EFT in a Time Dependent World

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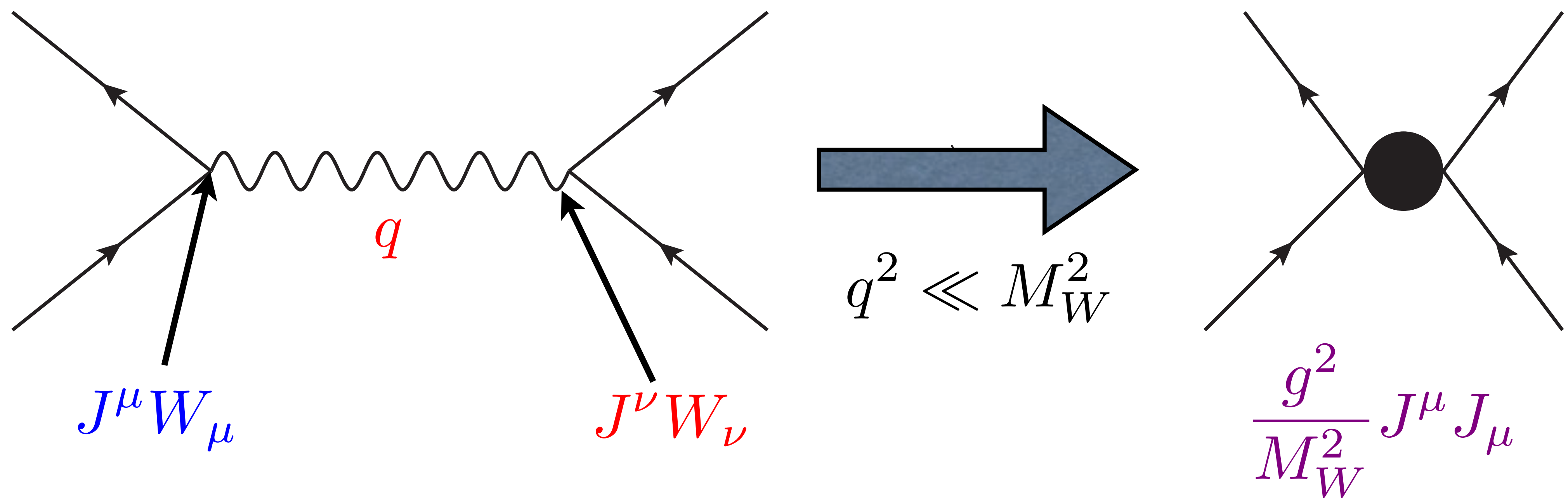
Introduction

Q: How do we integrate out the quantum fluctuations of a heavy field in a time dependent system?

Look back at how we do this for in-out amplitudes:



Matching:

$$S_{fi}^{\rm Full} \equiv \langle 0_H, f_L {\rm out} | 0_H, i_L {\rm in}\rangle = \langle 0_H, f_L {\rm in} | T(\exp -i \int_{-\infty}^{\infty} H_I(t)) | 0_H, i_L {\rm in}\rangle =$$

$$S_{fi}^{\rm Eff} = \langle f_L {\rm in} | T(\exp -i \int_{-\infty}^{\infty} H_I^{\rm Eff}(t)) | i_L {\rm in}\rangle$$

This defines the effective Hamiltonian involving only the light fields

How would the matching go for time dependent correlation functions?

Recall:

$$\langle\mathcal{O}\rangle(t) \equiv \mathrm{Tr}\left(\rho(t)\mathcal{O}\right), \quad i\frac{\partial\rho(t)}{\partial t} = [H,\rho(t)] \Rightarrow \rho(t) = U(t,t_0)\rho(t_0)U^\dagger(t,t_0)$$

Now consider an operator defined on the total Hilbert space for heavy and light fields acting only on the light fields:

$$\mathcal{H}_{\rm tot} = \mathcal{H}_L \otimes \mathcal{H}_H, \quad \mathcal{O} = \mathcal{O}_L \otimes \mathbb{I}_H$$

Then: $\mathrm{Tr}_{\mathcal{H}}(\rho\mathcal{O})) = \mathrm{Tr}_{\mathcal{H}_L}(\rho_{\rm red}\mathcal{O}_L), \quad \rho_{\rm red} \equiv \mathrm{Tr}_{\mathcal{H}_H}\rho$

How does $\rho_{\rm red}$ evolve in time?

IF: $\rho_{\rm red}(t) = U_{\rm red}(t,t_0)\rho_{\rm red}(t_0)U^\dagger_{\rm red}(t,t_0)$

THEN: We could define an effective Hamiltonian that would generate light field correlators via

$$i\frac{\partial U_{\rm red}(t,t_0)}{\partial t} = H_I^{\rm Eff}(t)U_{\rm red}(t,t_0)$$

BUT: This is not true in general!

The propagator for the reduced density matrix is the influence functional and it knows about the open channels the light field could lose energy into.

On the other hand: if it's too hard to excite the heavy modes, there may be some approximately unitary evolution and thus some sort of effective Hamiltonian.

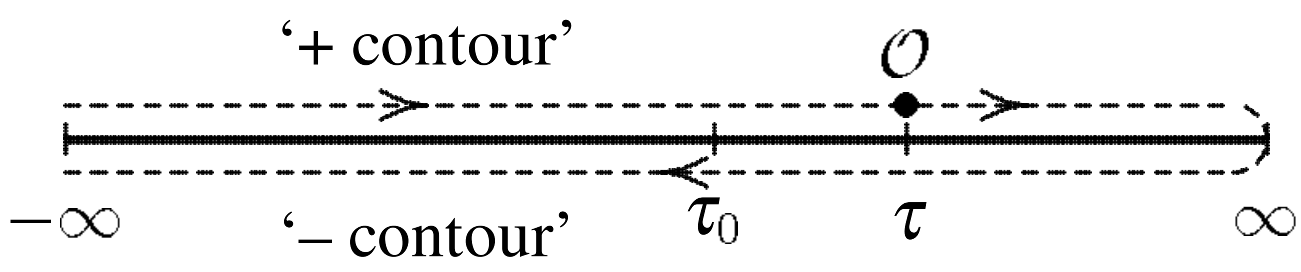
What actually happens to light field observables?

Interlude 1: The Schwinger-Keldysh Formalism

Want to compute time dependent expectation values

$$\langle \mathcal{O}(t) \rangle \equiv \mathrm{Tr}(\rho(t)\mathcal{O}(t)) = \mathrm{Tr}(\rho(t_0)U^\dagger(t,t_0)\mathcal{O}(t)U(t,t_0))$$

The two time evolution operators, with daggering lead to a doubling of the time contour



As usual, there is more than one way to evaluate expectation values

$$\mathrm{tr}\big(\rho(t)\mathcal{O}(t)\big) = \mathrm{tr}\Big[\rho(t_0)T_c\Big(\mathcal{O}^+(t)e^{-i\int_{t_0}^{\infty}dt'\,[H_I^+(t')-H_I^-(t')]}\Big)\Big]$$

$$\langle\mathcal{O}_S\rangle(t) = \int d\phi\,\langle\phi|U^\dagger(t,t_0)\mathcal{O}_S U(t,t_0)\rho(t_0)|\phi\rangle$$

$$= \int d\phi^+ d\phi^- \,\Big(\int_{\mathrm{bc}} \mathcal{D}\Phi^+\mathcal{D}\Phi^-\, e^{i(S[\Phi^+]-S[\Phi^-])}\mathcal{O}_S\Big)\rho(\phi^+,\phi^-;t_0)$$

$$\Phi^+(t_0,\vec{x}) = \phi^+(\vec{x}),\quad \Phi^-(t_0,\vec{x}) = \phi^-(\vec{x}),\quad \Phi^+(t,\vec{x}) = \Phi^-(t,\vec{x})$$

This last way suggests the generating functional for correlators

$$Z[J^+,J^-] = \int d\phi^+ d\phi^-\,\rho(\phi^+,\phi^-;t_0)\int_{\mathrm{bc}}\mathcal{D}\Phi^+\mathcal{D}\Phi^-\, e^{i(S[\Phi^+,J^+]-S[\Phi^-,J^-])}$$

$$S[\Phi^\pm,J^\pm] = \int_{t_0}^{t} dt'\int d^3\vec{x}\,\Big\{\mathcal{L}[\Phi^\pm(t',\vec{x})] + J^\pm(t',\vec{x})\Phi^\pm(t',\vec{x})\Big\},$$

The Green's functions are

$$G^{++}(t,\vec{x};t',\vec{y}) = \Theta(t-t')\,G^>(t,\vec{x};t',\vec{y}) + \Theta(t'-t)\,G^<(t,\vec{x};t',\vec{y})$$

$$G^{+-}(t,\vec{x};t',\vec{y}) = G^<(t,\vec{x};t',\vec{y})$$

$$G^{-+}(t,\vec{x};t',\vec{y}) = G^>(t,\vec{x};t',\vec{y})$$

$$G^{--}(t,\vec{x};t',\vec{y}) = \Theta(t'-t)\,G^>(t,\vec{x};t',\vec{y}) + \Theta(t-t')\,G^<(t,\vec{x};t',\vec{y})$$

Interlude 2: The initial State

Many ways to prepare the initial state:

- Start from the infinite past in free field vacuum and turn on interactions adiabatically up to the initial time we want to evolve from (Gell-Mann Low theorem),

- Start from the infinite past in some arbitrary density matrix that commutes with the free Hamiltonian and then turn on interactions at the initial time,

- Don't consider the infinite past. Time starts at the initial time and that state is just specified then, with no further construction details.

For simplicity: $\rho(t_0) \to |0(t_0)\rangle\langle 0(t_0)|$

with $|0(t_0)\rangle$ the free-field vacuum

A Toy Example

light

heavy

$$S = \int dt d^3\vec{x} \left\{ \frac{1}{2}\partial_\mu \Phi \partial^\mu \Phi - \frac{1}{2} m^2 \Phi^2 + \frac{1}{2}\partial_\mu \chi \partial^\mu \chi - \frac{1}{2} M^2 \chi^2 - \frac{1}{24}\lambda \Phi^4 - \frac{1}{24}\tilde{\lambda}\chi^4 - \frac{1}{2} g \Theta(t - t_0) \chi^2 \Phi^2 \right\}.$$

Initial State: prepared in the free vacuum for both fields.

What should we compute?

Try heavy corrections to the light field zero mode equations of motion.

Look at in-in heavy contributions to the light field tadpole

Procedure

- Expand light field on contour about zero mode:

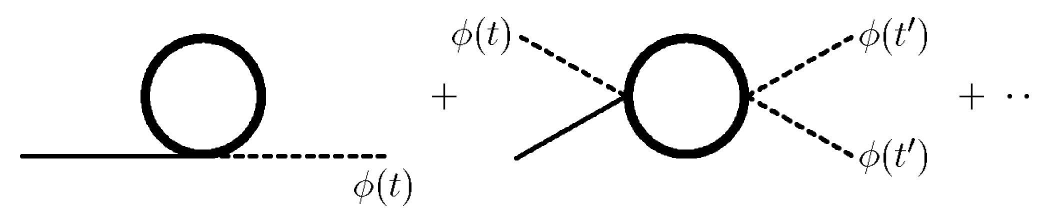
$$\Phi^{\pm}(\vec{x},t) = \phi(t) + \Psi^{\pm}(\vec{x},t), \quad \text{with} \quad \langle \Psi^{\pm}(\vec{x},t)\rangle = 0$$

- Construct the interaction Hamiltonian between the heavy field and light fluctuations:

$$H_I^{\pm}(t) = \int d^3\vec{x} \left\{ \tfrac{1}{2} g\phi^2 \chi^{\pm 2} + g\phi\varphi^{\pm}\chi^{\pm 2} + \tfrac{1}{2} g{\varphi^{\pm}}^2 \chi^{\pm 2} \right\}$$

- Use perturbation theory to compute the contributions to the light tadpole:

$\langle 0(t)|\varphi(t,\vec{x})|0(t)\rangle =$



$$\ddot{\phi}(t) + m^2\phi(t) + \frac{1}{6}\lambda\phi^3(t) +$$
$$\frac{1}{2}g^2\phi(t)\int_{t_0}^{t} dt'\,\phi^2(t')\int \frac{d^3\vec{k}}{(2\pi)^3}\,\underbrace{\frac{\sin\left[2\omega_k(t-t')\right]}{\omega_k^2}}_{\frac{1}{2}\frac{d}{dt}K(t-t')} + \cdots = 0$$

Integrate by parts 3 times:

$$\frac{1}{2}g^2\phi(t)\int_{t_0}^{t} dt'\,\phi^2(t')\int \frac{d^3\vec{k}}{(2\pi)^3}\,\frac{\sin\left[2\omega_k(t-t')\right]}{\omega_k^2} = \frac{1}{4}g^2\phi(t)\int_{t_0}^{t} dt'\,\phi^2(t')\frac{d}{dt'}K(t-t')$$
$$= \frac{1}{4}\underbrace{g^2\phi^3(t)K(0)}_{\text{quartic coupling renormalization}} - \frac{1}{4}\underbrace{g^2\phi(t)\phi^2(t_0)K(t-t_0)}_{\text{time dep mass term}} - \frac{1}{4}g^2\phi(t)\int_{t_0}^{t} dt'\,\underbrace{\frac{d\phi^2(t')}{dt'}K(t-t')}_{\text{dissipative term}}$$

To get the large M limit, keep integrating by parts and evaluate the resulting kernels via the saddle point method

$$\langle 0(t_1)|\varphi(t_1,\vec{x})|0(t_1)\rangle = \int_{t_0}^{t_1} dt\, \frac{\sin[m(t-t_1)]}{m}\Big\{\cdots\Big\} = 0$$

$$\begin{aligned}
\Big\{\cdots\Big\} = 0 = \ddot{\phi}(t) &+ m_{\rm phys}^2\,\phi(t) + \frac{1}{6}\lambda\,\phi^3(t) + \frac{1}{4}g^2 K_3^+(0)\,\phi^3(t) \\
&+ \frac{g^2}{16\pi^2}\,\phi(t)\sum_{n=1}^{\infty}\frac{(-1)^n}{2^n}\frac{(n-1)!}{(2n+1)!!}\frac{1}{M^{2n}}\frac{d^{2n}\phi^2(t)}{dt^{2n}} \qquad \longleftarrow \quad \text{local} \\
&\left.\begin{aligned}
&- \frac{1}{2}g^2\,\phi(t)\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2n+1}}K^+_{3+2n}(t-t_0)\frac{d^{2n}\phi^2(t')}{dt'^{2n}}\Big|_{t'=t_0} \\
&- \frac{1}{2}g^2\,\phi(t)\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{2^{2n}}K^-_{2+2n}(t-t_0)\frac{d^{2n-1}\phi^2(t')}{dt'^{2n-1}}\Big|_{t'=t_0} + \mathcal{O}(g^3)
\end{aligned}\right\}\ \text{non-local}
\end{aligned}$$

$$K^+_{3+2n}(t-t_0) = \frac{1}{2\sqrt{e\pi^3}}\frac{1}{M^{2n}}\frac{\cos\big[2M(t-t_0)+\frac{3\pi}{4}\big]}{[2M(t-t_0)]^{3/2}} + \mathcal{O}\big([M(t-t_0)]^{-5/2}\big)$$

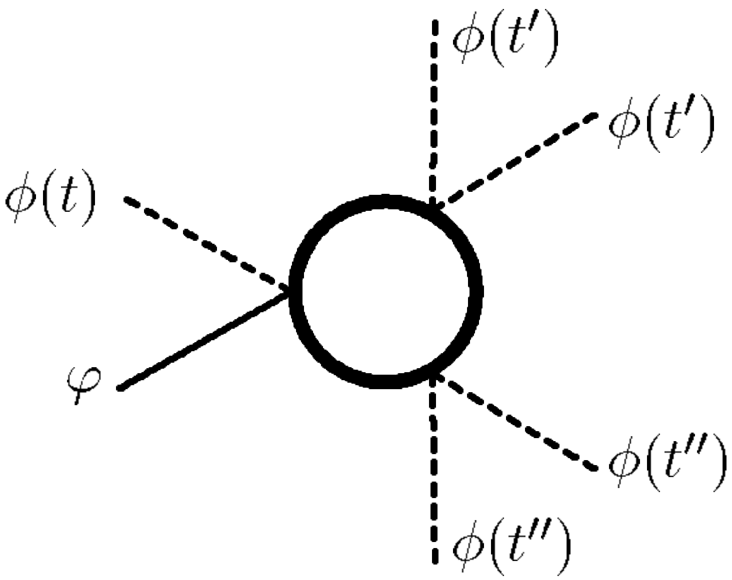
$$K^-_{2+2n}(t-t_0) = \frac{1}{2\sqrt{e\pi^3}}\frac{1}{M^{2n-1}}\frac{\sin\big[2M(t-t_0)+\frac{3\pi}{4}\big]}{[2M(t-t_0)]^{3/2}} + \mathcal{O}\big([M(t-t_0)]^{-5/2}\big)$$

We see that to this order in the coupling, we generate standard local terms, but then we also generate NON-local, transient mass terms.

Note that the transience is on the heavy time scale and that if the interactions had been turned on adiabatically in the far past, these terms would not have appeared.

Does anything new happen at higher orders?

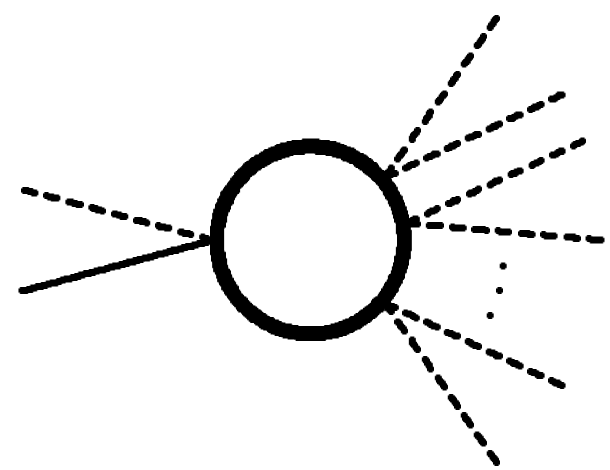
One-Loop, higher order:



$$
\begin{aligned}
&\ddot{\phi}(t) + m^2\phi(t) + \frac{1}{6}\lambda\phi^3(t) + \text{order } g^2 \text{ terms} \\
&+ \frac{1}{48\pi^2}\frac{g^3}{M^2}\phi^5(t) + \frac{1}{48\pi^2}\frac{g^3}{M^2}\phi^4(t_0)\phi(t) \\
&- \frac{1}{8}g^3 K_5^+(t-t_0)\phi^2(t_0)\phi^3(t) - \frac{1}{8}g^3 K_5^+(t-t_0)\phi^4(t_0)\phi(t) \\
&- \frac{1}{4}g^3\phi^2(t_0)K_4^-(t-t_0)\phi(t)\int_{t_0}^{t} dt'\,\phi^2(t') + \cdots = 0
\end{aligned}
$$

Could induce secular growth

First of a sequence of terms



$$\sim g^n\,\phi^{2n-1}.$$

$$\frac{g^n}{M^{2n-m-4}}\tilde{d}^{(n)}_{[k_i],m}(t-t_0)\phi(t)\int_{t_0}^{t} dt_1\,\phi^{2k_1}(t_1)\int_{t_0}^{t_1} dt_2\,\phi^{2k_2}(t_2)\cdots\int_{t_0}^{t_{m-1}} dt_m\,\phi^{2k_m}(t_m)$$

Solve eom perturbatively: $\ddot{\phi} + m^2\phi \approx 0 \Rightarrow \phi(t) = \phi_0 \cos(mt) + \mathcal{O}(g, \lambda)$

Insert this into nested integrals:

$$\frac{g^k}{M^k} \int_{t_0}^{t} dt_1\, \phi^2(t_1) \int_{t_0}^{t_1} dt_2\, \phi^2(t_2) \cdots \int_{t_0}^{t_{m-1}} dt_k\, \phi^2(t_k) = \frac{1}{2^k k!} \frac{g^k}{M^k} \phi_0^{2k} (t - t_0)^k + \cdots$$

Looks like secular growth! Much faster than kernel suppression.

But...
$$\exp\left[-\frac{g\phi_0^2}{2M}(t - t_0)\right] = \sum_{k=0}^{\infty} \frac{(-1)^k}{2^k k!} \frac{g^k}{M^k} \phi_0^{2k} (t - t_0)^k$$

Terms could resum to something mild

Conclusions and Further Thoughts

Some conclusions

- Standard local effective action from in-out does NOT give correct in-in correlation functions of light fields.
- Attribute this to non-unitary evolution of reduced density matrix for light system.
- Interesting terms appear in zero mode equation of motion that could have unexpected effects for inflationary backgrounds.
- DO get standard results in adiabatic limit; finite initial time and non-adiabatic preparation crucial for new effects. Effects transient with scale set by heavy sector.

Further Directions

- What about corrections to higher order correlators?
- How do we see the breakdown of EFT in situations where field amplitudes can grow, such as preheating?
- What are the systematics of the evolution of the reduced density matrix?
- How does this all work in inflationary backgrounds?