Lecture 1:

Over the past ~ 12 years, there has been remarkable progress in understanding the structure of scattering amplitudes in gauge theory and gravity. The progress is both on the calculational "practical front" (better, more efficient ways to calculate) and on the mathematical front (elucidating the mathematical structure of amplitudes).

Examples of new developments include:
- Spinfoam calculus formalism
- On-shell recursion relations
- Generalized unitarity (trees → loops)
- Geometrization of amplitudes (polytope picture, Grassmannians, amplituhedron)
- "Gravity = (gauge theory)"

and much more.

In these lectures, I will take a practical approach to introduce you to formalism that you may find useful to know in your work on QFT, pheno, SUSY.
This also serves as the background needed for delving into the more advanced subjects (which I may or may not have time to tell you more about).

Since this is, after all, the pre-SUSY summer school, I will take some SUSY relevant amplitudes as the primary examples of these lectures. Thus, today we will consider quarks to squarks

\[ q + q^* \rightarrow \tilde{q} + \tilde{q}^* \]

spin-0 w/ mass m.

and later we will compute gluons to squarks

\[ g + g \rightarrow \tilde{g} + \tilde{g}^* \]

We'll develop the tools needed as we go.

So let's start w/ quarks → squarks. We'll assume a high-energy limit so that

* partons are considered fundamental particles & we can use perturbative (super)QCD
* quarks can be approximated as massless, \( m_1 \ll E \)
* However, we'll keep the squark masses \( m \neq 0 \)
I will be introducing the necessary formalism to you in terms of Feynman diagrams. Then tomorrow we'll learn about recursion relations.

For \( q + \bar{q} \rightarrow q + \bar{q} \) we have

We will (for simplicity) set the Yukawa diagram to zero.

We will consider all particles outgoing (crossing can then take you back in, out)

so momenta:

\[ p_1^μ + p_2^μ + p_3^μ + p_4^μ = 0. \]

**Feynman Rules (outgoing)**

- external scalar: \( 1 \)

- external fermion \( \bar{U}^+(p) \)

- external antifermion \( V^+(p) \)

- internal gluon (Feynman gauge)

\[
\delta^{ab} \frac{\mu ν}{p^2} \]

**Vertices:**

\[
\langle q \bar{q} \rangle^a \rightarrow \frac{i g Y^μ}{\sqrt{2}} T^a_{ij} \]

\[
|Dq|^2 \rightarrow \frac{i g}{\sqrt{2}} (p_4 - p_3)^μ T^b_{kj} \]

\[
\text{Tr} (T^a T^b) = \delta^{ab} \]
\[ A_\mu(q_1^\mu q_2^\nu) = \bar{u}_h (\gamma^\mu \gamma^\nu) u_h T_{ij} \left( \frac{\delta^{ab}}{(p_1 + p_2)^2} - \frac{i g}{\sqrt{2}} (p_4 - p_3)^k T_{kl} \right) \]

Now all that is completely standard textbook QFT.

The normal procedure is now to compute \( \Sigma |A_\mu|^2 \) by working out some typically lengthy and boring helicity-canceling \( \gamma \)-matrix traces (recall \( \Sigma \bar{u}_h u_h \to \Sigma \)) to get the cross-section \( \frac{d\sigma}{d\Omega} \propto \Sigma |A_\mu|^2 \).

Here is where we deviate from "standard" practice.

The fermion wave functions are solutions to the Dirac eq. For the massless case

\[ \not{\! p} V_{\pm}(p) = 0 \quad \text{&} \quad \bar{u}_{\pm}(p) \not{\! p} = 0 \]

\( \not{\! p} \) helicity labels the 2 indep solutions

\[ V_+(p) = \begin{pmatrix} 1p^\mu a \\ 0 \end{pmatrix} \quad V_-(p) = \begin{pmatrix} 0 \\ 1p^\mu a \end{pmatrix} \]

\[ \bar{u}_-(p) = \begin{pmatrix} 0, \langle p \ell a \rangle \end{pmatrix} \quad \bar{u}_+(p) = \langle \ell p | 1^a, 0 \rangle \]

where \( a, \bar{a} = 1, 2 \).
These kets & bras are commuting 2-component spinors... they are the basis of the Spinor Helicity formalism.

Now let's look at their properties.

Define \( P_{a b} = p_\mu \sigma^\mu_{a b} \). Then, since \( p_\mu = (0, \sigma^\mu) \)
we have \( (P_{a b}) p^g = 0 \), \( [p_1^a, P_{a b}] = 0 \) etc.,
by the Dirac eq.

Also, \( |p> a = \epsilon^{a b} \cdot k |p_b> \) etc.

Can form Lorentz inv. (spinor) products:

\[
\langle p q \rangle = \langle p | l_a | q \rangle^a = - \langle q | p \rangle \\
[p q] = [p | l_a | q]_a = - [q | p] \\
\]

And we can form objects like (w/ a bit of abuse of notation)

\[
\langle p | \gamma^\mu | q \rangle = \langle p | l_a | (\sigma^\mu)^{a b} | q \rangle_b \\
\]

But note

\[
\langle p | \gamma^\mu | q \rangle = 0 = [p | \gamma^\mu | q] \\
\bar{u} \cdot \gamma^\mu u = (0, \langle p |) (0, \sigma^\mu) (0, \bar{u}) = (0, \langle p |) (0, \bar{u}) = 0 \\
\]

* Often will write \( \langle 12 \rangle = \langle p | p \rangle \) etc.
Now recall the spinor completeness relation:

\[ \sum_{s=\mp} u_s \bar{u}_s = -\rho \quad (m=0) \]

"Massless crossing": \( u_\pm = v_\mp \) etc.

This becomes

\[ |p\rangle [p| + |p| \langle p| = -\rho \]

\[ l|p\rangle_a [p|_a = -\rho a \quad etc. \]

So \( p^\mu \leftrightarrow (p_a \bar{c}, p^b \bar{\alpha}) \leftrightarrow l|p\rangle [p| & \langle p| \]

iff \( p^2 = 0 \)

Actually, \( p_{ab} = -|p\rangle [p| \) says that the 2x2 matrix has rank 2 (written in terms of 2 2-component "vectors"

True b/c if \( p^2 = -m^2 \) \( \Rightarrow \) \( \det \rho a = m^2 \).

Well, now we are in business for studying our quarks-squarks amplitude.

Immediately we see that the amplitude vanishes if the quarks have the SAME helicity

\[ A_{\mu} (q^1 q^2 \bar{q}^3 \bar{q}^4) = 0 \quad b/c \quad \langle 118^8 12 \rangle = 0 \]

Consider now opposite helicity.
\[ A_U \left( q_1^+ q_2^+ q_3^- q_4^+ \right) = - \frac{g^2}{\varepsilon} \left\langle 11 \left| y^1 \right| 2 \right\rangle \frac{1}{(p_1 + p_2)^2} \left( p_4 - p_3 \right) \mu \quad \begin{array}{c} T_9^\alpha \ T_9^\beta \end{array} \]

\[ (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 = \left\langle 12 \right\rangle \left[ 12 \right] \]

why? \[ \left\langle 12 \right\rangle \left[ 12 \right] = - \text{Tr} \left[ \left( 1 \right) \left( \left[ 1 \right] \left[ 2 \right] \left[ 2 \right] \right) \right] = - p_1 \mu p_2 \nu \left( \text{Tr} \left( \sigma_{\mu \nu} \right) \right) \]

\[ = 2 p_1 \cdot p_2 \]

\[ \left\langle 11 y^1 \left| 12 \right\rangle = \left\langle 11 \right\rangle \left( \phi_3 - \phi_3 \right) \right\rangle \left[ 12 \right] \]

\[ = \left\langle 11 \right\rangle \left( - \phi_1 - \phi_2 - 2 \phi_3 \right) \right\rangle \left[ 12 \right] \]

\[ = - 2 \left\langle 11 \right\rangle \left\langle \phi_3 \right\rangle \left[ 12 \right] \]

so

\[ A_U \left( - \right) = g^2 \frac{\left\langle 11 \right\rangle \left\langle 3 \right\rangle \left[ 12 \right]}{\left\langle 12 \right\rangle \left[ 12 \right]} T_9^\alpha T_9^\beta \]

nice & compact.

Now, for a moment assume that the squarks are massless too: \[ m_q = 0 \Rightarrow \rho^2 = 0 \Rightarrow \rho_3 = -13 \left[ 3 \right] \text{ etr} . \]

Then \[ \left\langle 11 \right\rangle \left[ 3 \right] \left\rangle - \left\langle 13 \right\rangle \left[ 3 \right] \left\rangle \left[ 2 \right] \right\rangle \]

\[ A_U = + g^2 \frac{\left\langle 13 \right\rangle \left[ 23 \right]}{\left\langle 12 \right\rangle \left[ 12 \right]} T_9^\alpha T_9^\beta . \]
\[ |A_{ij}|^2 \]

\[ p \text{ real } : \quad |p\rangle^* = [p_1] \quad \& \quad 1p_j^* = \langle p_1 \]

\[ \text{so} \quad \langle p_1 \rangle^* = [q_1 p] \]

\[ \text{Then } \sum |A_{ij}|^2 = q^4 \frac{\langle 13 \rangle \langle 1 \rangle [13] \langle 23 \rangle [23]}{\langle 12 \rangle^2 \langle 12 \rangle^2} \text{ tr}(T_{\alpha}T_{\beta}) \text{ tr}(T_{\alpha}^{-1}T_{\beta}) \]

\[ s = - (p_1 + p_2)^2 \]
\[ t = - (p_1 + p_3)^2 \]
\[ u = - (p_1 + p_4)^2 \]

\[ \text{Done. No nasty } g \text{-matrix traces!} \]

Was it simple just because we chose the quark mass \( m = 0 \)?

Well, let's put it back in. So we'll have

\[ |A_{ij}|^2 \leq \langle 11312 \rangle \langle 21311 \rangle = ? \]

\[ \langle 11 \gamma^\mu |2 \rangle \langle 21 \gamma^\nu |1 \rangle = A \gamma^{\mu \nu} + B (p_1^\mu p_2^\nu + p_2^\mu p_1^\nu) \]

Dot in \( p_1^\nu \) or \( p_2^\nu \):

\[ 0 = A p_1^\mu + B p_1^\mu (p_1 \cdot p_2) \Rightarrow A = - B (p_1^2) \]
Fierz id:

\[ \langle 11 \gamma^\mu 12 \rangle \langle 31 \gamma^\mu 14 \rangle = 2 \langle 13 \rangle \langle 24 \rangle \]

so contract w/ \( \gamma^\mu \)

\[ 2 \langle 12 \rangle \langle 21 \rangle = 4 A + B \cdot 2(p_1 p_2) = -2 B (p_1 p_2) \]

\[
\Rightarrow \quad B = 2
\]

so

\[ \langle 11 \gamma^\mu 12 \rangle \langle 21 \gamma^\mu 11 \rangle = -2 (p_1 \cdot p_2) \gamma^{\mu \nu} + 2 (p_1^\mu p_2^\nu + p_2^\mu p_1^\nu) \]

Hence

\[ \langle 11312 \rangle \langle 21311 \rangle = +4 (p_1 \cdot p_3) (p_2 \cdot p_3) - 2 (p_1 \cdot p_2) p_3^2 \]

\[
= \left[ (p_1 + p_3)^2 - p_3^2 \right] \left[ (p_2 + p_3)^2 - p_3^2 \right] - (p_1 + p_2)^2 p_3^2
\]

\[
= (-t + m^2) (-u + m^2) - s m^2
\]

\[
= t_1 u_1 - s m^2
\]

\[
\sum_{\text{color}} |A_y|^2 = 2 \times (N^2 - 1) g^4 \frac{u_1 t_1 - m^2 s}{s^2}
\]
So here we go ... I spent the first lecture on modern amplitude techniques calculating one amplitude for you... using Feynman diagrams.* Sl** You thought this would be modern, right?

Anyway, this served to introduce to you the spinor helicity formalism, which we will use heavily in the next two lectures. And you've seen it now in a concrete & relevant example, so hopefully $|p\rangle$ & $|p\rangle$ are a little less scary than an hour ago...

The plan is this:

Lecture 2: recursion relations & the power of spinor helicity.

Lecture 3: Apply to gluon squarks. (Outlook, loops, geometry & all that.)