On-Shell Constructions of the Non-linear Sigma Model

Based on 1904.12859, 1911.08490 and 2009.00008, collaborations with Ian Low and Laurentiu Rodina

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October 26, 2020
Outline

1. The local constructions
2. The soft bootstrap
3. The double copy
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2. The soft bootstrap
3. The double copy
The EFT for NGBs

NLSM: EFT that describes NGB’s
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- In the Lagrangian:

$$\mathcal{L} = f^2 \Lambda^2 \left( \text{Series of } \frac{\partial}{\Lambda}, \frac{\pi}{f} \right).$$
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- In the Lagrangian:

$$\mathcal{L} = f^2 \Lambda^2 \left( \text{Series of} \frac{\partial}{\Lambda}, \frac{\pi}{f} \right).$$

- In the amplitudes:

$$\mathcal{M}_{n,L} = \frac{1}{fn+2L-2} \left( \text{Series of} \frac{p}{\Lambda} \right).$$
The local constructions

The coset construction:

Callan, Coleman, Wess and Zumino, 1969

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\[-i\xi^\dagger \partial_\mu \xi = d^a_\mu X^a + E^i_\mu T^i\]
\[= d_\mu + E_\mu,\]

where $\xi = \exp(i\pi^a X^a / f)$
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  $M(\tau p) = O(\tau) \implies$ shift symmetry:
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\[d_\mu \to h d_\mu h^\dagger, E_\mu \to h E_\mu h^\dagger - ih \partial_\mu h^\dagger\]
We only consider symmetric cosets in this talk: $X^a \leftrightarrow -X^a$
Caveat

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Examples:

- The QCD chiral Lagrangian: $\text{SU}(N) \times \text{SU}(N)/\text{SU}(N)$, $N = 2, 3$
- The standard model: $\text{SO}(4)/\text{SO}(3)$
- The composite Higgs models: $\text{SO}(5)/\text{SO}(4)$
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We only consider representations $R$ that can be embedded into a symmetry coset: “Closure condition”

$$T_{ab}^i T_{cd}^i + T_{ac}^i T_{db}^i + T_{ad}^i T_{bc}^i = 0$$

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Universality in pNGB Higgs!

Low, 1412.2146; Liu, Low, ZY, 1805.00489, 1809.09126
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Otherwise, Adler’s zero no longer holds: $\mathcal{M}(\tau p) = \mathcal{O}(\tau^0)$

Kampf, Novotny, Shifman, Trnka, 1910.04766
Why on-shell?

- Convenience, e.g. soft bootstrap
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- Unveil hidden structures, e.g. double copy structures
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The idea

Constructing all amplitudes without $\mathcal{L}$ or Feynman diagrams
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Unitarity:

\[ P_I^2 = 0 \]
The Lagrangian

Soft bootstrap

Double copy

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Contact terms? Need more constraints:
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- Particle content, mass dimension etc.
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Constructing all amplitudes without $\mathcal{L}$ or Feynman diagrams

Unitarity:

\[ \mathcal{M} \rightarrow M_L \quad M_R \quad I \]

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- Gauge invariance, e.g. BCFW for gravity
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- Symmetry: linearly realized global symmetry (e.g. color ordering for gauge theory), SUSY
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Constructing all amplitudes without $\mathcal{L}$ or Feynman diagrams

Unitarity:

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- Particle content, mass dimension etc.
- Gauge invariance, e.g. BCFW for gravity
- Symmetry: linearly realized global symmetry (e.g. color ordering for gauge theory), SUSY
- Extra constraints of IR/UV, amplitude relations...
The machinery: recursion relations

- Deform the momenta: \( \hat{p}_i = \hat{p}_i(z) \), \( \hat{p}_i(0) = p_i \)
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$$\text{Res}_{z=z_I} \frac{\hat{M}_n(z)}{z} = - \frac{\hat{M}_L(z_I)\hat{M}_R(z_I)}{z_I} \text{Res}_{z=z_I} \frac{1}{\hat{P}_i^2(z)}.$$
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  \]

- Recursion relation
  \[
  \oint \frac{dz}{z} \hat{M}_n(z) = 0
  \]
  $\rightarrow$ $\mathcal{M}_n = \sum_I \sum_{z_I} \frac{\hat{M}_L(z_I)\hat{M}_R(z_I)}{z_I} \text{Res}_{z = z_I} \frac{1}{\hat{P}_I^2(z)}.$
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\[ \oint \frac{dz}{z} \hat{\mathcal{M}}_n(z) = 0 \rightarrow \lim_{z \to \infty} \hat{\mathcal{M}}(z) = 0. \]

Works fine for gravity and gauge theory because of gauge invariance.

NLSM starts at \( \mathcal{O}(p^2) \rightarrow \mathcal{O}(z^2) \)

Need to incorporate the shift symmetry!
The machinery: soft subtract recursion relations

\[ \oint \frac{dz}{z} \frac{\hat{M}_n(z)}{K(z)} = 0 \quad \rightarrow \quad \lim_{z \to \infty} \frac{\hat{M}(z)}{K(z)} = 0. \]
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For NLSM, Adler’s zero condition: if we take \( p_i \to \tau p_i \) and \( \tau \to 0 \),

\[ \mathcal{M}_n(\cdots, \tau p_i, \cdots) = \mathcal{O}(\tau). \]
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All line shift: $p_i \to \hat{p}_i = (1 - a_i z) p_i$; $\hat{p}_i = 0 \quad \rightarrow \quad z = 1/a_i.$
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Independence of \( a_i \leftrightarrow \) allowed theory space

Cheung, Kampf, Novotny, Shen, Trnka, 1509.03309, 1611.03137; Elvang, Hadjiantonis, Jones, Paranjape, 1806.06079
The machinery: flavor ordering

The leading order Lagrangian:

\[
    \mathcal{L}^{(2)} = \frac{f^2}{2} d^a d^{a\mu} = \frac{1}{2} \partial_\mu \pi^a \left[ \frac{\sin^2 \sqrt{T}}{T} \right]_{ab} \partial^\mu \pi^b,
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where \((T)_{ab} = \frac{1}{f^2} T^i_{ac} T^i_{db} \pi^c \pi^d\)
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where \((T)_{ab} = \frac{1}{f^2} T_i^{\text{ac}} T_i^{\text{db}} \pi^c \pi^d, \ T_i^{\text{ab}} \propto \text{tr} (T^i [X^a, X^b])\)
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Single trace flavor-ordering:

\[ M_{n a_1 a_2 \cdots a_n} = \sum_\sigma \text{tr} \left( X^{a_{\sigma(1)}} X^{a_{\sigma(2)}} \cdots X^{a_{\sigma(n)}} \right) M_n(\sigma) \]
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- Works for a general \( R \) of \( H \)
- Continue to work at \( \mathcal{O}(p^4) \) with single/double traces
Factorization of flavors?

We would like to construct the ordered amplitudes

\[ M_\sigma = - \sum_{l, \pm} \frac{1}{P_l^2} \frac{\hat{M}_L^{(l)}(z_l^\pm) \hat{M}_R^{(l)}(z_l^\pm)}{K_n(z_l^\pm)(1 - z_l^\pm / z_l^\mp)}, \]
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Convenient factorization of the ordering:

\[ \{\sigma\} = \{\sigma_L, \sigma_R\} \]
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Constraint on the flavor factor:

\[ C^{a_1 a_2 \cdots a_n} = C^{a_1 a_2 \cdots a_k a_l} C^{a_l a_{k+1} \cdots a_n}, \]
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Can be done for the adjoint of \( H = U(N) (SU(N)) \) in the trace basis: we have \( H \times H/H \approx H \),

\[ \text{tr} (\cdots X^a \cdots) \rightarrow \text{tr} (\cdots T^a \cdots), \quad (T^a)_{bc} (T^a)_{de} = \delta^{be} \delta^{cd}. \]
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However, the amplitudes up to \( \mathcal{O}(p^4) \) in the trace basis are universal!
$O(p^2)$

Soft blocks:
- Correct mass dimension and little group scaling
- Local
- Satisfy symmetry constraints: ordering, Adler’s zero…
$O(p^2)$

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Single-trace: $S_4^{(2)}(1, 2, 3, 4) = (c_0/f^2)s_{13}$, $s_{ij} = (p_i + p_j)^2$,
generates the general $O(p^2)$ single-trace amplitudes
\( \mathcal{O}(p^2) \)

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What about double-trace: \( S^{(2)}_4(1, 2|3, 4) = (d_0/f^2)s_{12} \)
Flavor factor: \( \text{tr}(X^{a_1}X^{a_2})\text{tr}(X^{a_3}X^{a_4}) = \delta^{a_1a_2}\delta^{a_3a_4} \).
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Result:

- \( S_4^{(2)}(1, 2|3, 4) \) generates the \( \mathcal{O}(p^2) \) pair basis amplitudes \( M(12|34|56|\cdots) \)
- “Mixed ordering” does not work
The leading order Lagrangian:

\[ \mathcal{L}^{(2)} = \frac{f^2}{2} d_{\mu}^a d_{a\mu} = \frac{1}{2} \partial_\mu \pi^a \left[ \frac{\sin^2 \sqrt{T}}{T} \right]_{ab} \partial^\mu \pi^b, \]

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Pair basis for N of SO(N)

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For N of SO(N): completeness relation

\[ T^i_{ab} T^i_{cd} = -\frac{1}{2} (\delta^{ac} \delta^{bd} - \delta^{ad} \delta^{bc}).\]
Pair basis for $\textbf{N}$ of SO($N$)

For $\textbf{N}$ of SO($N$): completeness relation

$$T^i_{ab} T^i_{cd} = -\frac{1}{2} (\delta^{ac} \delta^{bd} - \delta^{ad} \delta^{bc}).$$

The pair basis:

$$M^{a_1 \cdots a_n}_n = \sum_{\dot{\alpha}} \left( \prod_{j=1}^{n/2} \delta^{a_\dot{\alpha}(2j-1)} a_{\dot{\alpha}(2j)} \right) \times M_n(\dot{\alpha}(1), \dot{\alpha}(2)|\dot{\alpha}(3), \dot{\alpha}(4)| \cdots |\dot{\alpha}(2n-1), \dot{\alpha}(2n)).$$
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$$T^i_{ab} T^i_{cd} = -\frac{1}{2}(\delta^{ac} \delta^{bd} - \delta^{ad} \delta^{bc}).$$

The pair basis:

$$M^{a_1 \cdots a_n}_{n} = \sum_{\hat{\alpha}} \left( \prod_{j=1}^{n/2} \delta^{a_{\hat{\alpha}(2j-1)} a_{\hat{\alpha}(2j)}} \right)$$

$$\times M_n(\hat{\alpha}(1), \hat{\alpha}(2)|\hat{\alpha}(3), \hat{\alpha}(4)| \cdots |\hat{\alpha}(2n-1), \hat{\alpha}(2n)), $$

Factorization of the flavor factor:

$$C^{a_1 a_2 \cdots a_n} = C^{a_1 a_2 \cdots a_k a_l} C^{a_l a_{k+1} \cdots a_n},$$

Automatic!
To build the Lagrangian, we need traces of $d_\mu$, with $\nabla_\mu$ acting on them so that

$$\nabla_\mu d_\nu = \partial_\mu d_\nu + i [E_\mu, d_\nu].$$
\( \mathcal{O}(p^4) \)

The most general Lagrangian

\[
\mathcal{L}^{\text{NLSM}} = \mathcal{L}^{(2)} + \frac{f^2}{\Lambda^2} \left( \sum_{i=1}^{4} C_i O_i + C_\infty O_{\text{wzw}} \right) + \mathcal{O} \left( \frac{1}{\Lambda^4} \right),
\]

with

\[
\begin{align*}
O_1 &= \left[ \text{tr}(d_\mu d^{\mu}) \right]^2, \quad O_2 = \left[ \text{tr}(d_\mu d_\nu) \right]^2, \\
O_3 &= \text{tr}([d_\mu, d_\nu]^2), \quad O_4 = \text{tr}([d_\mu, d_\nu]^2), \\
S_{\text{wzw}} &\propto \int d^5 y \, \varepsilon^{\alpha\beta\gamma\delta\epsilon} \text{tr}(d_\alpha d_\beta d_\gamma d_\delta d_\epsilon) = \int d^4 x \, O_{\text{wzw}}
\end{align*}
\]
The most general Lagrangian

\[ \mathcal{L}^{\text{NLSM}} = \mathcal{L}^{(2)} + \frac{f^2}{\Lambda^2} \left( \sum_{i=1}^{4} C_i O_i + C_{-} O_{\text{wzw}} \right) + \mathcal{O} \left( \frac{1}{\Lambda^4} \right), \]

with

\[
\begin{align*}
O_1 &= \left[ \text{tr} (d_\mu d^{\mu}) \right]^2, \\
O_2 &= \left[ \text{tr} (d_\mu d_\nu) \right]^2, \\
O_3 &= \text{tr} ([d_\mu, d_\nu]^2), \\
O_4 &= \text{tr} (\{d_\mu, d_\nu\}^2),
\end{align*}
\]

\[ S_{\text{wzw}} \propto \int d^5 y \varepsilon^{\alpha\beta\gamma\delta\epsilon} \text{tr} (d_\alpha d_\beta d_\gamma d_\delta d_\epsilon) = \int d^4 x \ O_{\text{wzw}} \]

To get this:

- Total derivatives
- Symmetry, e.g. \( \nabla_{[\mu} d_{\nu]} = 0 \), \( [\nabla_\mu, \nabla_\nu] = [d_\mu, d_\nu] \)
- Equation of motion
The most general Lagrangian

\[ \mathcal{L}^{\text{NLSM}} = \mathcal{L}^{(2)} + \frac{f^2}{\Lambda^2} \left( \sum_{i=1}^{4} C_i O_i + C_{-} O_{\text{wzw}} \right) + \mathcal{O} \left( \frac{1}{\Lambda^4} \right), \]

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O_3 &= \text{tr}([d_\mu, d_\nu]^2), \\
O_4 &= \text{tr}([d_\mu, d_\nu]^2),
\end{align*}

\[ S_{\text{wzw}} \propto \int d^5 y \varepsilon^{\alpha\beta\gamma\delta\epsilon} \text{tr}(d_\alpha d_\beta d_\gamma d_\delta d_\epsilon) = \int d^4 x \ O_{\text{wzw}} \]

The on-shell way:

- Total derivatives $\rightarrow$ total momentum conservation
- Symmetry $\rightarrow$ orderings, Adler’s zero
- Equation of motion $\rightarrow$ on-shell condition
\( \mathcal{O}(p^4) \) soft bootstrap

\( \mathcal{O}(p^4) \) soft blocks:

\[
S_4^{(4)}(1, 2, 3, 4) = \frac{1}{f^2 \Lambda^2} (c_1 s_{13}^2 + c_2 s_{12} s_{23}),
\]

\[
S_4^{(4)}(1, 2|3, 4) = \frac{1}{f^2 \Lambda^2} (d_1 s_{12}^2 + d_2 s_{13} s_{23}),
\]

\[
S_5^{(4)}(1, 2, 3, 4, 5) = \frac{1}{f^2 \Lambda^3} c - \varepsilon_{\mu \nu \rho \gamma} p_1^\mu p_2^\nu p_3^\rho p_4^\gamma.
\]
$\mathcal{O}(p^4)$ soft bootstrap

$\mathcal{O}(p^4)$ soft blocks:

\[
\begin{align*}
S_4^{(4)}(1, 2, 3, 4) &= \frac{1}{f^2 \Lambda^2} (c_1 s_{13}^2 + c_2 s_{12}s_{23}), \\
S_4^{(4)}(1, 2|3, 4) &= \frac{1}{f^2 \Lambda^2} (d_1 s_{12}^2 + d_2 s_{13}s_{23}), \\
S_5^{(4)}(1, 2, 3, 4, 5) &= \frac{1}{f^2 \Lambda^3} c_\varepsilon \varepsilon_{\mu \nu \rho \gamma} p_1^\mu p_2^\nu p_3^\rho p_4^\gamma.
\end{align*}
\]

Degrees of freedom match exactly with the action:

Low, ZY, 1904.12859.
\( \mathcal{O}(p^4) \) soft bootstrap

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S_5^{(4)}(1, 2, 3, 4, 5) = \frac{1}{f^2 \Lambda^3} c_{-\varepsilon_{\mu\nu\rho\gamma}} p_1^\mu p_2^\nu p_3^\rho p_4^\gamma.
\]

Degrees of freedom match exactly with the action:

Low, ZY, 1904.12859.

- The general case, starting with \( S_4^{(2)}(1, 2, 3, 4) \):
  4 independent P-even operators and a WZW term
$O(p^4)$ soft bootstrap

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S_{4}^{(4)}(1, 2, 3, 4) = \frac{1}{f^2 \Lambda^2} (c_1 s_{12}^2 + c_2 s_{13} s_{23}),
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\]
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S_{5}^{(4)}(1, 2, 3, 4, 5) = \frac{1}{f^2 \Lambda^3} c_{-\varepsilon_{\mu\nu\rho\gamma}} p_{1}^{\mu} p_{2}^{\nu} p_{3}^{\rho} p_{4}^{\gamma}.
\]

Degrees of freedom match exactly with the action:

Low, ZY, 1904.12859.

- The general case, starting with $S_{4}^{(2)}(1, 2, 3, 4)$:
  4 independent P-even operators and a WZW term

- The special case, starting with $S_{4}^{(2)}(1, 2|3, 4)$:
  2 independent P-even operators for SO($N$), while the WZW term exists only if $N = 5$
Example: the WZW term vs. the pair basis

Consider the SO($N$) fundamental NLSM:

\[ O(p^2) : S(2)^4(1, 2 | 3, 4) = s_{12}, \]

\[ O(p^4) : S(4)^5(1, 2, 3, 4, 5) = c_{-\epsilon}(1234). \]

What we should have at 7-pt:

\[ M(1, 2, 3, 4, 5 | 6, 7). \]

Soft recursion:

\[ p_i \rightarrow \hat{p}_i = (1 - a_i z) p_i, \]

\[ M_7 = \hat{M}_7(0) = -\sum_{I, \pm 1} P_2 I \hat{S}_{5, (I \pm \pm I)} \hat{S}_{4, (I \pm \pm I)} F_7(z \pm I)(1 - z^{\pm I} / z^{\mp I}). \]

What if we actually have only 5 flavors:

\[ M(1, 2, 3, 4, \{5, 6, 7\}) = M(1, 2, 3, 4, 5 | 6, 7) + M(1, 2, 3, 4, 6 | 5, 7) + M(1, 2, 3, 4, 7 | 5, 6). \]

WZW term exists only if

\[ N = 5!^{15 / 21}. \]
Example: the WZW term vs. the pair basis

Consider the $\text{SO}(N)$ fundamental NLSM:

- $O(p^2)$: $S_4^{(2)}(1, 2|3, 4) = s_{12}$. 
Example: the WZW term vs. the pair basis

Consider the SO($N$) fundamental NLSM:

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- What we should have at 7-pt: $M(1, 2, 3, 4, 5|6, 7)$.
- Soft recursion: $p_i \to \hat{p}_i = (1 - a_i z) p_i$,

\[
M_7 = \hat{M}_7(0) = - \sum_{l, \pm} \frac{1}{P_l^2} \frac{S_5,(l)(z_l^\pm)\hat{S}_4,(l)(z_l^\pm)}{F_7(z_l^\pm)(1 - z_l^\pm/z_l^{\mp})}.
\]
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Consider the SO\((N)\) fundamental NLSM:

- \(\mathcal{O}(p^2): S_4^{(2)}(1, 2|3, 4) = s_{12}\).
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- Soft recursion: \(p_i \rightarrow \hat{p}_i = (1 - a_i z) p_i\),

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What if we actually have only 5 flavors?
Example: the WZW term vs. the pair basis

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What if we actually have only 5 flavors?

$$M(1, 2, 3, 4, \{5, 6, 7\}) = M(1, 2, 3, 4, 5|6, 7) + M(1, 2, 3, 4, 6|5, 7) + M(1, 2, 3, 4, 7|5, 6).$$
Example: the WZW term vs. the pair basis

Consider the $\text{SO}(N)$ fundamental NLSM:

- $O(p^2)$: $S_{4}^{(2)}(1,2|3,4) = s_{12}$.
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What if we actually have only 5 flavors?

$$M(1,2,3,4,\{5,6,7\}) = M(1,2,3,4,5|6,7) + M(1,2,3,4,6|5,7) + M(1,2,3,4,7|5,6).$$

WZW term exists only if $N = 5!$
Outline

1. The local constructions

2. The soft bootstrap

3. The double copy
The color-kinematics duality

Gauge theory:

\[ \mathcal{M}^\text{YM}_n = \sum_{g \in \{g_n\}} \frac{c_g}{d_g} n_g \]
The color-kinematics duality

Gauge theory:

\[ \mathcal{M}_{YM}^n = \sum_{g \in \{g_n\}} \frac{c_g n_g}{d_g} \]

- \( c_g \) satisfies anti-symmetry and the Jacobi identity

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
4 & & & \\
3 & & & \\
2 & & & \\
\end{array}
+ \begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & & & \\
3 & & & \\
1 & & & \\
\end{array}
+ \begin{array}{cccc}
1 & 2 & 3 & 4 \\
2 & & & \\
4 & & & \\
1 & & & \\
\end{array} = 0
\]
The color-kinematics duality

Gauge theory:

\[ \mathcal{M}_{n}^{YM} = \sum_{g \in \{g_n\}} \frac{c_g \ n_g}{d_g} \]

- \( c_g \) satisfies anti-symmetry and the Jacobi identity
- \( \exists n_g \) satisfying anti-symmetry and the Jacobi identity!

Bern, Carrasco, Johansson, 0805.3993
The color-kinematics duality

Gauge theory:

\[ \mathcal{M}_n^{YM} = \sum_{g \in \{g_n\}} \frac{c_g \ n_g}{d_g} \]

- \( c_g \) satisfies anti-symmetry and the Jacobi identity
- \( \exists n_g \) satisfying anti-symmetry and the Jacobi identity!
- Replace \( c_g \) with \( n_g \) leads to gravity amplitudes!

\[ \mathcal{M}_n^{GR} = \sum_{g \in \{g_n\}} \frac{n_g n_g}{d_g} \]

Bern, Carrasco, Johansson, 0805.3993
The color-kinematics duality

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- \(\exists n_g\) satisfying anti-symmetry and the Jacobi identity!

Replace \(c_g\) with \(n_g\) leads to gravity amplitudes!

\[ \mathcal{M}_{n}^{GR} = \sum_{g \in \{g_n\}} \frac{n_g n_g}{d_g} . \]

- Double copy: \(GR = YM \otimes YM\).

Review: Bern, Carrasco, Chiodaroli, Johansson, Roiban, 1909.01358
The flavor-kinematics duality

For NLSM at $\mathcal{O}(p^2)$,

$$\mathcal{M}_n^{(2)} = \sum_{g \in \{g_n\}} f_g \frac{n_g}{d_g}.$$
The flavor-kinematics duality

For NLSM at $O(p^2)$,

$$M_n^{(2)} = \sum_{g \in \{ g_n \}} \frac{f_g n_g}{d_g}.$$

- $f_g$ satisfies anti-symmetry and the “Jacobi identities”

$$T^i_{ab} T^i_{cd} + T^i_{ac} T^i_{db} + T^i_{ad} T^i_{bc} = 0, \quad [T^i, T^j] = i f^{ijk} T^k$$
The flavor-kinematics duality

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Chen, Du, 1311.1133; Du, Fu, 1606.05846; Carrasco, Mafra, Schlotterer, 1608.02569
The flavor-kinematics duality

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- Replace $f_g$ with $n_g$ leads to the special Galileon theory

$$\mathcal{M}_n^{\text{sGal}} = \sum_{g \in \{g_n\}} \frac{n_g n_g}{d_g}.$$
The flavor-kinematics duality

For NLSM at $O(p^2)$,

$$\mathcal{M}^{(2)}_n = \sum_{g\in\{g_n\}} \frac{f_g n_g}{d_g}.$$ 

- $f_g$ satisfies anti-symmetry and the "Jacobi identities"

$$T^i_{ab} T^i_{cd} + T^i_{ac} T^i_{db} + T^i_{ad} T^i_{bc} = 0, \ [T^i, T^j] = i f^{ijk} T^k$$

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Chen, Du, 1311.1133; Du, Fu, 1606.05846; Carrasco, Mafra, Schlotterer, 1608.02569

- Replace $f_g$ with $n_g$ leads to the special Galileon theory

$$\mathcal{M}_{n}^{sGal} = \sum_{g\in\{g_n\}} \frac{n_g n_g}{d_g}.$$ 

- Double copy: $sGal = NLSM \otimes NLSM, BI = NLSM \otimes YM...$
Flavor-kinematics at $\mathcal{O}(p^4)$?

At 4-pt, $\mathcal{O}(p^2)$:

$$\mathcal{M}_4 = \frac{f_s n_s}{s} + \frac{f_t n_t}{t} + \frac{f_u n_u}{u},$$

with $f_s = T^i_{a_1a_2} T^i_{a_3a_4}$, $n_s = s(t - u)$.

- Lie algebra/“closure condition”: $f_s + f_t + f_u = 0$
- Flavor-kinematic duality: $n_s + n_t + n_u = 0$
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- Lie algebra/“closure condition”: $f_s + f_t + f_u = 0$
- Flavor-kinematic duality: $n_s + n_t + n_u = 0$

At $\mathcal{O}(p^4)$, correcting $n_i$ fails

Elvang, Hadjiantonis, Jones, Paranjape, 1806.06079; González, Penco, Trodden, 1908.07531

New ideas: correcting the $f_g$!

Carrasco, Rodina, Yin, Zekioglu, 1910.12850
Flavor-kinematics at $O(p^4)$!

4 different ways to correct $f_i$ leads to $4 O(p^4)$ soft blocks

Low, ZY, 1911.08490
Flavor-kinematics at $\mathcal{O}(p^4)!$

4 different ways to correct $f_i$ leads to 4 $\mathcal{O}(p^4)$ soft blocks

- **Single-trace:**

  $$\hat{f}_s^{(1)} = f_t(u - s) - f_u(s - t),$$

  $$\hat{f}_s^{(2)} = d^{a_1a_2a_3a_4}(t - u)$$

  where

  $$d^{a_1a_2a_3a_4} \propto \sum_{\sigma} \text{tr}(X^{a_{\sigma(1)}}X^{a_{\sigma(2)}}X^{a_{\sigma(3)}}X^{a_{\sigma(4)}})$$

Low, ZY, 1911.08490
Flavor-kinematics at $\mathcal{O}(p^4)$!

4 different ways to correct $f_i$ leads to 4 $\mathcal{O}(p^4)$ soft blocks

**Single-trace:**

\[
\hat{f}_s^{(1)} = f_t(u - s) - f_u(s - t), \\
\hat{f}_s^{(2)} = d^{a_1 a_2 a_3 a_4} (t - u)
\]

where

\[
d^{a_1 a_2 a_3 a_4} \propto \sum_{\sigma} \text{tr}(X^{a_{\sigma(1)}} X^{a_{\sigma(2)}} X^{a_{\sigma(3)}} X^{a_{\sigma(4)}})
\]

**Double-trace:**

\[
\hat{f}_s^{(3)} = f_t'(u - s) - f_u'(s - t), \\
\hat{f}_s^{(4)} \propto \frac{1}{3!} \sum_{\sigma \in S_3} \delta^{a_{\sigma(1)} a_{\sigma(2)}} \delta^{a_{\sigma(3)} a_{\sigma(4)}} (t - u),
\]

where $f'_s = \delta^{a_1 a_3} \delta^{a_2 a_4} - \delta^{a_1 a_4} \delta^{a_2 a_3}$

Low, ZY, 1911.08490
Higher multiplicity

Proposal:

\[ \mathcal{M}_n^{(4)} = \sum_{g \in \{g_n\}} \hat{f}_g \frac{n_g}{d_g}, \]
Higher multiplicity

Proposal:

\[ M_n^{(4)} = \sum_{g \in \{g_n\}} \frac{\hat{f}_g}{d_g} n_g, \]

- Works for \( S_4^{(4)}(1, 2|3, 4) \propto s_{13}s_{23} \) if we assume local \( \hat{f} \)

Low, Rodina, ZY, 2009.00008
Higher multiplicity

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\mathcal{M}_n^{(4)} = \sum_{g \in \{g_n\}} \frac{\hat{f}_g \ n_g}{d_g},
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- Works for \( S_4^{(4)}(1, 2|3, 4) \propto s_{13} s_{23} \) if we assume local \( \hat{f} \)

- Replacing \( n_g \) with \( c_g \):

\[
\mathcal{M}_n^{YM+\phi^3} \supset \sum_{g \in \{g_n\}} \frac{\hat{f}_g \ c_g}{d_g}
\]

Low, Rodina, ZY, 2009.00008
Higher multiplicity

Proposal:

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Low, Rodina, ZY, 2009.00008

- Replacing \( n_g \) with \( c_g \):

\[ \mathcal{M}^{YM+\phi^3}_n \supset \sum_{g \in \{g_n\}} \frac{\hat{f}_g \ c_g}{d_g} \]

- Double copy: \( \text{NLSM}^{(4)} \subset \text{NLSM}^{(2)} \otimes (YM + \phi^3) \)
Summary and outlook

Soft bootstrap is efficient for counting degrees of freedom
Summary and outlook

Soft bootstrap is efficient for counting degrees of freedom

- Higher order terms in the chiral Lagrangian

Dai, Low, Mehen, Mohapatra, 2009.01819
Summary and outlook

Soft bootstrap is efficient for counting degrees of freedom

- Higher order terms in the chiral Lagrangian
  Dai, Low, Mehen, Mohapatra, 2009.01819

- Other EFTs, e.g. SMEFT
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  Intriguing structures in the amplitudes
Summary and outlook

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  Dai, Low, Mehen, Mohapatra, 2009.01819

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- Double copy structures may extend beyond $\mathcal{O}(p^2)$
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  Dai, Low, Mehen, Mohapatra, 2009.01819

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Intriguing structures in the amplitudes

- Double copy structures may extend beyond $O(p^2)$

- Soft theorems and extended theories
  Cachazo, Cha, Mizera, 1604.03893
  Low, ZY, 1709.08639, 1804.08629; ZY, 1810.07186; Low, Rodina, ZY, to appear
Summary and outlook

Soft bootstrap is efficient for counting degrees of freedom
  - Higher order terms in the chiral Lagrangian
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Intriguing structures in the amplitudes
  - Double copy structures may extend beyond $\mathcal{O}(p^2)$
  - Soft theorems and extended theories
    Cachazo, Cha, Mizera, 1604.03893
    Low, ZY, 1709.08639, 1804.08629; ZY, 1810.07186; Low, Rodina, ZY, to appear
  - Other theories: YMS, DBI, goldstini, the fundamental Higgs, SM...