Collider Physics
in The LHC Era And Beyond

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Contents:

**Lecture I:**
Basics of Collider physics
Physics at an $e^+e^-$ Collider

**Lecture II:**
Physics at Hadron Colliders
Perspectives Beyond the LHC
Prelude: LHC Run-II is in mission!

June 3, 2015: Run-II started at $E_{cm} = 6.5 \oplus 6.5 = 13$ TeV. New era in science has begun!

High Energy Physics IS at an extremely interesting time!

The completion of the Standard Model: With the discovery of the Higgs boson, for the first time ever, we have a consistent relativistic quantum-mechanical theory, weakly coupled, unitary, renormalizable, vacuum (quasi?) stable, valid up to an exponentially high scale!

Question: Where IS the next scale? $\mathcal{O}(1 \text{ TeV})$? $M_{\text{GUT}}$? $M_{\text{Planck}}$?
Large spread of masses for elementary particles:

Large hierarchy: Electroweak scale $\Leftrightarrow M_{Planck}$? Conceptual.
Little hierarchy: Electroweak scale $\Leftrightarrow$ Next scale at TeV? Observational.

Consult with the other excellent lectures.
That motivates us to the new energy frontier!

- **LHC** (300 fb$^{-1}$), **HL-LHC** (3 ab$^{-1}$) lead to way: 2015–2030
- **ILC** as a Higgs factory (250 GeV) and beyond: 2020–2030 (250/500/1000 GeV, 250/500/1000 fb$^{-1}$).
- **FCC$_{ee}$** (4×2.5 ab$^{-1}$)/CEPC as a Higgs factory: 2028–2035
- **FCC$_{hh}$/SPPC/VLHC** (100 TeV, 3 ab$^{-1}$) to the energy frontier: 2040–

*Nature News (July, 2014)*
(0). A Historical Count:

Rutherford’s experiments were the first to study matter structure:

\[
\frac{d\sigma}{d\Omega} = \frac{(\alpha Z_1 Z_2)^2}{4E^2 \sin^4 \theta/2}
\]

Gold foil target

discover the point-like nucleus:

SLAC-MIT DIS experiments

\[
\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4 \theta/2} \left( \frac{F_1(x, Q^2)}{m_p} \sin^2 \frac{\theta}{2} + \frac{F_2(x, Q^2)}{E - E'} \cos^2 \frac{\theta}{2} \right)
\]

Proton target

discover the point-like structure of the proton:

QCD parton model \( \Rightarrow 2xF_1(x, Q^2) = F_2(x, Q^2) = \sum_i x f_i(x) e_i^2 \).

Rutherford’s legendary method continues to date!
(A). High-energy Colliders:

To study the deepest layers of matter, we need the probes with highest energies.

Two parameters of importance:

1. The energy:

\[ s \equiv (p_1 + p_2)^2 = \left\{ \begin{align*}
(E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2, \\
(m_1^2 + m_2^2 + 2(E_1 E_2 - \vec{p}_1 \cdot \vec{p}_2)).
\end{align*} \right. \]

\[ E_{cm} \equiv \sqrt{s} \approx \left\{ \begin{align*}
2E_1 \approx 2E_2 & \quad \text{in the c.m. frame } \vec{p}_1 + \vec{p}_2 = 0, \\
\frac{2E_1 m_2}{\sqrt{2E_1 m_2}} & \quad \text{in the fixed target frame } \vec{p}_2 = 0.
\end{align*} \right. \]
2. The luminosity:

Colliding beam

\[ t = \frac{1}{f} \]

\[ \mathcal{L} \propto f n_1 n_2 / a , \]

\((a \text{ some beam transverse profile}) \text{ in units of } \text{#particles/cm}^2/\text{s}\)

\[ 10^{33} \text{ cm}^{-2}\text{s}^{-1} = 1 \text{ nb}^{-1} \text{s}^{-1} \approx 10 \text{ fb}^{-1}/\text{year}. \]

Current and future high-energy colliders:

<table>
<thead>
<tr>
<th>Hadron Colliders</th>
<th>(\sqrt{s}) (TeV)</th>
<th>(\mathcal{L}) (cm(^{-2}\text{s}^{-1}))</th>
<th>(\delta E/E)</th>
<th>(f) (MHz)</th>
<th>#/bunch (10(^{10}))</th>
<th>L (km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>LHC Run (I) II</td>
<td>(7,8) 13</td>
<td>((10^{32}) \times 10^{33}) 7 \times 10^{34}</td>
<td>0.01%</td>
<td>40</td>
<td>10.5</td>
<td>26.66</td>
</tr>
<tr>
<td>HL-LHC</td>
<td>14</td>
<td></td>
<td>0.013%</td>
<td>40</td>
<td>22</td>
<td>26.66</td>
</tr>
<tr>
<td>FCC(_{hh}) (SppC)</td>
<td>100</td>
<td>(1.2 \times 10^{35})</td>
<td>0.01%</td>
<td>40</td>
<td>10</td>
<td>100</td>
</tr>
</tbody>
</table>

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<thead>
<tr>
<th>(e^+e^-) Colliders</th>
<th>(\sqrt{s}) (TeV)</th>
<th>(\mathcal{L}) (cm(^{-2}\text{s}^{-1}))</th>
<th>(\delta E/E)</th>
<th>(f) (MHz)</th>
<th>polar.</th>
<th>L (km)</th>
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</thead>
<tbody>
<tr>
<td>ILC</td>
<td>0.5–1</td>
<td>(2.5 \times 10^{34})</td>
<td>0.1%</td>
<td>3</td>
<td>80, 60%</td>
<td>14 – 33</td>
</tr>
<tr>
<td>FCC(_{ee}/CEPC)</td>
<td>0.25–0.35</td>
<td>(4 \cdot 10^{35}/2 \cdot 10^{34})</td>
<td>0.13%</td>
<td>3</td>
<td>50-100</td>
<td></td>
</tr>
<tr>
<td>CLIC</td>
<td>3–5</td>
<td>(~ 10^{35})</td>
<td>0.35%</td>
<td>1500</td>
<td>80, 60%</td>
<td>33 – 53</td>
</tr>
</tbody>
</table>
(B). $e^+e^-$ Colliders

The collisions between $e^-$ and $e^+$ have major advantages:

- The system of an electron and a positron has zero charge, zero lepton number etc.,
  \( \Rightarrow \) it is suitable to create new particles after $e^+e^-$ annihilation.
- With symmetric beams between the electrons and positrons, the laboratory frame is the same as the c.m. frame,
  \( \Rightarrow \) the total c.m. energy is fully exploited to reach the highest possible physics threshold.
- With well-understood beam properties,
  \( \Rightarrow \) the scattering kinematics is well-constrained.
- Backgrounds low and well-undercontrol:
  For $\sigma \approx 10$ pb $\Rightarrow 0.1$ Hz at $10^{34}$ cm$^{-2}$s$^{-1}$.
- Linear Collider: possible to achieve high degrees of beam polarizations,
  \( \Rightarrow \) chiral couplings and other asymmetries can be effectively explored.
Disadvantages

• Large synchrotron radiation due to acceleration,

\[ \Delta E \sim \frac{1}{R} \left( \frac{E}{m_e} \right)^4. \]

Thus, a multi-hundred GeV $e^+e^-$ collider will have to be made a linear accelerator.

• This becomes a major challenge for achieving a high luminosity when a storage ring is not utilized; beamsstrahlung severe.

**CEPC/FCC$_{ee}$ Higgs Factory**

It has been discussed to build a **circular** $e^+e^-$ collider

50 – 100 km, \( E_{cm} = 245 \text{ GeV} - 350 \text{ GeV} \)

with multiple interaction points for very high luminosities.
LHC: the new high-energy frontier

“Hard” Scattering

Advantages

- Higher c.m. energy, thus higher energy threshold:
  \[ \sqrt{S} = 14 \text{ TeV:} \quad M_{new}^2 \sim s = x_1 x_2 S \quad \Rightarrow \quad M_{new} \sim 0.3 \sqrt{S} \sim 4 \text{ TeV.} \]
• Higher luminosity: \(10^{34}/\text{cm}^2/\text{s} \Rightarrow 100 \text{ fb}^{-1}/\text{yr.}\)
  Annual yield: \(1 \text{B } W^{\pm}; \ 100 \text{M } t\bar{t}; \ 10 \text{M } W^{+}W^{-}; \ 1 \text{M } H^{0}...\)

• Multiple (strong, electroweak) channels:
  \(q\bar{q}', gg, qg, b\bar{b} \rightarrow \) colored; \(Q = 0, \pm 1; \ J = 0, 1, 2 \) states;
  \(WW, WZ, ZZ, \gamma\gamma \rightarrow I_{W} = 0, 1, 2; \ Q = 0, \pm 1, \pm 2; \ J = 0, 1, 2 \) states.

  Disadvantages

• Initial state unknown:
  colliding partons unknown on event-by-event basis;
  parton c.m. energy unknown: \(E_{cm}^{2} \equiv s = x_{1}x_{2}S;\)
  parton c.m. frame unknown.
  \(\Rightarrow \) largely rely on final state reconstruction.

• The large rate turns to a hostile environment:
  \(\Rightarrow \) Severe backgrounds!

  Our primary job!
(D). Particle Detection:

The detector complex: Utilize the strong and electromagnetic interactions between detector materials and produced particles.
What we “see” as particles in the detector:  (a few meters)

For a relativistic particle, the travel distance:

\[ d = (\beta c \tau) \gamma \approx (300 \ \mu m) (\frac{\tau}{10^{-12} \text{s}}) \ \gamma \]

- **stable particles** directly “seen”:
  \[ p, \bar{p}, e^\pm, \gamma \]

- **quasi-stable particles** of a life-time \( \tau \geq 10^{-10} \text{ s} \) also directly “seen”:
  \[ n, \Lambda, K_L^0, ..., \mu^\pm, \pi^\pm, K^\pm... \]

- a life-time \( \tau \sim 10^{-12} \text{ s} \) may display a secondary decay vertex, “vertex-tagged particles”:
  \[ B^0,\pm, D^0,\pm, \tau^\pm... \]

- **short-lived** not “directly seen”, but “reconstructable”:
  \[ \pi^0, \rho^0,\pm... , Z, W^\pm, t, H... \]

- **missing particles** are weakly-interacting and neutral:
  \[ \nu, \tilde{\chi}^0, G_{KK}... \]
† For stable and quasi-stable particles of a life-time \( \tau \geq 10^{-10} - 10^{-12} \text{ s} \), they show up as

Theorists should know:

For charged tracks: \( \Delta p/p \propto p \),

- typical resolution: \( \sim p/(10^4 \text{ GeV}) \).

For calorimetry: \( \Delta E/E \propto \frac{1}{\sqrt{E}} \),

- typical resolution: \( \sim (10\%_{\text{ecal}}, 50\%_{\text{hcal}})/\sqrt{E/\text{GeV}} \).
† For *vertex-tagged particles* $\tau \approx 10^{-12} \text{ s}$, heavy flavor tagging: the secondary vertex:

Typical resolution: $d_0 \sim 30 - 50 \mu\text{m}$ or so

⇒ Better have two (non-collinear) charged tracks for a secondary vertex;

Or use the “impact parameter” w.r.t. the primary vertex.

For theorists: just multiply a “tagging efficiency”:

$\epsilon_b \sim 70\%$; $\epsilon_c \sim 40\%$; $\epsilon_\tau \sim 40\%$. 
† For short-lived particles \((Z, W^\pm, t, H...): \tau < 10^{-12} \text{ s or so,}\)
make use of final state kinematics to reconstruct the resonance.

† For missing particles:
make use of energy-momentum conservation to deduce their existence.

\[
p_1^i + p_2^i = \sum_{f} p_f + p_{\text{miss}}.
\]

But in hadron collisions, the longitudinal momenta unknown,
thus transverse direction only:

\[
0 = \sum_{f} \vec{p}_f \ T + \vec{p}_{\text{miss}} \ T.
\]

often called “missing \(p_T\) “\((\not{p}_T)\)” or (conventionally) “missing \(E_T\)” \((\not{E}_T)\).

Note: “missing \(E_T\)” \((\text{MET})\) is conceptually ill-defined!
It is only sensible for massless particles:

\[
\not{E}_T = \sqrt{\not{p}^{2}_{\text{miss}} \ T + m^2}.
\]
What we “see” for the SM particles
(no universality!)

<table>
<thead>
<tr>
<th>Leptons</th>
<th>Vetexing</th>
<th>Tracking</th>
<th>ECAL</th>
<th>HCAL</th>
<th>Muon Cham.</th>
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<td>$\times$</td>
<td>$\vec{p}$</td>
<td>$E$</td>
<td>$\times$</td>
<td>$\times$</td>
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<tr>
<td>$\mu^\pm$</td>
<td>$\times$</td>
<td>$\vec{p}$</td>
<td>$\sqrt{\vec{p}}$</td>
<td>$\sqrt{\vec{p}}$</td>
<td>$\mu^\pm$</td>
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<tr>
<td>$\tau^\pm$</td>
<td>$\sqrt{\times}$</td>
<td>$\sqrt{\vec{p}}$</td>
<td>$e^\pm$</td>
<td>$h^\pm$; $3h^\pm$</td>
<td>$\mu^\pm$</td>
</tr>
<tr>
<td>$\nu_e, \nu_\mu, \nu_\tau$</td>
<td>$\times$</td>
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<tr>
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<td>$\times$</td>
<td>$\sqrt{\vec{p}}$</td>
<td>$\sqrt{\vec{p}}$</td>
<td>$\sqrt{\vec{p}}$</td>
<td>$\mu^\pm$</td>
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<tr>
<td>$c \rightarrow D$</td>
<td>$\sqrt{\vec{p}}$</td>
<td>$\sqrt{\vec{p}}$</td>
<td>$e^\pm$</td>
<td>$h's$</td>
<td>$\mu^\pm$</td>
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<tr>
<td>$b \rightarrow B$</td>
<td>$\sqrt{\vec{p}}$</td>
<td>$\sqrt{\vec{p}}$</td>
<td>$e^\pm$</td>
<td>$h's$</td>
<td>$\mu^\pm$</td>
</tr>
<tr>
<td>$t \rightarrow bW^\pm$</td>
<td>$b$</td>
<td>$\sqrt{\vec{p}}$</td>
<td>$e^\pm$</td>
<td>$b + 2$ jets</td>
<td>$\mu^\pm$</td>
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<th>Gauge bosons</th>
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<tr>
<td>$\gamma$</td>
<td>$\times$</td>
<td>$\times$</td>
<td>$E$</td>
<td>$\times$</td>
<td>$\times$</td>
</tr>
<tr>
<td>$g$</td>
<td>$\times$</td>
<td>$\sqrt{\vec{p}}$</td>
<td>$\sqrt{\vec{p}}$</td>
<td>$\sqrt{\vec{p}}$</td>
<td>$\mu^\pm$</td>
</tr>
<tr>
<td>$W^\pm \rightarrow \ell^\pm \nu$</td>
<td>$\times$</td>
<td>$\vec{p}$</td>
<td>$e^\pm$</td>
<td>$x$</td>
<td>$\mu^\pm$</td>
</tr>
<tr>
<td>$\rightarrow q\bar{q}'$</td>
<td>$\times$</td>
<td>$\sqrt{\vec{p}}$</td>
<td>$\sqrt{\vec{p}}$</td>
<td>$2$ jets</td>
<td>$\times$</td>
</tr>
<tr>
<td>$Z^0 \rightarrow \ell^+\ell^-$</td>
<td>$\times$</td>
<td>$\vec{p}$</td>
<td>$e^\pm$</td>
<td>$x$</td>
<td>$\mu^\pm$</td>
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<tr>
<td>$\rightarrow q\bar{q}$</td>
<td>$(\bar{b}b)$</td>
<td>$\sqrt{\vec{p}}$</td>
<td>$\sqrt{\vec{p}}$</td>
<td>$2$ jets</td>
<td>$\times$</td>
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<th>the Higgs boson</th>
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<tr>
<td>$h^0 \rightarrow bb$</td>
<td>$\sqrt{\vec{p}}$</td>
<td>$\sqrt{\vec{p}}$</td>
<td>$e^\pm$</td>
<td>$h's$</td>
<td>$\mu^\pm$</td>
</tr>
<tr>
<td>$\rightarrow ZZ^*$</td>
<td>$\times$</td>
<td>$\vec{p}$</td>
<td>$e^\pm$</td>
<td>$\sqrt{\vec{p}}$</td>
<td>$\mu^\pm$</td>
</tr>
<tr>
<td>$\rightarrow WW^*$</td>
<td>$\times$</td>
<td>$\vec{p}$</td>
<td>$e^\pm$</td>
<td>$\sqrt{\vec{p}}$</td>
<td>$\mu^\pm$</td>
</tr>
</tbody>
</table>
Homework:

Exercise 1.1: For a $\pi^0$, $\mu^-$, or a $\tau^-$ respectively, calculate its decay length for $E = 10$ GeV.

Exercise 1.2: An event was identified to have a $\mu^+\mu^-$ pair, along with some missing energy. What can you say about the kinematics of the system of the missing particles? Consider both an $e^+e^-$ and a hadron collider.

Exercise 1.3: Electron and muon measurements: Estimate the relative errors of energy-momentum measurements for an electron by an electromagnetic calorimetry ($\Delta E/E$) and for a muon by tracking ($\Delta p/p$) at energies of $E = 50$ GeV and 500 GeV, respectively.

Exercise 1.4: A 125 GeV Higgs boson will have a production cross section of 20 pb at the 14 TeV LHC. How many events per year do you expect to produce for the Higgs boson with an instantaneous luminosity $10^{33}$/cm$^2$/s? Do you expect it to be easy to observe and why?
I-B. Basic Techniques
and Tools for Collider Physics

(A). Scattering cross section

For a $2 \rightarrow n$ scattering process:

$$\sigma(ab \rightarrow 1 + 2 + \ldots n) = \frac{1}{2s} \sum |M|^2 \, dPS_n,$$

$$dPS_n \equiv (2\pi)^4 \, \delta^4 \left( P - \sum_{i=1}^{n} p_i \right) \prod_{i=1}^{n} \frac{1}{(2\pi)^3} \frac{d^3 p_i}{2E_i},$$

$$s = (p_a + p_b)^2 \equiv P^2 = \left( \sum_{i=1}^{n} p_i \right)^2,$$

where $\sum |M|^2$: dynamics (dimension $4 - 2n$);

$dPS_n$: kinematics (Lorentz invariant, dimension $2n - 4$.)

For a $1 \rightarrow n$ decay process, the partial width in the rest frame:

$$\Gamma(a \rightarrow 1 + 2 + \ldots n) = \frac{1}{2M_a} \sum |M|^2 \, dPS_n.$$

$$\tau = \Gamma_{tot}^{-1} = (\sum_f \Gamma_f)^{-1}.$$
(B). Phase space and kinematics

One-particle Final State $a + b \rightarrow 1$:

$$dPS_1 \equiv (2\pi) \frac{d^3 \vec{p}_1}{2E_1} \delta^4(P - p_1)$$

$$= \pi |\vec{p}_1| d\Omega_1 \delta^3(\vec{P} - \vec{p}_1)$$

$$= 2\pi \delta(s - m_1^2).$$

where the first and second equal signs made use of the identities:

$$|\vec{p}|d|\vec{p}| = EdE, \quad \frac{d^3 \vec{p}}{2E} = \int d^4 p \, \delta(p^2 - m^2).$$

Kinematical relations:

$$\vec{P} \equiv \vec{p}_a + \vec{p}_b = \vec{p}_1, \quad E_1^{cm} = \sqrt{s} \text{ in the c.m. frame},$$

$$s = (p_a + p_b)^2 = m_1^2.$$

The “dimensionless phase-space volume” is $s(dPS_1) = 2\pi$.

Two-particle Final State \( a + b \rightarrow 1 + 2 \):

\[
dPS_2 \equiv \frac{1}{(2\pi)^2} \delta^4(P - p_1 - p_2) \frac{d^3\vec{p}_1}{2E_1} \frac{d^3\vec{p}_2}{2E_2}
\]

\[
= \frac{1}{(4\pi)^2} \frac{|\vec{p}_1^{\text{cm}}|}{\sqrt{s}} \ d\Omega_1 = \frac{1}{(4\pi)^2} \frac{|\vec{p}_1^{\text{cm}}|}{\sqrt{s}} \ d\cos \theta_1 d\phi_1
\]

\[
= \frac{1}{4\pi^2} \lambda^{1/2} \left( 1, \frac{m_1^2}{s}, \frac{m_2^2}{s} \right) dx_1 dx_2,
\]

\[
d\cos \theta_1 = 2dx_1, \ d\phi_1 = 2\pi dx_2, \ 0 \leq x_1, x_2 \leq 1,
\]

The magnitudes of the energy-momentum of the two particles are fully determined by the four-momentum conservation:

\[
|\vec{p}_1^{\text{cm}}| = |\vec{p}_2^{\text{cm}}| = \frac{\lambda^{1/2}(s, m_1^2, m_2^2)}{2\sqrt{s}}, \ E_1^{\text{cm}} = s + m_1^2 - m_2^2 \frac{2\sqrt{s}}{2\sqrt{s}}, \ E_2^{\text{cm}} = s + m_2^2 - m_1^2 \frac{2\sqrt{s}}{2\sqrt{s}}
\]

\[
\lambda(x, y, z) = (x - y - z)^2 - 4yz = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz.
\]

The phase-space volume of the two-body is scaled down with respect to that of the one-particle by a factor

\[
\frac{dPS_2}{s \ dPS_1} \approx \frac{1}{(4\pi)^2}.
\]

just like a “loop factor”.
Exercise 2.1: Assume that \( m_a = m_1 \) and \( m_b = m_2 \). Show that

\[
\begin{align*}
t &= -2p_{cm}^2(1 - \cos \theta_{a1}^*), \\
u &= -2p_{cm}^2(1 + \cos \theta_{a1}^*) + \frac{(m_1^2 - m_2^2)^2}{s},
\end{align*}
\]

\( p_{cm} = \lambda^{1/2}(s, m_1^2, m_2^2)/2\sqrt{s} \) is the momentum magnitude in the c.m. frame. Note: \( t \) is negative-definite; \( t \rightarrow 0 \) in the collinear limit.

Exercise 2.2: A particle of mass \( M \) decays to two particles isotropically in its rest frame. What does the momentum distribution look like in a frame in which the particle is moving with a speed \( \beta_z \)? Compare the result with your expectation for the shape change for a basket ball.
Three-particle Final State $a + b \rightarrow 1 + 2 + 3$:

$$dPS_3 \equiv \frac{1}{(2\pi)^5} \delta^4 (P - p_1 - p_2 - p_3) \frac{d^3\vec{p}_1 d^3\vec{p}_2 d^3\vec{p}_3}{2E_1 2E_2 2E_3}$$

$$= \frac{|\vec{p}_1|^2}{(2\pi)^3} \frac{d|\vec{p}_1| d\Omega_1}{2E_1} \frac{1}{(4\pi)^2} \frac{|\vec{p}^{(23)}_2|}{m_{23}} d\Omega_2$$

$$= \frac{1}{(4\pi)^3} \lambda^{1/2} \left(1, \frac{m_2^2}{m_{23}}, \frac{m_3^2}{m_{23}}\right) 2|\vec{p}_1| dE_1 \ dx_2 dx_3 dx_4 dx_5.$$ 

$$d \cos \theta_{1,2} = 2dx_{2,4}, \ d\phi_{1,2} = 2\pi dx_{3,5}, \ 0 \leq x_{2,3,4,5} \leq 1,$$

$$|\vec{p}^{cm}_1|^2 = |\vec{p}^{cm}_2 + \vec{p}^{cm}_3|^2 = (E^{cm}_1)^2 - m_1^2,$$

$$m_{23}^2 = s - 2\sqrt{s}E^{cm}_1 + m_1^2, \ |\vec{p}^{23}_2| = |\vec{p}^{23}_3| = \frac{\lambda^{1/2}(m_{23}^2, m_2^2, m_3^2)}{2m_{23}},$$

The particle energy spectrum is not monochromatic.

The maximum value (the end-point) for particle 1 in c.m. frame is

$$E_{1}^{max} = \frac{s + m_1^2 - (m_2 + m_3)^2}{2\sqrt{s}}, \ m_1 \leq E_1 \leq E_{1}^{max},$$

$$|\vec{p}_{1}^{max}| = \frac{\lambda^{1/2}(s, m_1^2, (m_2 + m_3)^2)}{2\sqrt{s}}, \ 0 \leq p_1 \leq p_{1}^{max}.$$
With $m_i = 10, 20, 30, \sqrt{s} = 100$ GeV.

More intuitive to work out the end-point for the kinetic energy, – recall the direct neutrino mass bound in $\beta$-decay:

$$K_{1}^{\text{max}} = E_{1}^{\text{max}} - m_1 = \frac{(\sqrt{s} - m_1 - m_2 - m_3)(\sqrt{s} - m_1 + m_2 + m_3)}{2\sqrt{s}}.$$
Recursion relation \( P \rightarrow 1 + 2 + 3 \ldots + n: \)

\[
dPS_n(P; p_1, \ldots, p_n) = dPS_{n-1}(P; p_1, \ldots, p_{n-1}, p_n) \frac{dm_{n-1,n}^2}{2\pi}.
\]

For instance,

\[
dPS_3 = dPS_2(i) \frac{dm_{prop}^2}{2\pi} dPS_2(f).
\]

This is generically true, but particularly useful when the diagram has an \( s \)-channel particle propagation.

Breit-Wigner Resonance, the Narrow Width Approximation

An unstable particle of mass $M$ and total width $\Gamma_V$, the propagator is

$$R(s) = \frac{1}{(s - M_V^2)^2 + \Gamma_V^2 M_V^2}.$$ 

Consider an intermediate state $V^*$

$$a \rightarrow bV^* \rightarrow b \ p_1p_2.$$ 

By the reduction formula, the resonant integral reads

$$\int (m^2_{\text{max}})^2 = (m_a - m_b)^2 \ dm^2_*.$$ 

$$\int (m^2_{\text{min}})^2 = (m_1 + m_2)^2 \ dm^2_*.$$ 

Variable change

$$\tan \theta = \frac{m^2_* - M_V^2}{\Gamma_V M_V},$$

resulting in a flat integrand over $\theta$

$$\int (m^2_{\text{max}})^2 \ dm^2_* = \int_{\theta_{\text{max}}}^{\theta_{\text{min}}} \frac{d\theta}{\Gamma_V M_V}.$$
In the limit
\[
(m_1 + m_2) + \Gamma_V \ll M_V \ll m_a - m_b - \Gamma_V,
\]

\[
\theta_{\min} = \tan^{-1} \frac{(m_1 + m_2)^2 - M_V^2}{\Gamma_V M_V} \to -\pi,
\]

\[
\theta_{\max} = \tan^{-1} \frac{(m_a - m_b)^2 - M_V^2}{\Gamma_V M_V} \to 0,
\]

then the Narrow Width Approximation
\[
\frac{1}{(m_2^* - M_V^2)^2 + \Gamma_V^2 M_V^2} \approx \frac{\pi}{\Gamma_V M_V} \delta(m_2^* - M_V^2).
\]

Exercise 2.4: Consider a three-body decay of a top quark, 
\(t \to bW^* \to b \, e\nu\). Making use of the phase space recursion relation and the narrow width approximation for the intermediate \(W\) boson, show that the partial decay width of the top quark can be expressed as
\[
\Gamma(t \to bW^* \to b \, e\nu) \approx \Gamma(t \to bW) \cdot BR(W \to e\nu).
\]
Properties of scattering amplitudes $T(s, t, u)$

- **Analyticity:** A scattering amplitude is analytical except:
  simple poles (corresponding to single particle states, bound states etc.);
  branch cuts (corresponding to thresholds).

- **Crossing symmetry:** A scattering amplitude for a $2 \rightarrow 2$ process is symmetric among the $s$-, $t$-, $u$-channels.

- **Unitarity:**
  S-matrix unitarity leads to:

  $$-i(T - T^\dagger) = TT^\dagger$$
Partial wave expansion for $a + b \rightarrow 1 + 2$:

$$M(s, t) = 16\pi \sum_{J=M}^{\infty} (2J + 1) a_J(s) d^J_{\mu\mu'}(\cos \theta)$$

$$a_J(s) = \frac{1}{32\pi} \int_{-1}^{1} M(s, t) d^J_{\mu\mu'}(\cos \theta) d \cos \theta.$$ 

where $\mu = s_a - s_b$, $\mu' = s_1 - s_2$, $M = \max(|\mu|, |\mu'|)$.

By Optical Theorem: $\sigma = \frac{1}{s} \text{Im} M(\theta = 0) = \frac{16\pi}{s} \sum_{J=M}^{\infty} (2J + 1) |a_J(s)|^2$.

The partial wave amplitude have the properties:

(a). partial wave unitarity: $\text{Im}(a_J) \geq |a_J|^2$, or $|\text{Re}(a_J)| \leq 1/2$,

(b). kinematical thresholds: $a_J(s) \propto \beta_{l_i}^{l_i} \beta_{l_f}^{l_f}$ ($J = L + S$).

$\Rightarrow$ well-known behavior: $\sigma \propto \beta_{l_f}^{2l_f+1}$.

Exercise 2.5: Appreciate the properties (a) and (b) by explicitly calculating the helicity amplitudes for

$$e_{L}^{-} e_{R}^{+} \rightarrow \gamma^{*} \rightarrow H^{-} H^{+}, \quad e_{L}^{-} e_{L,R}^{+} \rightarrow \gamma^{*} \rightarrow \mu_{L}^{-} \mu_{R}^{+}, \quad H^{-} H^{+} \rightarrow G^{*} \rightarrow H^{-} H^{+}.$$
(D). Calculational Tools

Traditional “Trace” Techniques in QFT:
Good for simple processes

Helicity Techniques:
technical simplification, necessary for multiple particles;
conceptual advancements.
(Henriette’s lectures)

Exercise 2.6: Calculate the squared matrix element for \( \sum |M(f\bar{f} \to ZZ)|^2 \), in terms of \( s, t, u \), in whatever technique you like.
Calculational packages:

- **Monte Carlo packages for phase space integration:**
  VEGAS by P. LePage: adaptive important-sampling MC
  http://en.wikipedia.org/wiki/Monte-Carlo_integration

- **Automated evaluation of cross sections:**

  (1) MadGraph/MadEvent and MadSUSY:
  Generate Fortran codes on-line! http://madgraph.hep.uiuc.edu
  (Now allows you to input new models.)

  (2) CompHEP/CalHEP: computer program for calculation of elementary particle processes in Standard Model and beyond. CompHEP has a built-in numeric interpreter. So this version permits to make numeric calculation without additional Fortran/C compiler. It is convenient for more or less simple calculations.
    — It allows your own construction of a Lagrangian model!
  http://theory.npi.msu.su/~kryukov
  (Now allows you to input new models.)
(3) SHERPA (F. Krauss et al.): (Gaining popularity) Generate Fortran codes on-line! Merging with MC generators (see next).
http://www.sherpa-mc.de/

• Cross sections at NLO packages: (Gaining popularity)
(1) MC(at)NLO (B. Webber et al.):
http://www.hep.phy.cam.ac.uk/theory/webber/MCatNLO/
Combining a MC event generator with NLO calculations for QCD processes.

(2) MCFM (K. Ellis et al.):
http://mcfm.fnal.gov/
Parton-level, NLO processes for hadronic collisions.

(3) BlackHat (Z.Bern, L.Dixon, D.Kosover et al.):
http://blackhat.hepforge.org/
Parton-level, NLO processes to combine with Sherpa
• **Numerical simulation packages:** Monte Carlo Event Generators
  
  Reading: [http://www.sherpa-mc.de/](http://www.sherpa-mc.de/)

  (1) **PYTHIA:**
  PYTHIA is a Monte Carlo program for the generation of high-energy physics events, i.e. for the description of collisions at high energies between $e^+, e^-, p$ and $\bar{p}$ in various combinations. They contain theory and models for a number of physics aspects, including hard and soft interactions, parton distributions, initial and final state parton showers, multiple interactions, fragmentation and decay.
  
  — It can be combined with MadGraph and detector simulations.
  
  [http://www.thep.lu.se/torbjorn/Pythia.html](http://www.thep.lu.se/torbjorn/Pythia.html)

  Already made crucial contributions to Tevatron/LHC.

  (2) **HERWIG**
  HERWIG is a Monte Carlo program which simulates $pp, p\bar{p}$ interactions at high energies. It has the most sophisticated perturbative treatments, and possible NLO QCD matrix elements in parton showing.
  
  [http://hepwww.rl.ac.uk/theory/seymour/herwig/](http://hepwww.rl.ac.uk/theory/seymour/herwig/)

• **Detector Simulations** “Pretty Good Simulation” (PGS):
  By John Conway: A simplified detector simulation,
mainly for theorists to estimate the detector effects.

PGS has been adopted for running with PYTHIA and MadGraph. (but just a "toy".)

**DELPHES:** A modular framework for fast simulation of a generic collider experiment.
http://arxiv.org/abs/1307.6346
Over all:

THEORY <-> EXPERIMENT Connection

Theory -> Feynman Rules -> Matrix Element -> Parton level event generator -> MC event generator -> Observables

LanHEP, FeynRules, SARAH, ...

CalcHEP/CompHEP/MicroMEGAs, FeynArts/FormCalc, MadGraph/MadEvent, Sherpa, Whizard/Omega, Golem, Herwig++, MCFM, MC@NLO, Alpgen, ...

PDF, mass spectra calculators, ...

Jet matching

PYTHIA, HERWIG, ISAJET, Sherpa, ...

FAST/FULL Detector Simulation

PGS, Delphes / CMSSW, ATHENA, ...

Analysis to find/exclude/modify theory:
Plots, PAW/Root, Fortran/C++ codes, Private codes, MasterCode, HEPMDB, ...

Alexander Belyaev
“Practical introduction into selected TOOLS for High Energy Physics”
I-C. Physics at an $e^+e^-$ Collider

(A.) Simple Formalism

Event rate of a reaction:

$$R(s) = \sigma(s)\mathcal{L}, \quad \text{for constant } \mathcal{L}$$

$$= \mathcal{L} \int d\tau \frac{dL(s,\tau)}{d\tau} \sigma(\hat{s}), \quad \tau = \frac{\hat{s}}{s}. $$

As for the differential production cross section of two-particle $a, b$,

$$\frac{d\sigma(e^+e^- \to ab)}{d\cos \theta} = \frac{\beta}{32\pi s} \sum |M|^2$$

where

- $\beta = \lambda^{1/2}(1, m_a^2/s, m_b^2/s)$, is the speed factor for the out-going particles in the c.m. frame, and $p_{cm} = \beta \sqrt{s}/2$,

- $\sum |M|^2$ the squared matrix element, summed and averaged over quantum numbers (like color and spins etc.)

- unpolarized beams so that the azimuthal angle trivially integrated out,
Total cross sections and event rates for SM processes:
(B). Resonant production: Breit-Wigner formula

\[
\frac{1}{(s - M_V^2)^2 + \Gamma_V^2 M_V^2}
\]

If the energy spread \(\delta \sqrt{s} \ll \Gamma_V\), the line-shape mapped out:

\[
\sigma(e^+e^- \rightarrow V^* \rightarrow X) = \frac{4\pi(2j + 1)\Gamma(V \rightarrow e^+e^-)\Gamma(V \rightarrow X)}{(s - M_V^2)^2 + \Gamma_V^2 M_V^2} \frac{s}{M_V^2},
\]

If \(\delta \sqrt{s} \gg \Gamma_V\), the narrow-width approximation:

\[
\frac{1}{(s - M_V^2)^2 + \Gamma_V^2 M_V^2} \rightarrow \frac{\pi}{M_V \Gamma_V} \delta(s - M_V^2),
\]

\[
\sigma(e^+e^- \rightarrow V^* \rightarrow X) = \frac{2\pi^2(2j + 1)\Gamma(V \rightarrow e^+e^-)BF(V \rightarrow X)}{M_V^2} dL(\hat{s} = M_V^2) \frac{d\sqrt{\hat{s}}}{d\sqrt{s}}
\]

Exercise 3.1: sketch the derivation of these two formulas, assuming a Gaussian distribution for

\[
\frac{dL}{d\sqrt{s}} = \frac{1}{\sqrt{2\pi} \Delta} \exp\left[\frac{-(\sqrt{\hat{s}} - \sqrt{s})^2}{2\Delta^2}\right].
\]
Note: Away from resonance

For an s-channel or a finite-angle scattering:

\[ \sigma \sim \frac{1}{s}. \]

For forward (co-linear) scattering:

\[ \sigma \sim \frac{1}{M_V^2} \ln^2 \frac{s}{M_V^2}. \]

• The simplest reaction

\[ \sigma(e^+e^- \rightarrow \gamma^* \rightarrow \mu^+\mu^-) \equiv \sigma_{pt} = \frac{4\pi\alpha^2}{3s}. \]

In fact, \( \sigma_{pt} \approx 100 \text{ fb}/(\sqrt{s}/\text{TeV})^2 \) has become standard units to measure the size of cross sections.
(C). Gauge boson radiation:

A qualitatively different process is initiated from gauge boson radiation, typically off fermions:

The simplest case is the photon radiation off an electron, like:

\[ e^+e^- \rightarrow e^+, \quad \gamma^*e^- \rightarrow e^+e^- \]

The dominant features are due to the result of a \( t \)-channel singularity, induced by the collinear photon splitting:

\[ \sigma(e^-a \rightarrow e^-X) \approx \int dx \ P_{\gamma/e}(x) \ \sigma(\gamma a \rightarrow X). \]

The so called the effective photon approximation.
For an electron of energy $E$, the probability of finding a collinear photon of energy $xE$ is given by

$$P_{\gamma/e}(x) = \frac{\alpha \, \ln \left( \frac{E^2}{m_e^2} \right)}{2\pi x^2} \frac{1 + (1 - x)^2}{x},$$

known as the Weizsäcker-Williams spectrum.

Exercise 3.3: Try to derive this splitting function.

We see that:

- $m_e$ enters the log to regularize the collinear singularity;
- $1/x$ leads to the infrared behavior of the photon;
- This picture of the photon probability distribution is also valid for other photon spectrum:

Based on the back-scattering laser technique, it has been proposed to produce much harder photon spectrum, to construct a “photon collider”...
(massive) Gauge boson radiation:

A similar picture may be envisioned for the electroweak massive gauge bosons, \( V = W^\pm, Z \).

Consider a fermion \( f \) of energy \( E \), the probability of finding a (nearly) collinear gauge boson \( V \) of energy \( xE \) and transverse momentum \( p_T \) (with respect to \( \vec{p}_f \)) is approximated by

\[
P_{V/f}^T(x, p_T^2) = \frac{g_V^2 + g_A^2}{8\pi^2} \frac{1 + (1 - x)^2}{x} \frac{p_T^2}{(p_T^2 + (1 - x)M_V^2)^2},
\]

\[
P_{V/f}^L(x, p_T^2) = \frac{g_V^2 + g_A^2}{4\pi^2} \frac{1 - x}{x} \frac{(1 - x)M_V^2}{(p_T^2 + (1 - x)M_V^2)^2}.
\]

Although the collinear scattering would not be a good approximation until reaching very high energies \( \sqrt{s} \gg M_V \), it is instructive to consider the qualitative features.
(D). Recoil mass technique:

One of the most important techniques, that distinguishes an $e^+e^-$ collisions from hadronic collisions.
Consider a process:

$$e^+ + e^- \rightarrow V + X,$$

where $V$: a (bunch of) visible particle(s); $X$: unspecified.

Then:

$$p_{e^+} + p_{e^-} = p_V + p_X, \quad (p_{e^+} + p_{e^-} - p_V)^2 = p_X^2,$$

$$M_X^2 = (p_{e^+} + p_{e^-} - p_V)^2 = s + M_V^2 - 2\sqrt{s}E_V.$$

One thus obtain the “model-independent” inclusive measurements

a. mass of $X$ by the recoil mass peak

b. coupling of $X$ by simple event-count at the peak
At peak cross section \( \approx 200 \text{ fb} \) with \( 5 \text{ ab}^{-1} \) \( \Rightarrow 1 \text{M } h^0! \)

The key point for a Higgs factory:

*Model-independent measurements on the ZZh coupling in a clean experimental environment.*
Consider: $e^+ + e^- \rightarrow f\bar{f} + h$.

$M_h^2 = (p_{e^+} + p_{e^-} - p_f - p_{\bar{f}})^2 = s + M_V^2 - 2\sqrt{s}E_{f\bar{f}}$.

Kinematical selection of "inclusive" signal events!

Marching to higher energies: 500 GeV – 1 TeV: