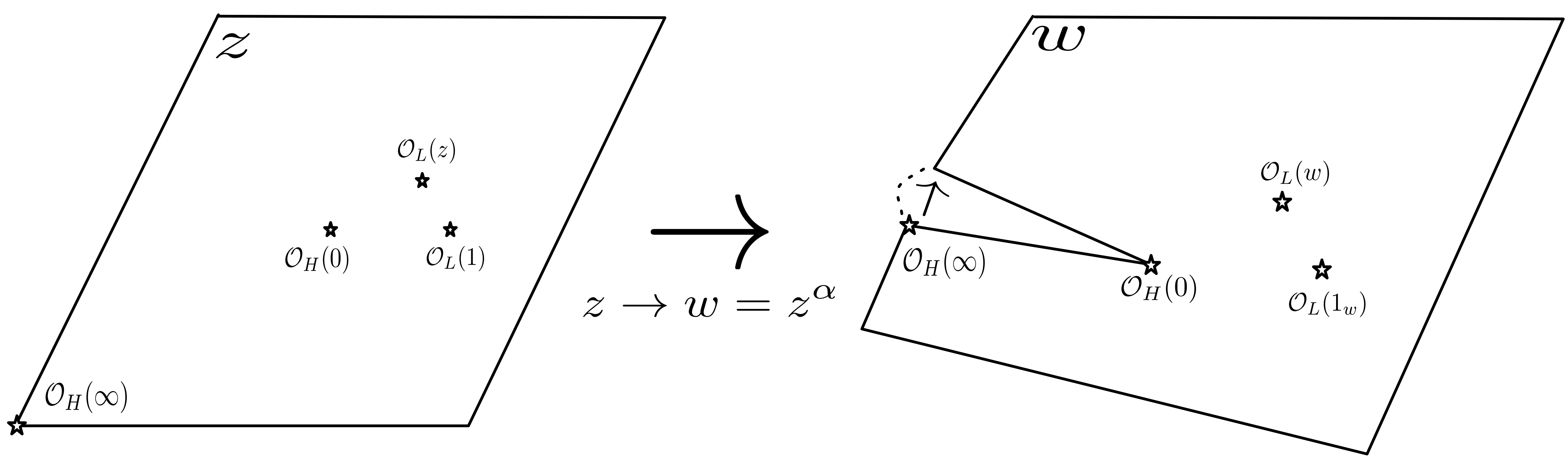
ALF, Kaplan, Walters

Large C and Classical Backgrounds

The coordinate transformation is very simple:

$$z \to w(z) = z^{\alpha} \qquad \alpha_{\text{rock}} = \sqrt{1 - \frac{12\Delta_{\text{rock}}}{c}}$$

Think of this as cutting out a wedge of space:

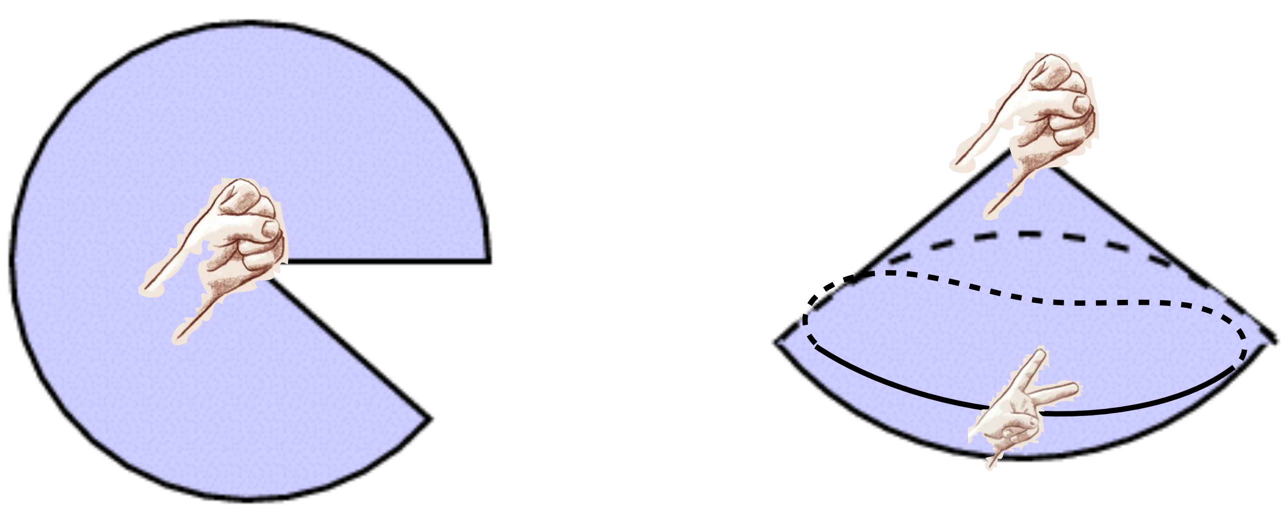


Relation to Gravity

In 3-dimensional AdS, there is a minimum threshold for black hole masses:

$$m_{\text{min}} = \frac{1}{8G_N} = \frac{c}{12}$$

Below this threshold, a local mass just makes a deficit angle singularity. It matches the CFT result!



$$\frac{\delta\phi}{2\pi} = \sqrt{1 - 8G_N m_{\text{rock}}} = \sqrt{1 - 12\frac{\Delta_{\text{rock}}}{c}}$$

Classical Background

What about $\Delta_{\rm rock} > \frac{c}{12}$? $\alpha_{\rm rock} = \sqrt{1 - \frac{12\Delta_{\rm rock}}{c}} \equiv 2\pi i T_{\rm rock}$

What is the  two-point function in this background?

The contribution from vacuum + T^n is

$\langle \phi_{\rm rock}|$  (t)  $(0)|\phi_{\rm rock}\rangle = \left(\frac{\pi T_{\rm rock}}{\sin(\pi T_{\rm rock} t)}\right)^{\Delta_{\rm sciss}}$ ← Exactly thermal!

~Eigenstate Thermalization Hypothesis

$|\phi_{\rm rock}\rangle =$ 

Relation to Gravity

What about $\Delta_{\text{rock}} > \frac{c}{12}$? $\quad (m > m_{\text{min}})$

Above this threshold, a black hole horizon forms in AdS

T_{rock} matches the temperature of the AdS black hole!

Future Directions

d≥3

ALF, Kaplan, Walters

In 2d, we used the vacuum expectation value of T to cancel the single-T contribution

$$\langle \phi_{\rm rock} \phi_{\rm rock} T \rangle \rightarrow \langle \phi_{\rm rock} \phi_{\rm rock} T \rangle - \langle \phi_{\rm rock} \phi_{\rm rock} \rangle \langle T \rangle$$

This can be done in 4d and 6d as well! The result depends only on the "a_d" anomaly coefficient.

New coordinates are periodic in time, with period

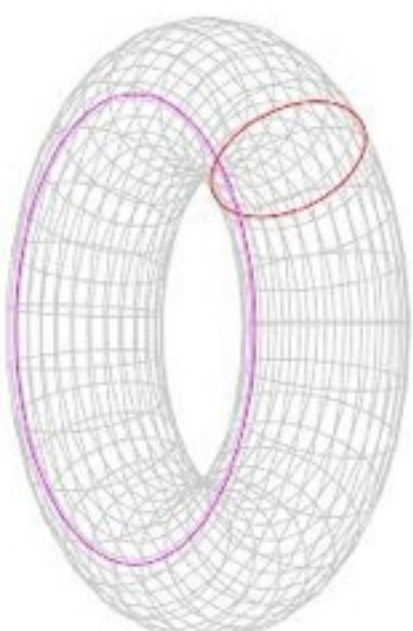
$$\beta^{-d} \sim \frac{\Delta_{\rm rock}}{a_d}$$

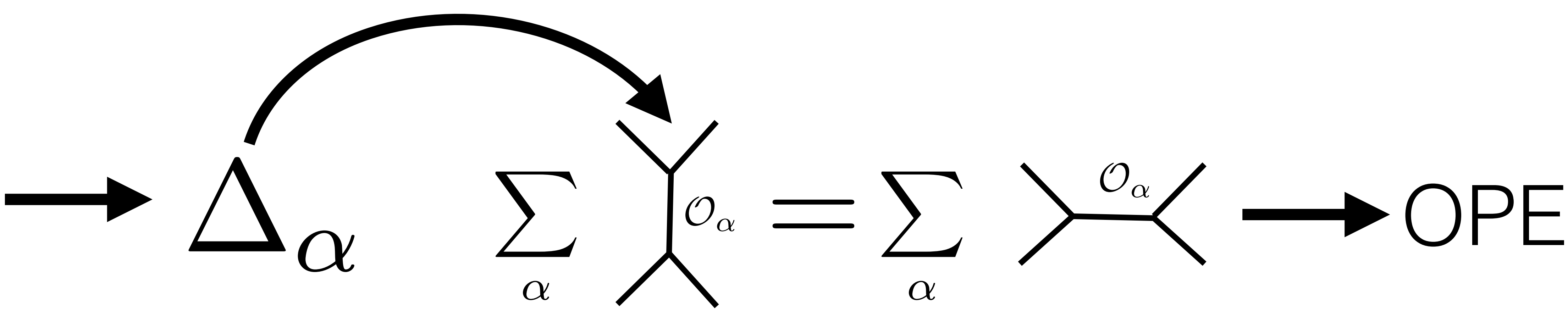
which is the right scaling in the limit of very large AdS black holes in Einstein gravity!

Future Directions

Going beyond the Semi-Classical Limit in 2D

Additional tool: put together with information on scaling dimensions from modular invariance?





Extract non-perturbative information about gravity using the bootstrap?

Conclusions

There is an enormous amount of information contained in the bootstrap equation.

Recent results on conformal blocks give us a sharp tool for extracting information.

The constraints of large C + *"only a few light operators"* seem to be far-reaching, and still have much to teach us.

Old dream: AdS_3 gravity may be simple enough to have tractable solutions - i.e. 2d QCD (`t Hooft model) for gravity

The End