The Borderline
between

Classical and Quantum Mechanics

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The 2-dimensional Ising Problem.
Bell’s inequalities.
The observation of two entangled particles by space-like separated observers, “Alice” and “Bob”.

\[ t = t_0 \]
\[ t = t_1 \]
\[ t = t_2 \]
\[ t = t_3 \]
\[ t = t_4 \]

![Diagram](image.png)
Photons: total spin is zero: $\vec{\sigma}_A + \vec{\sigma}_B = 0$. Therefore, if Alice and Bob use the same coordinate frame, $\langle \sigma^a_A \rangle = -\langle \sigma^a_B \rangle$.

If Alice and Bob have different unit vectors, $\vec{e}_A \neq \vec{e}_B$, their observations, $A = \langle \vec{\sigma}_A \cdot \vec{e}_A \rangle$ and $B = \langle \vec{\sigma}_B \cdot \vec{e}_B \rangle$, are correlated. As function of their angle:
Real numbers and integers

Imagine that, in contrast to appearances, the real world, at its most fundamental level, were not based on real numbers at all. We here consider systems where only integers describe what happens at a deeper level. Can one understand why our world appears to be based on real numbers?

A mapping exists of

*deteministic (or quantum) physics of a set of $2N$ integers $Q_i, P_i$ on quantum physics on $N$ real observables $q_i$ with $N$ associated momenta $p_i$.*

*Canonical Variables.* Our mapping replaces quantum operator sets $p_i$ and $q_i$ (with usual commutation relations) by sets of universally commuting integers $P_i$ and $Q_i$. 
Operators

Define \( \epsilon \equiv e^{2\pi} = 535.5 \)

Consider a Hilbert space spanned by the states
\(|Q\rangle\), \( Q = -\infty, \cdots, -2, -1, 0, 1, 2, \cdots, \infty \).

Introduce the operator \( \eta \), on the interval \(-\frac{1}{2} < \eta < \frac{1}{2}\), defined by:
\[ \epsilon^{iN\eta}|Q\rangle = |Q + N\rangle, \]
and Fourier transform the function \( \eta \)

\[ \eta = \sum_{N} \epsilon^{iN\eta} \int_{-\frac{1}{2}}^{\frac{1}{2}} \eta \, d\eta \, \epsilon^{-iN\eta} = \sum_{N \neq 0} \frac{i(-1)^N}{2\pi N} \epsilon^{iN\eta}, \]

\[ \langle Q_1 | \eta | Q_2 \rangle = \frac{i}{2\pi} (1 - \delta_{Q_1 Q_2}) \frac{(-1)^{Q_1 - Q_2}}{Q_1 - Q_2}. \]
Operators

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$$\langle Q_1 | \eta | Q_2 \rangle = \frac{i}{2\pi} (1 - \delta_{Q_1 Q_2}) \frac{(-1)^{Q_1 - Q_2}}{Q_1 - Q_2} .$$

$Q_1 |[\eta Q, Q]| Q_2 \rangle = \frac{i}{2\pi} \left( \delta_{Q_1 Q_2} - (-1)^{Q_2 - Q_1} \right) = \frac{i}{2\pi} \left( \mathbb{I} - |\psi\rangle \langle \psi| \right) .$

Find that $[\eta, Q] = \frac{i}{2\pi} \left( \mathbb{I} - |\psi\rangle \langle \psi| \right)$.

$|\psi\rangle$ is an *edge state*.
Make real number operators $-\infty < q < \infty$ as follows: $q = Q + \eta P$

There is a unitary transformation of states from one basis to another: $\langle Q, \eta P | \psi \rangle = \langle q | \psi \rangle$.

Then transform $\langle Q, \eta P | \psi \rangle = \sum_{P = -\infty}^{\infty} \epsilon^{-iP\eta} \langle Q, P | \psi \rangle = \langle q | \psi \rangle$

Alternatively, find the $p$ basis: $\langle q | p \rangle = \epsilon^{ipq}$
Matrix elements

(mathematical detail is skipped here: make the mapping $P \leftrightarrow Q$ symmetric)

In Hilbert space $\{|Q, P\rangle\}$, we have

$$q = Q + a_Q, \quad p = P + a_P,$$

$$\langle Q_1, P_1 | a_Q | Q_2, P_2 \rangle = \frac{(-1)^{P+Q+1} iP}{2\pi (P^2 + Q^2)}$$

$$\langle Q_1, P_1 | a_P | Q_2, P_2 \rangle = \frac{(-1)^{P+Q} iQ}{2\pi (P^2 + Q^2)}.$$

From these:

$$[q, p] = \frac{i}{2\pi} (1 - |\psi_{edge}\rangle \langle \psi_{edge}|), \quad \text{with} \quad \langle Q, P | \psi_{edge} \rangle = (-1)^{Q+P}$$
How does this work in QFT?

In ordinary QFT, the splitting \( \phi(\vec{x}, t) \rightarrow Q(x, t) + \eta_P(x, t) \) does not survive the field equations, because splitting numbers into “integer part” and “fractional part” is non-linear! It \textit{does} work with real numbers, if the field equations just \textit{interchange} them. In 1+1 dimensions, we have \textit{left movers} and \textit{right movers}:
Free massless bosons in $1 + 1$ dimensions

$$(\partial_x^2 - \partial_t^2)\phi(x, t) = (\partial_x + \partial_t)(\partial_x - \partial_t)\phi(x, t) = 0 \rightarrow 
\phi(x, t) = \phi_L(x + t) + \phi_R(x - t) .$$

$$[\phi(x, t), p(y, t)] = \frac{i}{2\pi}\delta(x - y) ; \quad H = \pi \int dx (p(x)^2 + (\partial_x \phi)^2).$$

Temporary: put $x$ and $t$ on a lattice.

$$\phi_{x,t} \equiv \phi(x, t) ; \quad [\phi_{x,t}, p_{x',t}] = \frac{i}{2\pi}\delta_{x,x'}.$$

We have: $\phi(x, t + a) + \phi(x, t - a) = \phi(x - a, t) + \phi(x + a, t)$.

How to map this model one-to-one on the cellular automaton:

$$Q(x, t + a) + Q(x, t - a) = Q(x - a, t) + Q(x + a, t) ,$$

where $Q$ are integers.
\[ p(x, t) = \frac{1}{2} a^L(x + t) + \frac{1}{2} a^R(x - t) . \]

\[ a^L = p + \partial_x \phi ; \quad a^R = p - \partial_x \phi . \]

Now,
\[ H = \frac{1}{2}(p^2 + (\partial_x \phi)^2) = \frac{1}{4}(a^L^2 + a^R^2) , \]
\[ [a^L, a^R] = 0 ; \quad [a^L(x), a^L(y)] = [a^R(y), a^R(x)] = i \pi \partial_x \delta(x - y) ; \]

Our cellular automaton will be on a lattice: \((x, t) \in \mathbb{Z} \). Therefore, replace commutator by

\[ [\phi(x), p(y)] = i \frac{2}{2 \pi} \delta_{x,y} \] (1)
\[ [a^L(x), a^L(y)] = \pm i \frac{2}{2 \pi} \text{ if } y = x \pm 1 . \]

Replace real valued operators \( a^{L,R}(x) \) by integer valued operators \( A^{L,R}(x) \) and their associated operators \( \eta^{L,R}_A(x) \) : 
\[ a^L(x) = A^L(x) + \eta^L_A(x - 1) . \]
This splitting survives the evolution law: $a^L, A^L$, and $\eta_A^L$ all move to the left, and $a^R, A^R,$ and $\eta_A^R$ move to the right.

Use the quantum Hamiltonian (its space-lattice version) to describe the evolution of this classical automaton. $H = H^L + H^R$.

In momentum space:

$$H^L = \frac{1}{2} \int_0^{1/2} dk \; a^L(k) a^L(-k) M(k) ; \quad M(\kappa) = \frac{\pi \kappa}{\sin(2\pi \kappa)} .$$

This Hamiltonian turns $a^L(x)$ into a pure left-mover, and $a^R(x)$ into a right-mover:

$$A^L(x + t) = Q(x, t + 1) - Q(x - 1, t)$$
The (super) string is a 1+1 dimensional theory.

Here, the quantized field is the set of (super) string coordinates. They are now replaced by the integer valued left- and right-movers $A^L,R(x \pm t)$.

Re-inserting the units gives a surprise: these coordinates form a discrete lattice with lattice length $a$ that is independent of the lattice chosen on the world sheet. Even if you send the world sheet to a continuum, the space-time lattice length $a$ is

$$a = 2\pi\sqrt{\alpha'}.$$ 

Furthermore, as we will see later, the string constant $\rho$ is not freely adjustable.
Fermions

A fermionic system can be handled the same way. Assume a Majorana fermionic field $\psi_A$ with $\psi_A = \psi_A^\dagger$, $A = 1, 2$ (or, $A = L, R$). Dirac equation: $(\gamma_+ \partial_- + \gamma_- \partial_+) \psi = 0$.

One finds that $\psi_A^\mu(x, t) = \begin{pmatrix} \psi_L^\mu(x + t) \\ \psi_R^\mu(x - t) \end{pmatrix}$.

The corresponding classical theory now has Boolean degrees of freedom, $\sigma(x, t) = \pm 1$, obeying the equations:

$$\sigma(x, t + 1) = \sigma(x - 1, t) \sigma(x - 1, t) \sigma(x, t - 1).$$

This also splits up into left- and right-movers:

$$\sigma(x, t) = \sigma_L(x + t) \sigma_R(x - t).$$
Superstring theories contain $D - 2$ independent bosonic fields (coordinates) and $D - 2$ Majorana fermion species. All these can be mapped onto deterministic models processing integers as well as ±1’s (Boolean variables) classically.

So-far, we only handled strings of infinite length. We need to add: (periodic) boundary conditions, interactions, and constraints. The constraints give us the remaining two longitudinal coordinates, needed to investigate Lorentz invariance.

The constraints only need to be imposed on the quantum side of the theory, as is done in superstrings. As is standard in Superstring theory, this restricts us to $D = 10$.

In Superstring Theory, both bosons and fermions obey gauge conditions and constraints, which should determine $\psi_A^\pm$ in terms of $a^{tr}_{L,R}$, and so also $\sigma^0_{L,R}$ and $\sigma^{D-1}_{L,R}$ should be determined by the transverse $\sigma^a_{L,R}$.
The quantum – classical mapping in string theory is not free of problems:

How do the longitudinal modes $A_{L,R}^{\pm}(x, t)$ behave in the deterministic model?

String theory wants us to pick a gauge such as $A_{L,R}^{+}(x, t) = 1$. Then

$$A_{L,R}^{-}(x, t) = \sum_{i=1}^{D-2} A_{L,R}^i(x, t)^2 \rightarrow A_{L,R}^{-}(x, t) \geq 1 ,$$

which does allow us to use $X^+$ as our time coordinate, but violates Lorentz invariance.

How to do this better: is there a better gauge? What is then our time coordinate?
Lorentz transformations?

1. The deterministic theory only has manifest $O(D - 2, \mathbb{Z})$ invariance (the transverse modes).
2. The quantum theory has manifest $O(D - 2, \mathbb{R})$ invariance.
3. In the quantum theory, we impose the constraints to obtain the longitudinal coordinates and the remaining parts of $D(D - 1, 1, \mathbb{R})$ invariance, as usual.

In principle, Lorentz invariance is only needed in the quantum formalism. In terms of the CA variables, Lorentz transformations now seem to be quite complicated operators.
String Interactions

**Conjecture:**

One can write down a *classical and unique* interaction among these classical strings: if two strings hit the same spacetime point $Q^\mu$, two arms are exchanged:

This is also deterministic if the string coupling constant $g_s$ is fixed to $g_s = 1$, and the strings must be oriented.

This generates closed, interacting, oriented strings.

But our conjecture requires a good, deterministic, definition of $X_{L,R}^\pm(x,t)$, which we do not have at present.
Conclusions

Superstring theory is a quantum theory that can be mapped onto a cellular automaton. The automaton puts the system on a space-time lattice with lattice length \( a = 2\pi \sqrt{\alpha'} \). Obviously, this is finite, but, as yet, we could prove this only for the bulk of the superstring, not for the (interacting) finite-size excited modes.

The lattice on the world sheet has to be sent to the continuum limit. This seems to be a question of gauge-fixing rather than a physical limit, but it still is a delicate procedure, to be investigated further.

According to the CA interpretation of QM, the collapse of the wave function and the Born rule are automatic consequences of the Schrödinger equation itself. They need not be put in by hand afterwards.

An apparent fundamental difficulty: Bell’s theorems. Do they apply here?
Bell’s inequalities

Theorem (Bell):
In any deterministic theory intended to reproduce quantum behavior, (for instance when Einstein-Podolsky-Rosen photons are observed through two spacelike separated filters, $\vec{a}$ and $\vec{b}$), one will have to allow superluminal signals between $\vec{a}$ and $\vec{b}$.

... since we should be allowed to modify the settings $\vec{a}$ and/or $\vec{b}$ any time, at free will...

But there is no “free will” in a deterministic theory (Super-determinism).

Theorem: even so, you cannot avoid Bell’s inequalities!
unless you accept “ridiculous fine-tuning”, or “conspiracy”

Today’s claim: we never need actual signals going backwards in time or faster than light. All we need is non-locally correlated vacuum fluctuations.

Vacuum fluctuations are ubiquitous in QFT vacua.
In the Bell experiment, at \( t = t_0 \), one must demand that those degrees of freedom that later force Alice and Bob to make their decisions, and the source that emits two entangled particles, have 3 - body correlations of the form

\[
\langle a \ b \ c \rangle \propto |\sin(a - b - c)|
\]

(or worse)
As for “conspiracy”: the ontological nature of a physical state is conserved in time. If a photon is observed, at late times, to be in a given polarization state, it has been in exactly the same state the moment it was emitted by the source. The conspiracy argument now demands that the “ontological basis” be unobservable! (as it is in string theory)

Shut up and calculate!

THE END

arXiv: 1204.4926
arXiv: 1205.4107
arXiv: 1207.3612
and to be published.