Intro to Shape Dynamics

Henrique Gomes

Physics , University of California, Davis

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In collaboration with Tim Koslowski, Sean Gryb and Flavio Mercatti

Notational warning!!

This will be a talk focused on 3+1 formulations of gravity.

3D vs 4D

Here g is a 3D Riemannian metric (the spatial metric) that evolves in time. If we ever need to use 4D Lorentzian metric, we'll use h.

- Over a closed (compact without boundary) 3-dimensional manifold Σ!
- We will avoid indices as much as we can, but they are there!
- When we talk about a conformal transformation, we mean Weyl transformations, as $g \mapsto \alpha g$.

What is Shape Dynamics?

What it is

A Hamiltonian formulation of gravity with the following proeminent features:

- Possesses the same canonical variables as Hamiltonian GR: (g, π).
- Does *not* possess refoliation invariance (boosts).
- Trades that symmetry for foliation preserving conformal transformations (Weyl) $+$ unique, non-local global Hamiltonian.

What it is not

- York's approach to the initial value problem (and its related constant-mean-curvature gauge for GR).
- Barbour et al's CS+V re-derivation of York.

Both very useful and necessary for Shape Dynamics, but neither has conformal symmetry manifest in the dynamics.

Roadmap

Roadmap of this talk

- Brief review of Hamiltonian GR.
- Main ideas in Shape Dynamics.
- Adding matter and large volume expansion
- Conclusion.

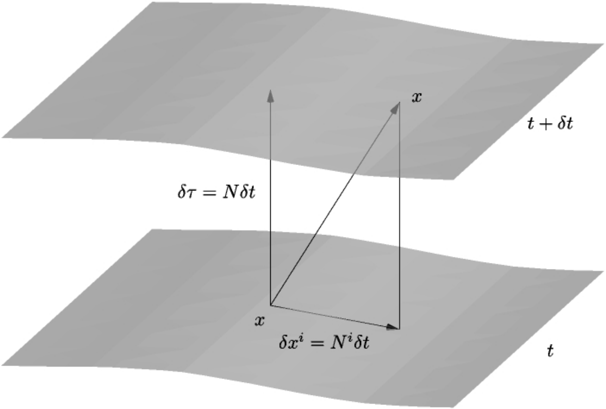
Purpose

Purpose of this talk is to familiarize the audience with the construction and main features of Shape Dynamics. Useful for following talks.

Outline

Hypersurface foliation

- Assume global hyperbolic: $M \simeq \Sigma \times \mathbb{R}$



Intermezzo: Dirac analysis

Constraints are a convenient way to encode over-parametrization of physical degrees of freedom.

Let $\phi_i(q,p) = 0, i \in I$ denote constraints. They define surfaces and flow in phase space, and can have different degrees of mutual "conservation":

- **Compatible.**
  These are called first class constraints. They arise when the dynamical flow generated by one constraint conserves the set: $\delta_{\phi_i}\phi_j = \{\phi_i, \phi_j\} = a^k\phi_k$
- **Impose further constraints.** This occurs when the flow is only conserved on some subsurface:

  $\delta_{\phi_i}\phi_j = f(p,q) \Rightarrow f = 0$ must now be added to the list of constraints.
- **Second class.** These arise when the two constraints are conjugate. $\delta_{\phi_i}\phi_j = 1$. Must either find a coordinate system where they don't appear, or project dynamics to constraint surface.

Canonical framework: ADM

- Use Gauss-Codazzi relations + Einstein equations: ${}^4R \mapsto (R, K)$
- Over-parametrization: relations that extrinsic curvature have to satisfy.
- Legendre: $(g, \dot{g}) \mapsto (g, \pi)$
  - Constraints: ensure relations hold.

    $H_a(x) = \pi_a{}^b{}_{;b} = 0$

    $S(x) = \mathrm{tr}(\pi \cdot \pi) - \frac{1}{2}(\mathrm{tr}\pi)^2 - R = 0$

$S(x)$ and $H_a(x)$ are *first class.* They generate compatible symmetries (on the constraint surface).

Total Hamiltonian: $H_{\mathrm{ADM}} = \int_\Sigma d^3x \, (N(x)S(x) + \xi^a(x)H_a(x))$
Is “pure constraint” . This is the ADM system.

Momentum and scalar constraints

- $\mathcal{H}_a(x)$ generates 3-diffeomorphisms.
  True (infinite-dimensional) Lie algebra.
- $S(x)$ generates time refoliation (and thus evolution).

  In contrast to the action of 3-diffeomorphisms:

  Enormous difficulty in giving meaning to GR's physical degrees of freedom!

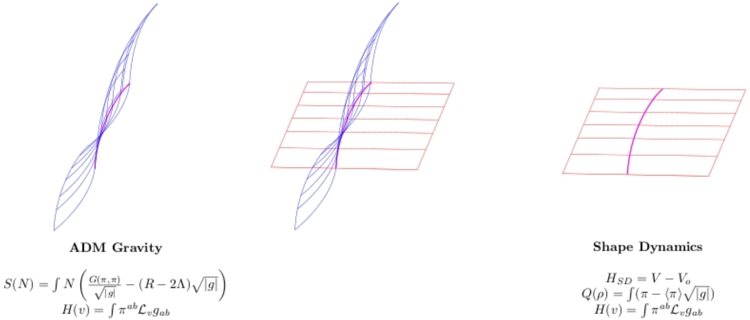
  Not subalgebra (commutation relations involve 3-diffeomorphisms), and entire constraint algebra is "soft" (structure functions).

Introduces many, many difficulties in quantization.

- $S(x)$ quadratic in momenta: $\frac{\delta}{\delta g(x)}\frac{\delta}{\delta g(x)}$: ill-defined.
- Constraints imposed at the quantum level: $\hat{S}\Psi[g] = 0$.
  Klein-Gordon type equation. Inner product?

Outline

Motivation: "pure constraint" systems such as GR may have observables coinciding with that of systems with different symmetries.



Refoliation invariance has famous gauge fixing exploring spatial conformal transformations ([York]).

Preliminary: intrinsic constant mean curvature gauge

What is the constant mean curvature condition?

- The trace of the extrinsic curvature of each leaf is spatially constant.

  Roughly, means that observers use the Hubble constant as a clock.

- Mathematically: set $\mathrm{tr}\pi = \frac{1}{V}\int \mathrm{tr}\pi =: \langle \mathrm{tr}\pi \rangle$.

  Note that $\mathrm{tr}\pi$ generates conformal transformations. I.e.

  - $\{\mathrm{tr}\pi(\epsilon), g\} = \epsilon g$
  - $\{\mathrm{tr}\pi(\epsilon), \pi\} = -\epsilon\pi$
  - $\mathrm{tr}\pi - \frac{1}{V}\int \mathrm{tr}\pi$ generates volume-preserving conformal transformations.

Shape Dynamics: Main message (words)

- On a certain region in phase space, there exists a very special system dynamically equivalent to ADM.
    - Region is that of constant-mean curvature (CMC) foliable Einstein spacetimes (with closed Σ). (see Isenberg's talk for counter-examples)
- System is one that does not possess refoliation symmetry.
    - Instead it possesses local 3D scale invariance. Symmetry trading!
    - All constraints linear in momenta.
    - Individual sets of constraints form subalgebras. Easy to quotient. Physical degrees of freedom clear.
    - Exists in the original ADM phase space (g, π) with the canonical Poisson bracket.
    - Possesses one global Hamiltonian which depends only on (g, π) (no explicit "time" dependence).

Shape Dynamics: Main message

ADM ($\Sigma \times \mathbb{R}$)

Local 1st class constraints:

- 3-diffeomorphisms
- refoliations

Shape Dynamics

Local 1st class constraints

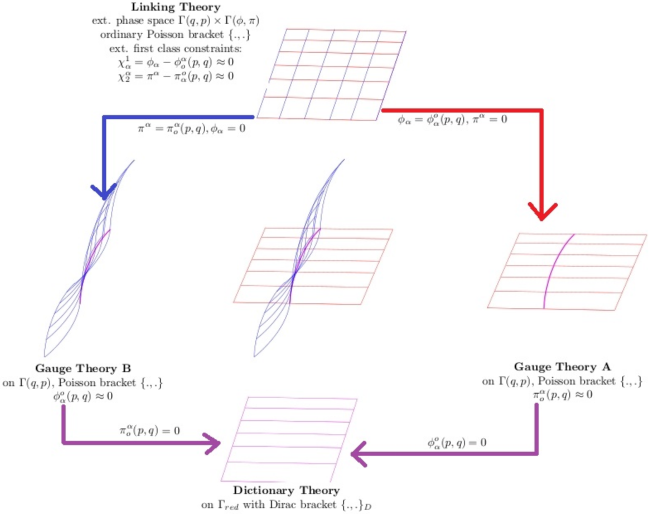
- 3-diffeomorphisms
- Conformal transformations

$$H_{\text{ADM}} = \int d^3x (N(x)S(x) + \xi^a(x)H_a(x))$$

$$H_{\text{dual}} = \mathcal{H}_{\text{gl}} + \int d^3x[\lambda(x)D(x) + \xi^a(x)H_a(x)]$$

- $H_a(x)$: momentum constraint (one per x).
- $S(x)$: Scalar constraint (one per x).
- $D(x) = 4(\pi - \langle\pi\rangle\sqrt{g})(x)$: conformal constraint (one per x).
- $\mathcal{H}_{\text{gl}}$: Global Shape Dynamics Hamiltonian.

How is it constructed? Figure.



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4. Will get that $S(x)$ separates into 1 first class constraint (evolution) and the rest second class.
5. Solve 2nd class constraints *for extra Stuckelberg variables*.
6. Get back to *original phase space* g, π *with canonical Poisson bracket.*

   Leftover constraints first class, generating diffeomorphisms, local 3d conformal transformations, and global evolution.

Take away message from SD.

ADM

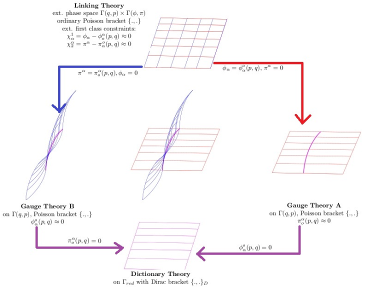
Local symmetries:

- 3-diffeomorphisms
- refoliations

Shape Dynamics

One Hamiltonian + local symmetries:

- 3-diffeomorphisms
- Conformal transformations



Shape Dynamics is to York CMC

as Electromagnetism is to
transverse gauge of vector potential.

Outline

Coupling other fields.

Question: how should fields scale $\psi \to e^{n\hat{\phi}}\psi$?

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Two problems:

1. Foliation depends on the field for scaling $n \neq 0$, not geometric (or worse, for YM depends on the gauge)

   Solution: only metric variables scale ("neutral coupling")

2. Uniqueness of global Hamiltonian: involves invertibility of elliptic 2nd order diff. op. Requires:

   $\frac{1}{2}\left(\frac{\delta H_{\rm m}}{\delta g_{ab}} g_{ab} - \frac{1}{2} H_{\rm m}\right) \leq \frac{1}{12}\langle\pi\rangle^2 + \sigma^2$

- Both issues solved with neutral coupling for Yang-Mills (and gauge invariance respected) and massless scalars.

  But invertibility (point 2) doesn't work always for massive scalars: bound on the field magnitude (e.g. bound on the cosmological constant).

Tractability: Large-volume expansion.

- Global Hamiltonian is non-local. Solve order by order in a large volume expansion. First few terms:

$$\mathcal{H}_{\text{gl}} = 2(\Lambda - \frac{1}{12}\langle\pi\rangle^2) - \frac{R_o}{V^{2/3}} + \frac{1}{V^2}\langle\sigma^2\rangle + \mathcal{O}(V^{-8/3})$$

Here R_o is the unique constant scalar curvature in the conformal class of R (Yamabe gauge).

Global Hamiltonian can be seen as reparametrization constraint: for large volume reparam. invariance implies full conformal invariance.

- Also a Hamilton-Jacobi expansion for the on-shell action:

$$\langle\pi\rangle \to \frac{\delta S}{\delta V}\ , \quad \pi^{ab} \to \frac{\delta S}{\delta g_{ab}}$$

$$S = S_0 V + S_1 V^{1/3} + S_2 V^{-1/3} + \mathcal{O}(V^{-1})$$
$$= \pm\left(\sqrt{\frac{16\Lambda}{3}}\,V - \sqrt{\frac{3}{\Lambda}}\,R_o\,V^{1/3} + \left(\frac{3}{\Lambda}\right)^{3/2}\left(R_o{}^2 - \frac{8}{3}\langle R_o^{ab}R^o_{ab}\rangle\right)V^{-1/3} + \dots\right)$$

Outline

Possible advantages

Classically matches GR over (g, π) that satisfy $\mathrm{tr}\pi = c$ (gauge choices in each) but

- Advantage over CMC gauge-fixed ADM in that variables *and* constraints on the dofs are "*local*".
- Different method to find solutions. Different symmetries. Different gauges.
  - Maybe find different solutions and go back to ADM gauge (and covariantize)?
  - First try: finding a solution for "KSdS" without imposing the ADM constraints.
- ADM cosmological perturbation theory complicated (because we can't separate evolution from constraints). Perturbations must satisfy all constraints at each level.
  - Here, introduce perturbations that only need to satisfy the local constraints, and use unperturbed global Hamiltonian to evolve?

Issues and outlook

The elephant in the room: global Hamiltonian is non-local.

- We saw some ways around it: large-volume expansions. Other expansions?
- Theory is non-local because we include a volume-preserving condition on conformal transfs.
  - This is necessary to have a non-trivial leftover Hamiltonian in Shape Dynamics. I.e. to match ADM trajectories with Shape Dynamics trajectories (to just match Cauchy data for a conformal theory and ADM, no such problem arises).
- If we are interested in the pure quantum theory, so what if we don't match trajectories?
  - BRST: A modification of Shape Dynamics possesses full Weyl and special conformal symmetry (no diffeos) and serves as a complete gauge-fixing fermion for the BRST-extended ADM.
  - The gauge-fixed ADM BRST-extended Hamiltonian possesses a hidden symmetry: "symmetry doubling". (Koslowski's talk)

THANK YOU

APPENDIX

Outline

Some details I: Extended phase space

Trivially embedd $(g, \pi) \mapsto (g, \pi, \phi, \pi_\phi)$.

$\phi = 0 \Rightarrow$ extra constraint: $\pi_\phi = 0$.

Canonical transf.: $F := \int d^3x \left(g_{ab}(x) e^{4\hat{\phi}(x)} \Pi^{ab}(x) + \phi \Pi_\phi \right)$.

- Variables transform as:

$$t_\phi g = e^{4\hat{\phi}(x)} g$$

$$t_\phi \pi = e^{-4\hat{\phi}} \left(\pi - \frac{1}{3} \langle \mathrm{tr} \pi \rangle_g \left(1 - e^{6\hat{\phi}} \right) g^{-1} \sqrt{|g|} \right).$$

$$t_\phi \pi_\phi = \pi_\phi - 4 \left(\mathrm{tr}\pi(x) - \sqrt{g}(x) \langle \mathrm{tr}\pi \rangle_g \right) = 0$$

- New set of constraints:

$$t_\phi H^a \ , \ t_\phi S \ , \ t_\phi \pi_\phi$$

Some details II: Gauge fixings in extended theory

GR(3+1):

- Set $\phi = 0$ again.

Shape Dynamics:

- Gauge-fixing $\pi_\phi = 0$ surface in Γ_{extended}.

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- $\{t_\phi H^a, \pi_\phi\} = 0$
- $\{t_\phi S, \pi_\phi\} \neq 0$
- We can separate $t_\phi S$ into:
  - $t_\phi(S(N_0))$ for $N_0(x) \in C^\infty(M)$ (one 1st class)
  - $K(x) := t_\phi S(x) - t_\phi(S(N_0))\sqrt{g}$ ("$\infty - 1$" 2nd classes)

Some details III: Second class constraint

What is meant by "purely second class" (maximally symplectic) ?

- The bracket $\{K, \pi_\phi\}$ has to be invertible.
- What to do if it is invertible? Say its 1. Have to somehow project down to surface again. One way is to find intrinsic coordinates.

Turns out (after quite a bit of work) that invertiblity of $\{K, \pi_\phi\}$ relies on the operator

$$\Delta := \nabla^2 - \frac{1}{12}\langle \mathrm{tr}\pi \rangle^2 - \bar{\sigma}^{ab}\bar{\sigma}_{ab}$$

being invertible (for vacuum). It is.

Reduction to Shape Dynamics

Furthermore, it can be now shown that $K = 0$ can be solved as a function of ϕ. Setting $\phi = \phi_o[g, \pi]$:

$$K(\phi_o, g, \pi) = 0$$

Locally, just implicit function theorem:

$\{K, \pi_\phi\} = \frac{\partial K}{\partial \phi}$ invertible $\Rightarrow$ there is a unique function ϕ_o above
Easy to check that Dirac bracket reverts to the canonical Poisson bracket in the original phase space.

Got rid of extra variables whilst *solving 2nd class constraints*; reduced dynamical system:

Shape Dynamics constraints

$$t_{\phi_o}(S(N_0)), H_a(x), (\mathrm{tr}\pi - \langle \mathrm{tr}\pi \rangle \sqrt{g})(x)$$

Everything dependent only on (g, π), no leftover dependence on the unphysical variables.

Shape Dynamics recap

ADM ($\Sigma \times \mathbb{R}$)

Local 1st class constraints:

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Shape Dynamics

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- $\mathcal{H}_{\text{gl}}$: Global Shape Dynamics Hamiltonian.

Construction of Doubly General Relativity

Extending Shape Dynamics

- fixed CMC condition $Q(x) = \pi(x) + \lambda\sqrt{|g|}$
- conformal spatial harmonic gauge
  $F^k(x) = (g^{ab}\delta^k_c + \frac{1}{3}g^{ak}\delta^b_c)e^c_\alpha(\nabla_a - \hat{\nabla}_a)e^\alpha_b$
- First class system: $\{Q(x), Q(y)\} = 0 = \{F^i(x), F^j(y)\}$
  as well as $\{Q(x), F^i(y)\} = F^i(y)\delta(x,y)$

Interpretation as "local conformal system"

Q generates spatial dilatations and Poisson brackets resemble $C(3)$ at each point

Gauge fixing ADM

- gauge fixing operator is elliptic and invertible in a region R
- out side R: meager set with finite dimensional kernel
  $\Rightarrow$ expect poles in ghost propagator

The Papers

Papers:

"Einstein gravity as a 3D conformally invariant theory" Class. Quant. Grav., 2011, 28; by HG, Gryb, S. ; and Koslowski, T.

"The Link between General Relativity and Shape Dynamics", gr-qc/1101.5974, to appear in CQG; by HG, and Koslowski, T.

"Coupling Shape Dynamics to matter and Spacetime", gr-qc/1110.3837, to appear in GRG, HG, and Koslowski, T.

"Coupling of Shape Dynamics to matter", gr-qc/1112.0374, to appear in J. Phys., HG

"Non-uniqueness of the Shape Dynamics Hamiltonian", gr-qc/1201.3969, submitted to Comm. Math. Phys. , HG

+ 2 PhD thesis, work in 2+1, dS/CFT, ...