Utrecht University

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The Borderline

between

Classical and Quantum Mechanics

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Davis,

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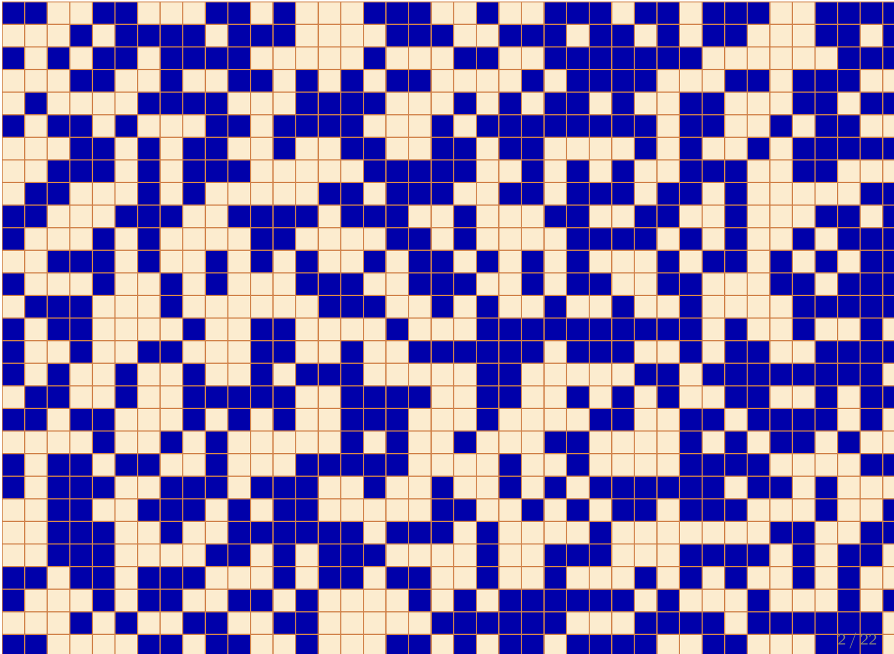
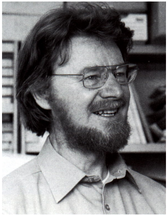
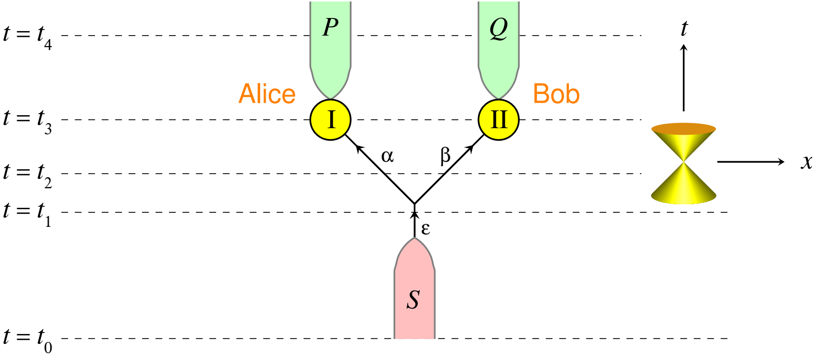
The 2-dimensional Ising Problem.

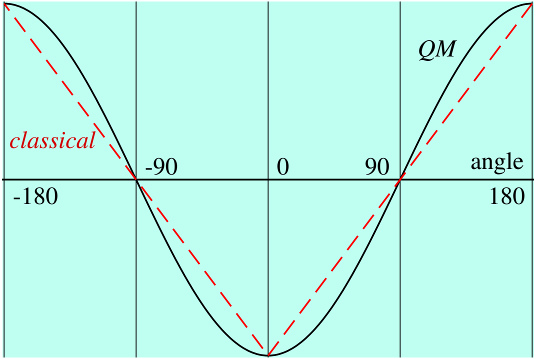


Bell's inequalities.
The observation of two entangled particles by space-like separated observers, "Alice" and "Bob".





Photons: total spin is zero: $\vec{\sigma}_A + \vec{\sigma}_B = 0$. therefore, if Alice and Bob use the same coordinate frame, $\langle\sigma^a_A\rangle = -\langle\sigma^a_B\rangle$.
If Alice and Bob have different unit vectors, $\vec{e}_A \neq \vec{e}_B$, their observations, $A = \langle\vec{\sigma}_A \cdot \vec{e}_A\rangle$ and $B = \langle\vec{\sigma}_B \cdot \vec{e}_B\rangle$, are *correlated*.
As function of their angle:



Real numbers and integers

Imagine that, in contrast to appearances, the real world, at its most fundamental level, were *not* based on real numbers at all. We here consider systems where *only* integers describe what happens at a deeper level. Can one understand *why* our world *appears* to be based on real numbers?

A *mapping* exists of

deterministic (or quantum) physics of a set of $2N$ integers Q_i, P_i	onto	quantum physics on N real observables q_i with N associated momenta p_i

Canonical Variables. Our mapping replaces quantum operator sets p_i and q_i (with usual commutation relations) by sets of universally commuting integers P_i and Q_i.

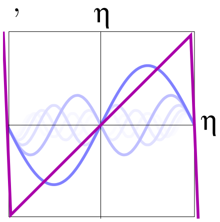
Operators

Define $\epsilon \equiv e^{2\pi} = 535.5$

Consider a Hilbert space spanned by the states
$|Q\rangle$, $Q = -\infty, \cdots, -2, -1, 0, 1, 2, ,\cdots, \infty.$

Introduce the *operator* η , on the interval $-\frac{1}{2} < \eta < \frac{1}{2}$, defined by:
$\epsilon^{iN\eta}|Q\rangle = |Q+N\rangle$, and Fourier transform the function η

$$\eta = \sum_N \epsilon^{iN\eta} \int_{-\frac{1}{2}}^{\frac{1}{2}} \eta \, \mathrm{d}\eta \, \epsilon^{-iN\eta} = \sum_{N \neq 0} \frac{i(-1)^N}{2\pi N} \epsilon^{iN\eta} ,$$



$$\langle Q_1|\eta|Q_2\rangle = \frac{i}{2\pi}(1 - \delta_{Q_1 Q_2}) \frac{(-1)^{Q_1 - Q_2}}{Q_1 - Q_2} .$$

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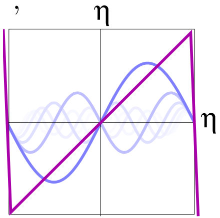
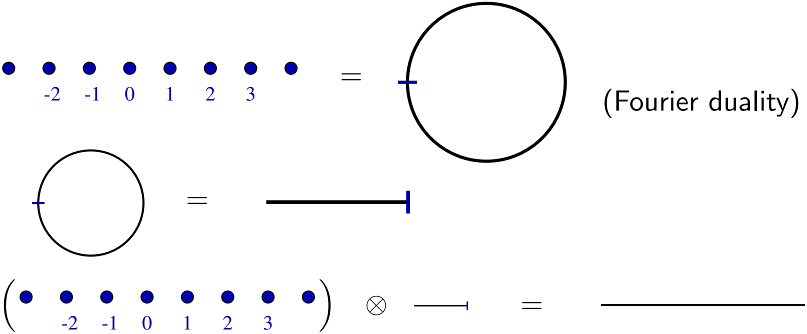


$$\langle Q_1|\eta|Q_2\rangle = \frac{i}{2\pi}(1-\delta_{Q_1Q_2})\frac{(-1)^{Q_1-Q_2}}{Q_1-Q_2} .$$

$$Q_1|[\eta_Q, Q]|Q_2\rangle = \frac{i}{2\pi}\left(\delta_{Q_1Q_2} - (-1)^{Q_2-Q_1}\right) = \frac{i}{2\pi}(\mathbb{I} - |\psi\rangle\langle\psi|) .$$

Find that $\quad [\eta, Q] = \frac{i}{2\pi}(\mathbb{I} - |\psi\rangle\langle\psi|) . \quad |\psi\rangle$ is an *edge state*



Make real number operators $-\infty < q < \infty$ as follows: $q = Q + \eta_P$

There is a unitary transformation of states from one basis to another: $\langle Q,\, \eta_P | \psi \rangle \;=\; \langle q | \psi \rangle$.

Then transform $\langle Q,\, \eta_P | \psi \rangle = \sum_{P=-\infty}^{\infty} \epsilon^{-iP\eta_P} \langle Q,\, P | \psi \rangle \;=\; \langle q | \psi \rangle$

Alternatively, find the p basis: $\langle q | p \rangle = \epsilon^{ipq}$

Matrix elements

(mathematical detail is skipped here: make the mapping $P \leftrightarrow Q$ symmetric)

In Hilbert space $\{|Q,\ P\rangle\}$, we have

$$q = Q + a_Q \quad , \quad p = P + a_P \ ,$$

$$\langle Q_1, P_1 | a_Q | Q_2, P_2 \rangle = \frac{(-1)^{P+Q+1}\, iP}{2\pi(P^2+Q^2)}$$

$$\langle Q_1, P_1 | a_P | Q_2, P_2 \rangle = \frac{(-1)^{P+Q}\, iQ}{2\pi(P^2+Q^2)} \ .$$

From these:

$[q, p] = \frac{i}{2\pi}(1 - |\psi_{\rm edge}\rangle\langle\psi_{\rm edge}|)$, with $\langle Q, P|\psi_{\rm edge}\rangle = (-1)^{Q+P}$

How does this work in QFT?

In ordinary QFT, the splitting $\phi(\vec{x}, t) \;\to\; Q(x, t) + \eta_P(x, t)$ does not survive the field equations, because splitting numbers into "integer part" and "fractional part" is non-linear! It *does* work with real numbers, if the field equations just *interchange* them. In 1+1 dimensions, we have *left movers* and *right movers*:

Free massless bosons in $1+1$ dimensions

$$(\partial_x^2 - \partial_t^2)\phi(x,t) = (\partial_x + \partial_t)(\partial_x - \partial_t)\phi(x,t) = 0 \ \rightarrow$$
$$\phi(x,t) = \phi_L(x+t) + \phi_R(x-t) \ .$$

$$[\phi(x,t), p(y,t)] = \tfrac{i}{2\pi}\delta(x-y) \ ; \qquad H = \pi \int \mathrm{d}x (p(x)^2 + (\partial_x\phi)^2) \ .$$

Temporary: put x and t on a lattice.

$$\phi_{x,t} \equiv \phi(x,t) \ ; \qquad [\phi_{x,t},\ p_{x',t}] = \tfrac{i}{2\pi}\delta_{x,x'} \ .$$

We have: $\phi(x,t+a) + \phi(x,t-a) = \phi(x-a,t) + \phi(x+a,t)$.

How to *map* this model one-to-one on the cellular automaton:

$$Q(x,t+a) + Q(x,t-a) = Q(x-a,t) + Q(x+a,t) \ ,$$

where Q are integers.

$$p(x,t) = \tfrac{1}{2}a^L(x+t) + \tfrac{1}{2}a^R(x-t)\ .$$
$$a^L = p + \partial_x\phi\ ; \quad a^R = p - \partial_x\phi\ .$$

Now, $$H = \tfrac{1}{2}(p^2 + (\partial_x\phi)^2) \ = \ \tfrac{1}{4}({a^L}^2 + {a^R}^2)\ ,$$
$$[a^L,\, a^R] = 0\ ; \quad [a^L(x),\, a^L(y)] = [a^R(y),\, a^R(x)] = \tfrac{i}{\pi}\partial_x\delta(x-y)\ ;$$

Our cellular automaton will be on a lattice: $(x,t) \in \mathbb{Z}$. Therefore, replace commutator by

$$[\phi(x),\, p(y)] = \tfrac{i}{2\pi}\delta_{x,y} \qquad (1)$$
$$[a^L(x),\, a^L(y)] = \pm\tfrac{i}{2\pi} \ \text{ if } \ y = x \pm 1 \quad .$$

Replace real valued operators $a^{L,R}(x)$ by integer valued operators $A^{L,R}(x)$ and their associated operators $\eta_A^{L,R}(x)$:

$$a^L(x) = A^L(x) + \eta_A^L(x-1)\ .$$

This splitting survives the evolution law:
$a^L,\ A^L,$ and η_A^L all move to the left, and $a^R,\ A^R,$ and η_A^R move to the right.

Use the *quantum hamiltonian* (its space-lattice version) to describe the evolution of this classical automaton. $H = H^L + H^R$.

In *momentum space* :

$$H^L = \tfrac{1}{2}\int_0^{1/2} \mathrm{d}k\, a^L(k)a^L(-k)\, M(k) \quad ; \quad M(\kappa) = \frac{\pi\kappa}{\sin(2\pi\kappa)} \ .$$

This hamiltonian turns $a^L(x)$ into a pure left-mover, and $a^R(x)$ into a right-mover:
$A^L(x+t) = Q(x, t+1) - Q(x-1, t)$

The (super) string is a 1+1 dimensional theory.

Here, the quantized field is the set of (super) string coordinates. They are now replaced by the integer valued left- and right-movers $A^{L,R}(x \pm t)$.

Re-inserting the units gives a surprise: these coordinates form a discrete lattice with lattice length a that is independent of the lattice chosen on the world sheet. Even if you send the world sheet to a continuum, the space-time lattice length a is

$$a = 2\pi\sqrt{\alpha'} \ .$$

Furthermore, as we will see later, the string constant ρ is not freely adjustable.

Fermions

A fermionic system can be handled the same way. Assume a Majorana fermionic field ψ_A with $\psi_A = \psi_A^\dagger$, $A = 1, 2$ (or, $A = L, R$). Dirac equation: $(\gamma_+\partial_- + \gamma_-\partial_+)\psi = 0$.

$$\text{One finds that} \quad \psi_A^\mu(x,t) = \begin{pmatrix} \psi_L^\mu(x+t) \\ \psi_R^\mu(x-t) \end{pmatrix} \ .$$

The corresponding classical theory now has Boolean degrees of freedom, $\sigma(x,t) = \pm 1$, obeying the equations:

$$\sigma(x,\ t+1) = \sigma(x-1,\ t)\ \sigma(x-1,\ t)\ \sigma(x,\ t-1)\ .$$

This also splits up into left- and right-movers:

$$\sigma(x,t) = \sigma_L(x+t)\ \sigma_R(x-t)\ .$$

Superstring theories contain $D-2$ independent bosonic fields (coordinates) and $D-2$ Majorana fermion species. All these can be mapped onto deterministic models processing integers as well as ±1's (Boolean variables) classically.

So-far, we only handled strings of infinite length. We need to add: (periodic) boundary conditions, interactions, and *constraints.* The constraints give us the remaining two *longitudunal* coordinates, needed to investigate *Lorentz invariance*.

The constraints only need to be imposed on the quantum side of the theory, as is done in superstrings.
As is standard in Superstring theory, this restricts us to $D=10$.

In Superstring Theory, both bosons and fermions obey gauge conditions and constraints, which should determine $\psi_A^{\pm}$ in terms of $a_{L,R}^{\mathrm{tr}}$, and so also $\sigma_{L,R}^{0}$ and $\sigma_{L,R}^{D-1}$ should be determined by the transverse $\sigma_{L,R}^{a}$

The quantum – classical mapping in string theory is *not* free of problems:

How do the *longitudinal* modes $A^{\pm}_{L,R}(x,\ t)$ behave in the deterministic model?

String theory wants us to pick a gauge such as $A^{+}_{L,R}(x,t)=1$. Then

$$A^{-}_{L,R}(x,t)=\sum_{i-1}^{D-2}A^{i}_{L,R}{}^{2}(x,t)\quad\rightarrow\quad A^{-}_{L,R}(x,t)\geq 1\ ,$$

which does allow us to use X^{+} as our time coordinate, but violates Lorentz invariance.

How to do this better: is there a better gauge? What is then our time coordinate?

Lorentz transformations ?

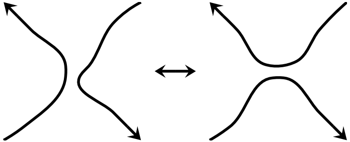
1. The deterministic theory only has manifest $O(D-2,\mathbb{Z})$ invariance (the transverse modes)
2. The quantum theory has manifest $O(D-2,\mathbb{R})$ invariance.
3. In the quantum theory, we impose the constraints to obtain the longitudinal coordinates and the remaining parts of $D(D-1,1,\mathbb{R})$ invariance, as usual.

In principle, Lorentz invariance is only needed in the quantum formalism. In terms of the CA variables, Lorentz transformations now seem to be quite complicated operators.

String Interactions

Conjecture :

One can write down a *classical and unique* interaction among these classical strings: if two strings hit the same spacetime point Q^μ, two arms are exchanged:



This is also deterministic if the string coupling constant g_s is fixed to $g_s = 1$, and the strings must be oriented.

This generates closed, interacting, oriented strings.

But our conjecture requires a good, deterministic, definition of $X^{\pm}_{L,R}(x,t)$, which we do not have at present.

Conclusions

Superstring theory is a quantum theory that can be mapped onto a cellular automaton. The automaton puts the system on a space-time lattice with lattice length $a = 2\pi\sqrt{\alpha'}$. Obviously, this is finite, but, as yet, we could prove this only for the bulk of the superstring, not for the (interacting) finite-size excited modes.

The lattice on the *world sheet* has to be sent to the continuum limit. This seems to be a question of gauge-fixing rather than a physical limit, but it still is a delicate procedure, to be investigated further.

According to the CA interpretation of QM, the collapse of the wave function and the Born rule are **automatic consequences of the Schrödinger equation itself**. They need not be put in by hand afterwards.

An apparent fundamental difficulty: Bell's theorems.
Do they apply here?

Bell's inequalities

Theorem (Bell):
In any deterministic theory intended to reproduce quantum behavior, (for instance when Einstein-Podolsky-Rosen photons are observed through two spacelike separated filters, $\vec{a}$ and $\vec{b}$), one will have to allow superluminal signals between $\vec{a}$ and $\vec{b}$.

... since we should be allowed to modify the settings $\vec{a}$ and/or $\vec{b}$ any time, at free will...

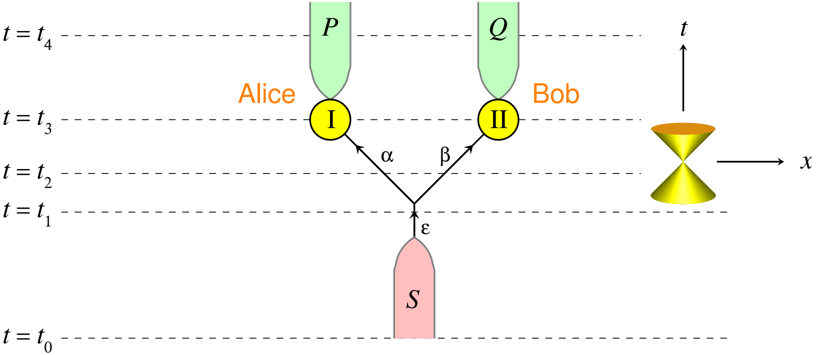
But there is no "free will" in a deterministic theory (Super-determinism).

Theorem: even so, you cannot avoid Bell's inequalities!

unless you accept "ridiculous fine-tuning", or "conspiracy"

Today's claim: we never need actual signals going backwards in time or faster than light. All we need is **non-locally correlated** *vacuum fluctuations.*

Vacuum fluctuations are ubiquitous in QFT vacua.



In the Bell experiment, at $t = t_0$, one must demand that those degrees of freedom that later force Alice and Bob to make their decisions, and the source that emits two entangled particles, have 3 - body correlations of the form

$$\langle\, a \;\, b \;\, c \,\rangle \propto |\sin(a - b - c)| \qquad \text{(or worse)}$$

As for “conspiracy”: the ontological nature of a physical state is *conserved in time*. If a photon is observed, at late times, to be in a given polarization state, it has been in *exactly the same state* the moment it was emitted by the source. The conspiracy argument now demands that the “ontological basis” be *unobservable!* (as it is in string theory)

Shut up and calculate!

THE END

arXiv: 1204.4926
arXiv: 1205.4107
arXiv: 1207.3612
and to be published.