Bohr-Sommerfeld Quantization of a Grain of Space

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In collaboration with Eugenio Bianchi

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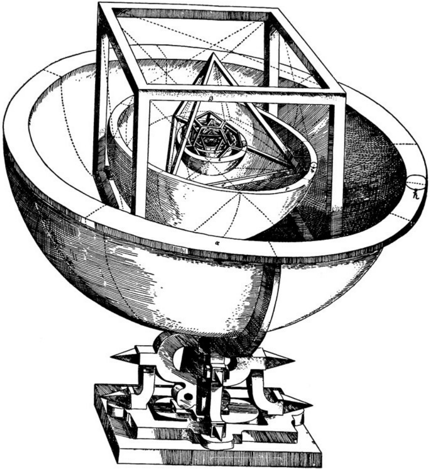
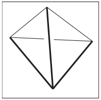
Kepler’s Cosmos



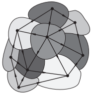
Kepler’s model of the cosmos

Outline

1. Bohr-Sommerfeld Quantization of Geometry



2. Overview of Loop Gravity



3. Volume Operator in Loop Gravity $\hat{V}$

4. Comparisons & Conclusions

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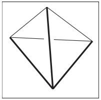

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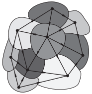

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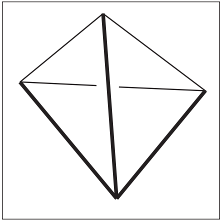
Overview

Strategy: Bohr-Sommerfeld Quantization



A tetrahedral grain of space

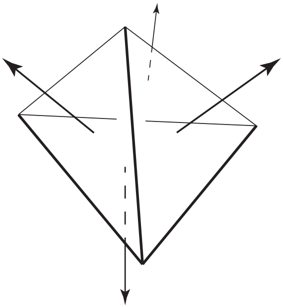
Need: A classical dynamical system,

- kinematics (phase space and Poisson brackets $\{f, g\}$)
- dynamics H.

Kinematics: Minkowski

The area vectors of a tetrahedron determine its shape:

$$\vec{A}_1 + \vec{A}_2 + \vec{A}_3 + \vec{A}_4 = 0.$$



Kinematics: Penrose

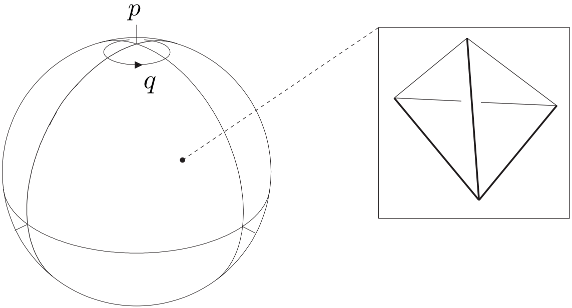
- Physical input: $\vec{A}_1, \ldots, \vec{A}_4$ are angular momenta

Angular momenta have Poisson brackets,

$$\{f, g\} = \sum_{l=1}^{4} \vec{A}_l \cdot \left(\frac{\partial f}{\partial \vec{A}_l} \times \frac{\partial g}{\partial \vec{A}_l} \right).$$

Kinematics II: Kapovich & Millson

$\vec{A}_1, \ldots, \vec{A}_4$ angular momenta



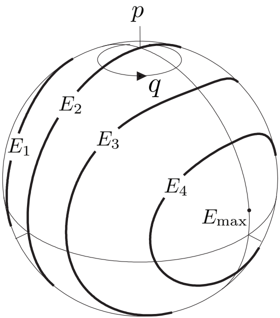
$$p = |\vec{A}_1 + \vec{A}_2| \qquad q = \text{Angle of rotation generated by } p\text{:}$$

$$\{q, p\} = 1$$

Dynamics

Take as Hamiltonian the volume:

$$H = V = \sqrt{|V^2|} = \frac{\sqrt{2}}{3}\sqrt{|\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)|}.$$



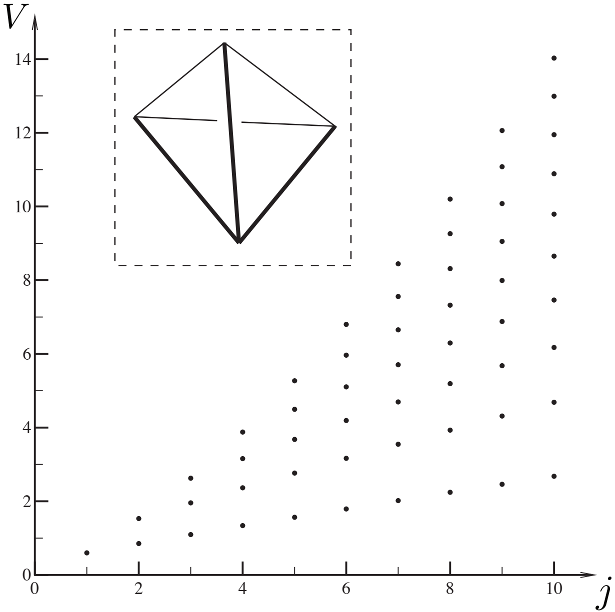
Bohr-Sommerfeld Quantization

Area of orbits given in terms of elliptic functions

$$S(E) = \left(\sum_{i=1}^{4} a_i \, K(m) - \sum_{i=1}^{4} b_i \, \Pi(\alpha_i^2, m) \right) E.$$

Require Bohr-Sommerfeld quantization condition,

$$S = (n + 1/2) 2\pi\hbar.$$



$$A_1 = j + 1/2$$

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$$A_4 = j + 3/2$$

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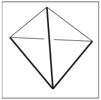

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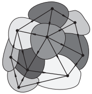

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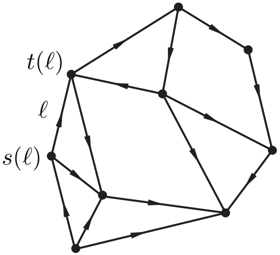
Loop quantum gravity

First task, construct the Hilbert space of LQG:

$$\mathcal{H}\,.$$

Similarities to Fock space of QED and to lattice gauge theory (e.g. QCD). Built on graphs:



Graph Γ

L "links" ℓ
N "nodes" n

source and target:
$s : \ell \mapsto s(\ell)$ and $t : \ell \mapsto t(\ell)$.

Fock Space

Massive scalar field:

- One particle: $\mathcal{H}_1 = L^2(M)$, M the Lorentz hyperboloid.
- n particles,
  $$\mathcal{H}_n = L^2(M^n)/\sim$$
  with $\sim$ permutations. Factorization symmetrizes states.
- All states up to N particles
  $$\mathcal{H}_N = \bigoplus_{n=0}^{N} \mathcal{H}_n.$$

Fock space

$$\mathcal{H}_{\text{Fock}} = \lim_{N\to\infty} \mathcal{H}_N.$$

Lattice Gauge Theory

Lattice Γ with L links ℓ, N nodes n and gauge group G

$$\tilde{\mathcal{H}}_\Gamma = L^2(G^L).$$

States $\psi(h_\ell) \in \tilde{\mathcal{H}}_\Gamma$ acted on by gauge transformations

$$\psi(h_\ell) \to \psi(g_{s(\ell)} h_\ell g^{-1}_{t(\ell)}) \qquad g_n \in G.$$

Gauge invariant Hilbert space is

$$\mathcal{H}_\Gamma = L^2(G^L/G^N).$$

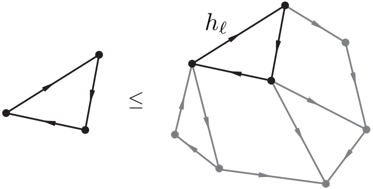
Loop Quantum Gravity

General graph Γ, called a spin network,

$$\tilde{\mathcal{H}}_\Gamma = L^2(SU(2)^L/SU(2)^N),$$

an $SU(2)$ lattice gauge theory.

If $\Gamma' \leq \Gamma$ then $\tilde{\mathcal{H}}_{\Gamma'} \subset \tilde{\mathcal{H}}_\Gamma$.



$$\mathcal{H}_\Gamma = \tilde{\mathcal{H}}_\Gamma / \sim$$

$$\mathcal{H} = \lim_{\Gamma \to \infty} \mathcal{H}_\Gamma$$

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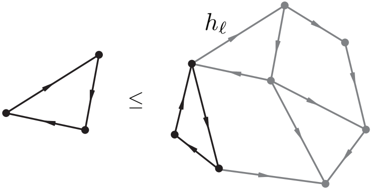

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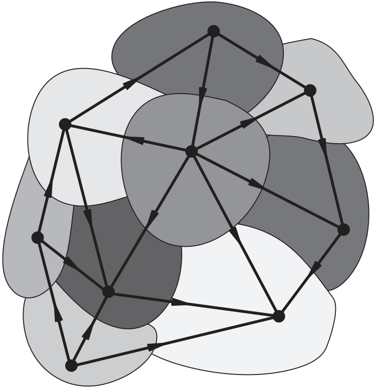
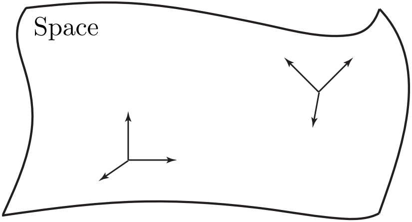
Physical Picture



Quanta of gravity are “grains” or “chunks” of space

Variables: gravitational electric field



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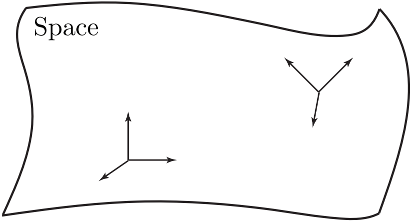

$$E_{i\,bc} = \epsilon_{ijk} e^j_b e^k_c \qquad (b,c = 1,2,3) \quad (i,j,k = 1,2,3)$$

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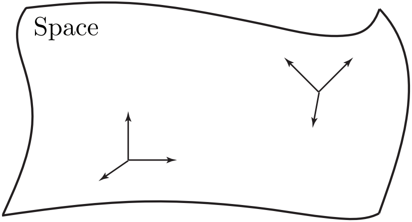

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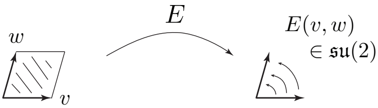
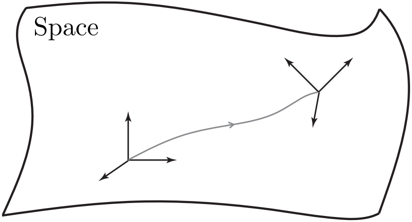


Inf. parallelogram → inf. rotation, mag.=area parallelogram.

Variables: Ashtekar-Barbero connection



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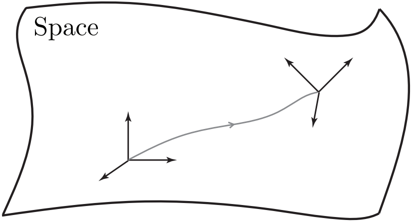


Ashtekar-Barbero connection is an $SU(2)$ gauge field,

$$A_a^i = \Gamma_a^i - \gamma K_a^i$$

Levi-Civita connection

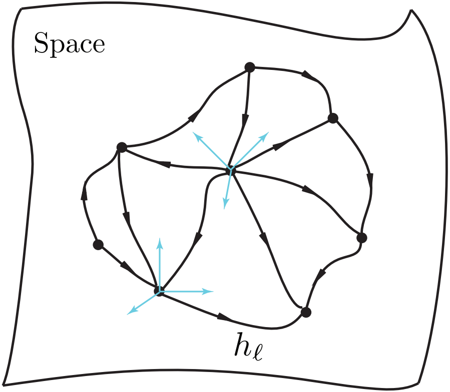
extrinsic curvature

Barbero-Immirzi parameter

Variables: Overview

$$A_a^i \sim \text{“position”} \qquad E_{jbc} \sim \text{“momentum”}$$

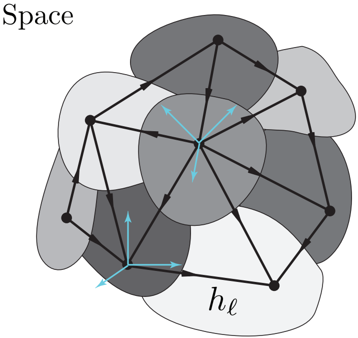
$$\{A_a^i(x), E_{jbc}(y)\} = 8\pi\ G\ \gamma\ \delta_j^i\ \epsilon_{abc}\ \delta(x-y)$$



Variables: Overview

$A^i_a \sim$ "position" $\qquad E_{jbc} \sim$ "momentum"

$$\{A^i_a(x), E_{jbc}(y)\} = 8\pi\ G\ \gamma\ \delta^i_j\ \epsilon_{abc}\ \delta(x-y)$$



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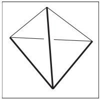

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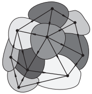

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Operators in Loop Gravity

$$\hat{x}\psi(x) = x\psi(x) \qquad \hat{p}\psi(x) = \frac{\hbar}{i}\frac{d}{dx}\psi(x)$$

A natural derivative operator $\sim$ momentum on $SU(2)$

$$J_i\psi(h) \equiv i\frac{d}{dt}\,\psi\left(he^{t\tau_i}\right)\Big|_{t=0} \qquad (i = 1, 2, 3)$$

with $\tau_i = -(i/2)\sigma_i$. Casimir, $J^2 = \vec{J}\cdot\vec{J}$, eigenvalue $j(j+1)$.

Geometrical operators of LQG built out of J_i, e.g.

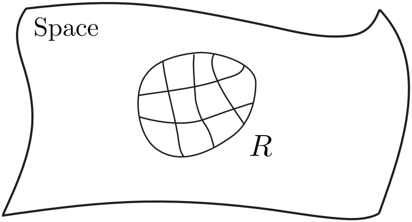
$$E_i = \int \epsilon_{ijk} e^j_b e^k_c dx^b dx^c = 8\pi\gamma\ell^2_{\mathrm{Pl}} J_i$$

Volume Operator

Volume operator from

$$V = \int_R d^3x \; h$$

by regularizing and quantizing. Regularization procedures complex.

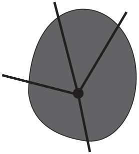


Volume Operator

Up to technicalities, all proposals agree for the 4-valent case:

$$\hat{V} = \frac{\sqrt{2}}{3}\sqrt{|\varepsilon_{ijk} J_1^i J_2^j J_3^k|}.$$

Space



Units: $G = c = 1$ set $8\pi\gamma = 1$. Areas have units of $\ell_{\rm Pl}^2 = \hbar$.

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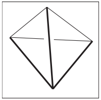

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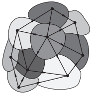

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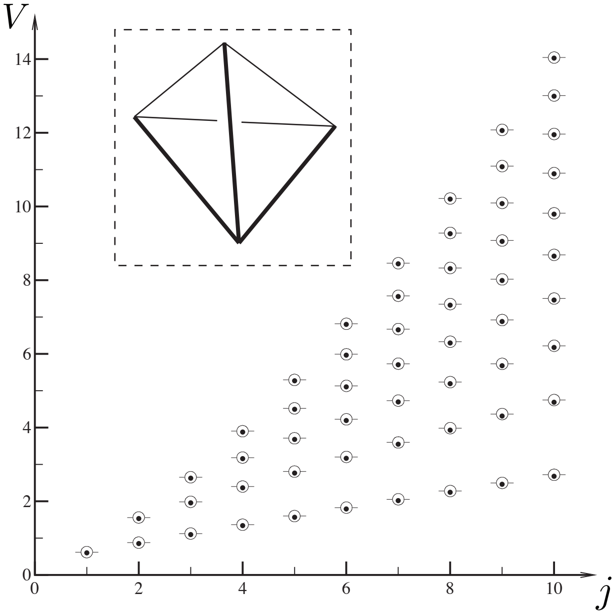
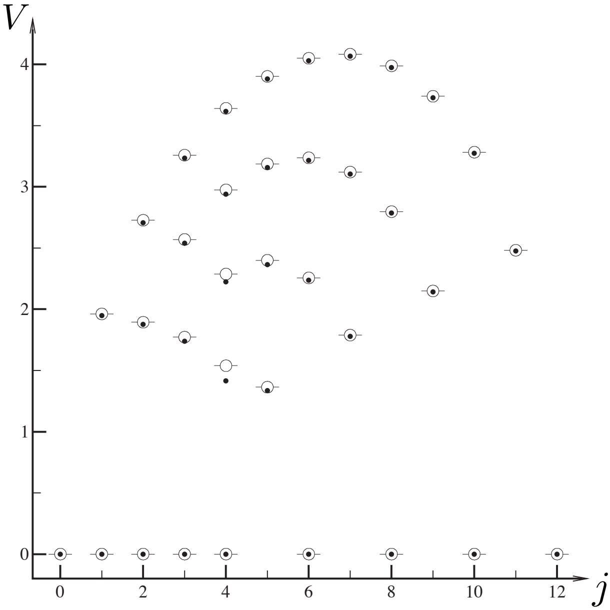


$$A_1 = j + 1/2$$

$$A_2 = j + 1/2$$

$$A_3 = j + 1/2$$

$$A_4 = j + 3/2$$



$A_1 = 9/2$

$A_2 = 9/2$

$A_3 = 9/2$

$A_4 = j + 1/2$

Table: Volume spectrom

$j_1 j_2 j_3 j_4$	Loop gravity	Bohr-Sommerfeld	Accuracy
$\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}$	0.3102	0.2523	0.19
$1\ 1\ \frac{1}{2}\ \frac{1}{2}$	0.3964	0.3440	0.13
$\frac{3}{2}\ \frac{3}{2}\ \frac{1}{2}\ \frac{1}{2}$	0.4638	0.4061	0.12
$\frac{3}{2}\ 1\ 1\ \frac{1}{2}$	0.4984	0.4584	0.08
1 1 1 1	0	0	0
	0.6204	0.5658	0.09
$2\ 2\ \frac{1}{2}\ \frac{1}{2}$	0.5216	0.4581	0.12
$2\ \frac{3}{2}\ 1\ \frac{1}{2}$	0.5773	0.5354	0.07
2 1 1 1	0.6204	0.5975	0.04
$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{1}{2}$	0.6204	0.5975	0.04

Table: Volume spectrom

$j_1 j_2 j_3 j_4$	Loop gravity	Bohr-Sommerfeld	Accuracy
…			
$2\ \frac{3}{2}\ \frac{3}{2}\ 1$	0	0	0
	0.9036	0.8676	0.04
$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$	0.5372	0.4521	0.16
	0.9929	0.9473	0.05
…			
6 6 6 7	1.8276	1.7949	0.018
	3.2039	3.1618	0.013
	4.2249	4.1895	0.008
	5.1328	5.1053	0.005
	5.9891	5.9673	0.004
	6.8173	6.7994	0.003

Conclusions & Acknowledgements

- Remarkably simple road to the quantization of geometry.
- Strengthens previous arguments for operator $\hat{V}$.
- The truncation of the degrees of freedom of the gravitational field to a graph is like a piecewise-linear approximation of space by polyhedra.

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