THE HOLOGRAPHIC S-MATRIX

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1107.1499 - Fitzpatrick, JK, Penedones, Raju, Van Rees
1111.6972 - Fitzpatrick, JK (Analyticity)
1112.4845 - Fitzpatrick, JK (Unitarity)
We seem to be on the cusp of a Higgs discovery, awaiting information on its production/decay, as well as the results of other searches for new physics!
Important work to be done improving searches, understanding implications, focusing in on leftover (new?) model space.

But amidst the LHC, also other exciting topics for Field Theorists...
Motivations: Improved Understanding of CFTs

- Want to write CFT correlation functions in a form that clarifies the physics.
- Hope to greatly simplify calculation of correlators, OPE coefficients, etc.
- Why do CFTs have local AdS duals?
Key Point: argue that `Mellin space' is the analog of momentum space for Conformal Field Theories.
Motivations: Holographic Theory of Flat Spacetime

- Scattering Amplitudes are very simple.
- In gravity, local observables not gauge invariant, need boundary observables.
- Want a holographic theory of the flat space S-Matrix where bulk locality and unitarity emerge simply and robustly.
**Motivations: Holographic Theory of Flat Spacetime**

Key Point: \( M(\delta_{ij}) \xrightarrow{R \to \infty} S(s, t, u) \)

Mellin Amplitude becomes the flat space S-Matrix!
Motivations: Hawking Evaporation

• Expect generic BHs decay via Hawking Radiation
• BH formation/decay = a Scattering process
• Thermodynamics from statistics of S-Matrix
• Compute S-Matrix from flat limit of AdS/CFT!

(an ATLAS picture of BH production and decay.)
**Simple, Sharp Questions about Black Holes?**

Expect Transplanckian S-Matrix has

\[ \langle n_{out} \rangle \approx \frac{E}{T_{BH}} \approx \left( \frac{E}{M_{pl}} \right)^{\frac{D-2}{D-3}} \]

Only gravity has scattering amplitudes like this. Reproducing it with AdS/CFT is a sharp question that should have a **generic** understanding.

Planck scale should emerge as a dimension in the CFT.

**A first step: we will derive a new bound on CFT correlators due to BH intermediate states.**
Outline

I. Mellin Space as Momentum Space for CFTs, or CFT correlators as scattering amplitudes
II. Mellin Amplitude as Holographic S-Matrix
III. Analyticity (locality!?) from Mellin-space Meromorphy, some loop level examples
IV. S-Matrix Unitarity from CFT Unitarity
V. S-Matrix program as the Bootstrap program and a peak at Black Holes
AdS/CFT Preliminary

With AdS in Global Coordinates

\[ ds^2 = \frac{1}{\cos^2 \rho} \left( dt^2 + d\rho^2 + \sin^2 \rho d\Omega^2 \right) \]

the Dilatation Operator generates time translations.
Let's Try to think of CFT Correlators as Scattering Amplitudes.
What is the CFT Analog of “Free Particles”? 

Scattering amplitudes involve states composed of particles that are asymptotically free.

The CFT analog is the large N expansion, because given operators $\mathcal{O}_1$ and $\mathcal{O}_2$, there must exist

“$\mathcal{O}_1 \mathcal{O}_2$”

with dimension $\approx \Delta_1 + \Delta_2$
How should we compute Correlators?

Previous computations in AdS used position space. Analogous to computing Feynman diagrams as...

$$\int d^d x D_F(x_1 - x) D_F(x_2 - x) D_F(x_3 - x) D_F(x_4 - x)$$

Even the 4-pt function is a box integral!!

In AdS, computations have been even worse, with very few results beyond 4-pt.

(We will see how to compute at n-pt, easily.)
In flat space we go to momentum space, which has several familiar advantages.

Eq. of Motion become algebraic

\[ \nabla^2 = -p^2 \]

because the Laplacian acts very simply on the momentum space representation.

We find a similar simplification in Mellin space, because the Conformal Casimir acts nicely.
Factorization and Momentum Space

Also, flat space scattering amplitudes Factorize

\[ M(p_i) \rightarrow M_L(p_{iL}, P_L) \frac{1}{P_L^2} M_R(-p_L, p_{iR}) \]

Involves analyticity and unitarity, since factorization poles follow from the exchange of single-particle states.

Also, there are purely algebraic Feynman Rules.

So position space obscures a lot of physics!
So what is the Mellin Amplitude?

A CFT Correlator written in Mellin Space (Mack):

\[ A_n(x_i) = \int [d\delta] M_n(\delta_{ij}) \prod_{i<j} (x_i - x_j)^{-2\delta_{ij}} \Gamma(\delta_{ij}) \]

\[ \sum_{j \neq i} \delta_{ij} = \Delta_i \]

Roughly speaking, the \( \delta_{ij} \) variables are a space of relative scaling dimensions between operators.

The Mellin Amplitude for scalar operators is **Conformally Invariant**.
Mellin Space ~ Space of Mandelstam Invariants

$\delta_{ij}$ are symmetric, and with $\delta_{ii} = 0$

You can always pretend that $\delta_{ij} = "p_i \cdot p_j"$ with

$$\sum_{i=1}^{n} p_i = 0 \quad \text{and} \quad p_i^2 = \Delta_i$$

(fictitious!) momentum conservation and on-shell-ness

We will often see combinations in propagators such as

$$\sum_{i,j=1}^{K} \delta_{ij} = (p_1 + \ldots + p_K)^2$$
A Simple Example

\(\mu \phi^3\) theory in AdS at tree level in 3-D

\[M(\delta_{ij}) = \frac{R^3 \mu^2}{2(4\pi)^3} \left( \frac{1}{\delta_{12} - 1} + \frac{1}{\delta_{13} - 1} + \frac{1}{\delta_{14} - 1} \right)\]

The pole prescription for the contour is

\[\frac{1}{1 - \delta_{12} - \delta_{13}}\]

\[\frac{1}{\delta_{12} - 1}\]

\[\frac{1}{1 - \delta_{12} - \delta_{13}}\]

\[\frac{1}{\delta_{13} - 1}\]
How does the Mellin Amplitude mimic Scattering Amplitudes?
In Mellin Space: The Functional Equation

Find a finite difference equation for Mellin amp:

\[(\delta_{12} - a_1)(\delta_{12} - a_2)M(\delta_{12}) = (\delta_{12} - a_3)(\delta_{12} - a_4)M(\delta_{12} - 1) - M_0\]
The CFT Analog of Factorization

Factorization occurs in CFTs, but is obscure in position space. Insert 1, organize with symmetry:

\[ \langle \mathcal{O}_1 \mathcal{O}_2 \left( \sum_{\alpha} |\alpha\rangle\langle\alpha| \right) \mathcal{O}_3 \mathcal{O}_4 \rangle \]

By operator-state correspondence, this decomposition is just a sum over exchanges of operators:

\[ = \sum_{\alpha} \mathcal{O}_\alpha \]

Mellin Amp displays this conformal block decomp as a sum over factorization channels. Why?
Why Do Mellin Amplitudes Factorize?

Look at the exchange of operators more carefully:

\[ A_n(x_i) \sim \sum_p \int d^d y \left\langle \prod_{i=1}^k \mathcal{O}_i(x_i) \mathcal{O}_p(y) \right\rangle \left\langle \tilde{\mathcal{O}}_p(y) \prod_{i=1+k}^n \mathcal{O}_i(x_i) \right\rangle \]

Each \( \mathcal{O}_p \) in the sum has a definite dimension, so each term scales as a definite power law.

Mellin space = the space of these powers.

Thus in Mellin space each term gives a pole, with a residue that is the product of lower correlators.
A Factorization Formula for ADS/CFT

Obtain an explicit AdS/CFT factorization formula:

\[ M = \sum_{m=0}^{\infty} \frac{\text{Res}(m)}{\delta_{LR} - \Delta - 2m} \]

\[ \text{Res}(m) \propto \left[ L_m(\delta_{ij}) R_m(\delta_{ij}) \right] \delta_{LR} = \Delta + 2m \]

where

\[ \delta_{LR} = \sum_{i,j \leq k} \delta_{ij} = "(p_1 + \ldots + p_k)^2" \]
Mellin Amplitudes Are Meromorphic

In general, expect Mellin amplitudes must always be meromorphic functions to get an OPE.

In fact, expect only simple poles, and that all poles will lie on the real axis for a unitarity CFT.

Provides a hint of analyticity for later...
We have a factorization formula, and we can factorize on any propagator, and reason to believe that Mellin amplitudes are basically just rational functions, so it would be surprising if there wasn’t a constructive method for generating Mellin Amps.
Diagrammatic Rules

\[ V_{abc}(m_a, m_b, m_c) \]

Conserve fictitious "momentum" at all vertices.

\[ S_a(m_a) \]

\[ \frac{\delta_{a-m_a}}{\delta_{a-m_a}} \]

n-pt scalar vertices are Lauricella functions; proven with our finite difference equation (nice form for vertices found by Paulos, 1107.1504).
So We Can Compute!

AdS/CFT Witten
Diagrams such as this
can be computed
straightforwardly.

Previously, very few computations beyond 4-pt!!
Relation to Flat Space S-Matrix?
the Flat Space Limit

AdS

$\pi R$

CFT

$2\tau$

$\psi_{\text{out}}$

$\psi_{\text{in}}$

Monday, January 30, 2012
**The Flat Space Limit**

- Recall Bulk Energy = CFT Dimension
- Flat Space Limit requires
  \[ E_{\text{bulk}} R_{\text{AdS}} \rightarrow \infty \]
- This means that we must study CFT states of very large dimension, while
  \[ N^2 \propto (M_{d+1} R_{\text{AdS}})^{d-1} \rightarrow \infty \]
The Flat Space Limit

But we know that $\delta_{ij} \sim$ dimension.

Natural to guess (and Penedones did) that

$$\lim_{R \to \infty} M(\delta_{ij} = R^2 s_{ij}) \sim T(s_{ij})$$

And it works! Checked explicitly for theories of scalars at tree level for any number of particles, and some 1-loop examples. More precisely...
The Flat Space Limit

The exact relation for massless external states:

\[ T(s_{ij}) = \Gamma(\Delta_{\Sigma} - h) \lim_{R \to \infty} \int_{-i\infty}^{i\infty} d\alpha e^{\alpha} \alpha^{h-\Delta_{\Sigma}} M \left( \delta_{ij} = \frac{R^2 s_{ij}}{2\alpha}, \Delta_a = Rm_a \right) \]

A one-dimensional contour integral applied to the (meromorphomorphic) Mellin Amplitude.

Note that as one might expect, single trace \(-->\) single particle.

How is it derived?
Create in and out states by CFT operator smearing:

\[ |\omega, \hat{v}\rangle = \int_{-\frac{\pi R}{2} - \tau}^{\frac{\pi R}{2} + \tau} dt e^{i\omega t} \mathcal{O}(t, -\hat{v}) |0\rangle \]

Single-trace Operator = Single Particle
Integrating CFT Correlator against plane waves:

\[ T(s_{ij}) = \lim_{R, \tau \to \infty} \int [d\delta] \int^{\tau+\frac{\pi R}{2}}_{-\tau+\frac{\pi R}{2}} dt_i e^{i(\omega_i - \Delta_i) t_i} M(\delta_{ij}) \prod_{i<j} \left( \cos \left( \frac{t_i - t_j}{R} \right) - \hat{p}_i \cdot \hat{p}_j \right)^{-\delta_{ij}} \Gamma(\delta_{ij}) \]

Time differences small: \(|t_i - t_j| \ll R\)
leading to approximately Gaussian time integrals.

\[ \delta_{ij} \text{ integrals can be evaluated via stationary phase in the flat space limit of Gamma functions:} \]

\[ \int [d\epsilon] M(\delta_{ij}) \exp \left[ \sum_{ij} R^2 s_{ij} \left( \frac{1}{\alpha + \epsilon_{ij}} \right) \log \left[ R^2 \left( \frac{1}{\alpha + \epsilon_{ij}} \right) \right] \right] \]
The S-Matrix from the Mellin Amplitude

\( \delta_{ij} \) variables align with \( s_{ij} \), leaving us with:

\[
T(s_{ij}) = \Gamma(\Delta_{\Sigma} - h) \lim_{R \to \infty} \int_{-i\infty}^{i\infty} d\alpha e^{\alpha h - \Delta_{\Sigma}} M \left( \delta_{ij} = \frac{R^2 s_{ij}}{2\alpha}, \Delta_a = R m_a \right)
\]

Our factorization formula and Feynman rules for the Mellin amplitude reduce to the factorization formula and Feynman rules of the tree-level scattering amplitudes.
Analyticity
and the
Holographic
S-Matrix

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**Locality = Analyticity?**

Only precise notion of locality (I’m aware of) is via analyticity and boundedness of S-Matrix.

The Scattering Amplitudes are given by a simple integral transform of the Mellin Amp.

The Mellin Amplitude is a meromorphic function with only simple poles, in any CFT.

Is this how we should think of locality emerging from a CFT!?
Analyticity in the Flat Space Limit

\[ T(s_{ij}) = \Gamma(\Delta_\Sigma - h) \lim_{R \to \infty} \int_{-i\infty}^{i\infty} d\alpha e^{\alpha h - \Delta_\Sigma} M \left( \delta_{ij} = \frac{R^2 s_{ij}}{2\alpha}, \Delta_a = Rm_a \right) \]

For finite R, just contour integral of meromorphic function, so obviously analytic.

Flat Space Limit just expands near infinity. We get branch cuts and imaginary parts from the coalescence of poles.
Flat Space Limit of a Bulk Exchange

Let’s see how branch cuts etc obtain, leaving a general analysis of locality for the future.

Taking flat space limit, a bulk propagator becomes:

$$\sum_m \frac{R(\Delta, m)}{\delta - (\Delta + m)} \to \frac{1}{s + \Delta^2}$$

(The Mellin amplitude is dominated by poles where $m \approx \Delta^2$, when we take the flat space limit.)

Loops?
We can also compute AdS loop diagrams

Using an AdS version of Kallen-Lehman, which makes it possible to write 2-point functions of local operators as a positive integral over free propagators.
At 1-loop, can write bubble diagram using:

\[
\begin{align*}
\Delta &\quad = \quad \sum_{n=0}^{\infty} N_{\Delta}(n) \quad 2\Delta + 2n \\
\end{align*}
\]

or

\[
G_{\Delta}(X, Y)^2 = \sum_{n=0}^{\infty} N_{\Delta}(n) G_{2\Delta+2n}(X, Y)
\]

We use an inner product obeyed by the propagators to compute this decomposition.
This gives a Kallen-Lehman-esq Mellin Amplitude:

\[ M(\delta) = \sum_n N(n) \sum_m \frac{R(2\Delta + 2n, m)}{\delta - (2\Delta + 2n + m)} \]

Represents the exchange of double-trace primary states of dimension \( 2\Delta + 2n \).
**Branch Cuts**

In the flat space limit, we find the integral:

$$
M(\delta) \rightarrow \int_0^\infty dn \frac{N(n)}{s + (2\Delta + 2n)^2}
$$

Circling in the complex plane gives a branch cut.
Branch Cuts from Mellin Amplitudes

\[ M(\delta) \rightarrow \int_0^\infty dn \frac{N(n)}{s + (2\Delta + 2n)^2} \]  
with \( N(n) \propto n^{d-2} \)

for \( \lambda \phi^4 \) theory. Gives branch cut! Discontinuity:

\[ \frac{N(\sqrt{s})}{\sqrt{s}} \propto \sqrt{s}^{d-3} \]

Correct for theory in \( d+1 \) dimensions.
Unitarity of the Holographic S-Matrix
The standard optical theorem with $S = 1 + iT$

$$-i(T - T^\dagger) = T^\dagger T$$

looks reminiscent of the Conformal Block decomp:

$$\quad = \sum_\alpha \quad \alpha \quad \frac{\alpha}{\alpha} \quad \frac{\alpha}{\alpha}$$

[From using conformal symmetry to organize]

$$\langle \mathcal{O}_1 \mathcal{O}_2 \left( \sum_\alpha |\alpha\rangle \langle \alpha| \right) \mathcal{O}_3 \mathcal{O}_4 \rangle$$

since operators = states in the CFT.]
Conformal Blocks and the OPE

We can apply the Operator Product Expansion

\[ \mathcal{O}_1(x_1)\mathcal{O}_2(x_2) = \sum_{\Delta,\ell} c^{12}_{\Delta,\ell} \mathcal{O}_{\Delta,\ell}(x) \]

to a 4-pt correlation function to find

\[ \sum_{n,\ell} \left( \begin{array}{c} \mathcal{O}_1 \\ \mathcal{O}_2 \end{array} \right) \left( [\mathcal{O}_a\mathcal{O}_b]_{n,\ell} \right) \left( \begin{array}{c} \mathcal{O}_1 \\ \mathcal{O}_2 \end{array} \right) B_{n,\ell} \]

This is a formula for the conformal block coefficients.
To compute need to **conglomerate** single trace operators into one multi-trace:

\[
\begin{align*}
O_1 & \quad O_a \\
O_2 & \quad O_b
\end{align*}
\]

\[
[O_a O_b]_{n,\ell}
\]

Can differentiate, but extremely cumbersome.

Easy in Mellin space, convolve with wavefunction.
Something like the Optical Theorem...

![Diagram](image)

\[
\sum_{n, \ell} \left( \begin{array}{c}
\mathcal{O}_1 \\
\mathcal{O}_2
\end{array} \right) \left( \begin{array}{c}
[\mathcal{O}_a \mathcal{O}_b]_{n, \ell} \\
\mathcal{O}_2
\end{array} \right) \left( \begin{array}{c}
\mathcal{O}_1 \\
\mathcal{O}_2
\end{array} \right) B_{n, \ell}
\]

We can get info about next order in perturbation theory!
Let's take the flat space limit of these CFT Unitarity Operations
What is the flat space limit of a conformal block?

\[ B_{\Delta_\alpha} \rightarrow \delta \left( s - \Delta_\alpha^2 \right) \]

“Obvious”, since blocks have definite angular momentum and definite dimension = energy.

\[ M_4(\delta_{ij}) = \sum_{\alpha} N_B(\Delta_\alpha) B_{\Delta_\alpha}(\delta_{ij}) \]

becomes (when we take the flat space limit)

\[ M(s, t) = N_B(s, t) \]
A 1-Loop Example

Figure 3: This figure provides a schematic depiction of how a 1-loop Witten diagram in AdS decomposes via the conformal block decomposition in the dual CFT. For illustrative purposes, the bulk theory has both a \( \phi \) and a \( g \) interaction. The dashed lines indicate 'cuts'; the central cut highlighted in purple provides the familiar imaginary contribution to the optical theorem. The conformal block decomposition also includes the 'edge cuts' on the left and right, which have no analog in discussions of the cutting rules. These edge cuts are very important in order to obtain the full correlator but in the flat space limit they only contribute to the real part of the Sw_matrix and so they drop out of the optical theorem.

Now that we have identified multi-trace operators as the analogue of multi-particle states, we can construct an interesting 'Sw_matrix' by taking inner products between these states at different times. Since these multi-trace operators are not eigenstates of the full Dilatation operator, this matrix will be non-trivial. The time \( \tau \) is really the logarithm of the scale factor in the CFT. The difference between the in and out states can be arbitrary, but choosing a time difference of order one could be particularly interesting because the different descendants of a given operator differ in dimension by 1. It turns out that a time difference of exactly \( \lambda \) leads us to the Sw_matrix of the dual bulk theory in the flat space limit.

\[ \begin{align*}
\left( \mathcal{O}_\phi \right)_{1} & \left( \mathcal{O}_\psi \right)_{\lambda g} \\
\left( \mathcal{O}_\phi \right)_{1} & \left( \mathcal{O}_\psi \right)_{\lambda g} \\
\left( \mathcal{O}_\phi \right)_{1} & \left( \mathcal{O}_\psi \right)_{\lambda g} \\
\left( \mathcal{O}_\phi \right)_{1} & \left( \mathcal{O}_\psi \right)_{\lambda g} \\
\end{align*} \]
Consequences of the Holographic S-Matrix?
CFT Bootstrap VERSUS S-Matrix Program

S-Matrix program used Unitarity and Analyticity, the latter being a formalization of locality.

The Bootstrap program for CFTs instead uses Unitarity and Crossing Symmetry, along with assumptions or data about the spectrum.

The Bootstrap Program naturally allows us to relax the assumption of bulk locality!
Black Holes as Intermediate States?

\[ S(s, t) = N_B(s, t) \]

But on very general grounds, expect that

\[ S(s) \sim \exp \left[ -\frac{1}{2} S_{BH}(s) \right] = \exp \left[ -\frac{1}{8} \left( G_D s^{\frac{D-2}{2}} \right)^{\frac{1}{D-3}} \right] \]

This gives a concrete prediction for the OPE and the conformal block decomposition of any CFT with a gravity dual where effective field theory applies!
Some Future Directions

- Mellin diagrammatic rules for loops, higher spin particles, twistors/spinor-helicity, SUSY, compactifications, dS/CFT, beloved theories...
- bolster recent progress on CFT Bootstrap?
- broken conformal invariance (eg QCD), flows between CFTs??
- sharpen criterion for analyticity = bulk locality?
- do all Gravitational S-Matrices come from CFTs??
- Find a CFT description of Hawking Evaporation, or at least see its simple and robust features!?
Conclusion

• Mellin Space = "Momentum Space for CFTs", conceptually and computationally
• Mellin Amplitude -> Holographic S-Matrix
• Analyticity follows from Meromorphy
• the OPE implies Unitarity, Cutting Rules
• Expect scattering through BHs is a robust ingredient in CFT dynamics, so we should attempt to understand it!
The End
Resonances

To see how this loop diagram gives Breit-Wigner

\[
\mu \phi^2 \chi
\]

we need to perform the resummation:

\[
\cdots + \text{bubble diagram} + \text{bubble diagram} + \cdots = \frac{(\cdot \cdot)}{(1 - \text{bubble})}
\]
Resonances

With a discrete spectrum, can view as mixing

\[
m_{\text{eff}}^2 = \begin{pmatrix}
\Delta_{\chi}^2 & R^2 \lambda_{\text{eff}}(0) & R^2 \lambda_{\text{eff}}(1) & R^2 \lambda_{\text{eff}}(2) & \cdots \\
R^2 \lambda_{\text{eff}}(0) & (2\Delta_\phi)^2 & 0 & 0 & \cdots \\
R^2 \lambda_{\text{eff}}(1) & 0 & (2\Delta_\phi + 2)^2 & 0 & \cdots \\
R^2 \lambda_{\text{eff}}(2) & 0 & 0 & (2\Delta_\phi + 4)^2 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

with

\[
\lambda_{\text{eff}}(n) \equiv \lambda \sqrt{\frac{N_{2\Delta_\phi}(n)}{R^{2h-1}}}
\]

This is a mixing between \( \chi \) particle and the various \( 2\phi \) states.
**Resonances**

By diagonalizing, one can compute the Mellin amp:

\[ M(\delta_{ij}) = \sum_a S_{1a} D_a(\delta_{LR}) S_{a1}^T, \]

\[ D_a(\delta_{LR}) = \sum_m \frac{R_m(\Delta_a)}{\delta_{LR} - (\Delta_a + 2m)} \]

We find that roughly \( \frac{\lambda_{eff} R}{m_\chi} \) 2-particle states contribute an eigenvalue proportional to \( \lambda_{eff} \), giving

\[
\frac{1}{s - m_\chi^2 + i\lambda^2 m_\chi^{D-4}}
\]

near the pole at weak coupling, as expected.
S-Matrix Unitarity from CFT Unitarity

Conformal Block Decomposition

\[ A(x_i) = \sum_{\Delta} c_{\Delta}^2 B_{\Delta}(x_i) \]

Cuts through diagram vs. cuts at edge:

Internal operators

\[ O^\prime \]

Double-trace operators

\[ O_1 O_2 \]
**S-Matrix Unitarity from CFT Unitarity**

\[ A(x_i) = \sum_{\Delta} c_{\Delta}^2 B_{\Delta}(x_i) \]

Flat-space limit of a conformal block is a delta function

\[ B_{\Delta} \rightarrow N_{\Delta} \delta(s - \Delta^2) \]

OPE coefficients are just factorized amplitudes times phase space!

\[ c_{\Delta} \sim M_{12 \rightarrow \Delta} \]

“Internal cuts” are just RHS of usual optical theorem!

\[ 2\text{Im}(M) \sim \sum_{\Delta} 2\text{Im}(N_{\Delta}) |c_{\Delta}|^2 \sim \int d\text{LIPS} |M_{12 \rightarrow \Delta}|^2 \]
What about Double-Trace Cuts?

Cuts through edge of diagram are “double-traces”, which contribute a total derivative

\[ \mathcal{A}_{d.t.}(x_i) = \sum_n \frac{\partial}{\partial n} (c_n^2 \gamma(n) B_n(x_i)) \]

Imaginary part is smooth, so in flat-space this becomes the integral of a total derivative!

\[ 2Im(\mathcal{M}_{d.t.}) \approx \int dn \frac{\partial}{\partial n} (\ldots) = 0 \]
A 1-Loop Example

One can check directly that the sum over all multi-trace CFT operators at a given dimension reproduces a phase space integral in the flat space limit.
Let’s Check It at 1-Loop

We need to compute both sides from the CFT.

\[ \text{Im} \left[ \begin{array}{cc} \text{In} & \text{Out} \\ \text{In} & \text{Out} \end{array} \right] = \sum_{\text{states}} \left| \begin{array}{c} \text{In} \\ \text{Out} \end{array} \right|^2 \]

The goal is to see that both are determined by a specific conformal block coefficient in \( \lambda \phi^4 \).

First let’s compute the left side, using the 1-loop result we discussed.
Recall that at 1-loop, branch cuts came from:

\[ M(\delta) \rightarrow \int_{0}^{\infty} dn \frac{N_{W}(n)}{s + (2\Delta + 2n)^2} \quad \Rightarrow \quad \text{disc} = \frac{N_{W}(\sqrt{s})}{\sqrt{s}} \]

where we had defined (a la Kellan-Lehman)

\[ G_{\Delta}(X, Y)^2 = \sum_{n=0}^{\infty} N_{W}(n) G_{2\Delta + 2n}(X, Y) \]

But the contribution of bulk exchange implies the exchange of a primary operator in the conformal block decomposition.
Conformal Blocks and the Imaginary Piece

In other words, we see that the conformal block decomposition determines the left side of

\[ \text{Im} \left[ \begin{array}{c}
\text{states}
\end{array} \right] = \sum_{\text{states}} | \text{out} \rangle^2 \]

Now we will compute the right side.
To compute need to conglomerate single trace operators into one multi-trace:

\[ O_1 \quad O_a \quad O \quad O_1 \]
\[ O_2 \quad O_b \quad O_2 \]

Easy in Mellin space, convolve with wavefunction.

By operator-state correspondence, this picks a state in the CFT (the state appearing in cutting rules!).
Unitarity Relation Determined by Blocks

\[ M_4(\delta_{ij}) = \sum_{\alpha} \left( \begin{array}{c} \mathcal{O}_1 \\ \mathcal{O}_2 \end{array} \right) \left( \begin{array}{c} [\mathcal{O}_a \mathcal{O}_b]_{n,\ell} \\ [\mathcal{O}_a \mathcal{O}_b]_{n,\ell} \end{array} \right) \left( \begin{array}{c} \mathcal{O}_1 \\ \mathcal{O}_2 \end{array} \right) B_{\Delta \alpha}(\delta_{ij}) \]

Coeff of Block at a given Dimension/Energy is the square of 2 --> X amp, summed over states!

\[ \text{Im} \left[ \begin{array}{c} \mathcal{O}_1 \\ \mathcal{O}_2 \end{array} \right] = \sum_{\text{states}} \left| \begin{array}{c} \mathcal{O}_1 \\ \mathcal{O}_2 \end{array} \right|_\text{out}^2 \]

Sum on CFT states = phase space integral in Flat Limit.

Both sides compute the same Conformal Block Coeff!
Unitarity Relation Determined by Blocks

\[ M_4(\delta_{ij}) = \sum_{\alpha} \left( \begin{array}{c} O_1 \\ O_2 \end{array} \right) \left[ O_{a_1} O_{b_2} \right]_{n,\ell} \left( \begin{array}{c} O_1 \\ O_2 \end{array} \right) \left( B_\Delta (\delta_{ij}) \right) \]

As with dispersion relations, one order in perturbation theory gives info about the next.

\[ M_4(\delta_{ij}) = \sum_{\alpha} N_B(\Delta_\alpha) B_\Delta (\delta_{ij}) \]

Gives a distinct way to compute coefficients in the conformal block expansion.
Conformal Blocks
from 3-pt Correlators

\[ M_4(\delta_{ij}) = \sum_{\alpha} N_B(\Delta_\alpha) B_{\Delta_\alpha}(\delta_{ij}) \]

Coefficients of each block come from 3-pt correlators

\[ N_B(\Delta_\alpha) = \frac{C_3(1, 2, \alpha) C_3(\alpha, 3, 4)}{C_2(\alpha, \alpha)} \]

Where the coefficients multiply universal functions

\[ \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_\alpha \rangle = \frac{C_3(1, 2, \alpha)}{x_{12}^{\Delta_\alpha_{12}} x_{2\alpha}^{\Delta_\alpha_{2\alpha}} x_{\alpha 1}^{\Delta_\alpha_{\alpha 1}}} \]
To compute need to conglomerate single trace operators into one multi-trace:

\[
\begin{align*}
\mathcal{O}_1 & \quad \mathcal{O}_a \\
\mathcal{O}_2 & \quad \mathcal{O}_b
\end{align*}
\rightarrow
\begin{align*}
\mathcal{O}_1 & \quad [\mathcal{O}_a \mathcal{O}_b]_{n,\ell} \\
\mathcal{O}_2
\end{align*}
\]

Easy in Mellin space, convolve with wavefunction.

By operator-state correspondence, this picks a state in the CFT (the state appearing in cutting rules!).
Bulk Exchange Leads to Operator Exchange

\[ G_\Delta(X,Y)^2 = \sum_{n=0}^{\infty} N_W(n)G_{2\Delta+2n}(X,Y) \]

implies that we must have terms in the conformal block decomposition:

\[ N_B(2\Delta + 2n) = N_W(n) \]

where the decomposition is defined by

\[ M_4(\delta_{ij}) = \sum_{\alpha} N_B(\Delta_\alpha)B_{\Delta_\alpha}(\delta_{ij}) \]
We showed that our factorization formula for the Mellin amplitude reduces to factorization of the tree-level scattering amplitudes, and that our Feynman rules reduce to the flat space rules.

\[ T(s_{ij}) = \Gamma (\Delta \Sigma - h) \lim_{R \to \infty} \int_{-i\infty}^{i\infty} d\alpha e^{\alpha} \alpha^{h-\Delta \Sigma} M \left( \delta_{ij} = \frac{R^2 s_{ij}}{2\alpha}, \Delta_a = Rm_a \right) \]

The \( i\epsilon \) prescription comes from CFT prescription.

We showed that our factorization formula for the Mellin amplitude reduces to factorization of the tree-level scattering amplitudes, and that our Feynman rules reduce to the flat space rules.
Point-source at the boundary = plane wave in the center of AdS, energy set by frequency:

\[ |\omega, \hat{v}\rangle = \int_{-\frac{\pi R}{2} - \tau}^{-\frac{\pi R}{2} + \tau} dt e^{i\omega t} O(t, -\hat{v}) |0\rangle \]

(an example of a wave packet state)