Chern-Simons theory: That obscure object of desire

University of California @ Davis
Davis, April 14, 2008

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1977 film about a neurotic relationship between a middle aged man and a beautiful young woman who drives him crazy.

She seduces and promises but never yields to the guy’s wishes.

The situation repeats itself endlessly, but with a new surprising twist every time.

It is frustrating and nerve-wracking, but it’s also addictive.
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A typical Yang-Mills action is something like this:

\[ I[A] = \frac{1}{4g} \int_M \sqrt{g} g^\mu_\alpha g^\nu_\beta \gamma_{ab} F_\mu\nu F^b_{\alpha\beta} d^4x \]

A typical Chern-Simons action is something like this:

\[ I[A] = \kappa \int_{M^3} A \wedge dA + \frac{2}{3} A \wedge A \wedge A \]

or this…

\[ I[A] = \kappa \int_{M^{2n+1}} \left( A \wedge (dA)^n + c_1 A^3 \wedge (dA)^{n-1} + \cdots + c_n A^{2n+1} \right) \]
Chern-Simons lagrangians define gauge field theories in a different class:

• They are explicit functions of the connection $(A)$, not local functions of the curvature $(F)$.

• Yet, they yield gauge-invariant field equations.

• Related to homotopic/topological invariants on fiber bundles: **characteristic classes**.

• They require no metric; just a Lie algebra (not necessarily semisimple); no adjustable parameters, conformally invariant. *More fundamental(?)*
• They are very sensitive to the dimension.

• They naturally couple to branes.

• Their quantization corresponds to sum over holonomies in an embedding space.

• CS theories are not exotic but a rather common occurrence in nature: Anomalies, quantum Hall effect, 11D supergravity (CJS), superconductivity, $2+1$ gravity, $2n+1$ gravity, all of classical mechanics,...
1. CS action in $0+1$ dimensions
E-M coupling

\[ I[A, z] = \int j^\mu A_\mu d^D x \]

\[ j^\mu (x) = e \int_{-\infty}^{+\infty} \dot{z}^\mu (\tau) \delta(x - z(\tau)) d\tau \]

\[ I[A(z)] = \int eA_\mu (z) dz^\mu = e \int_{\Gamma} A(z) \]

Gauge invariance \( A_\mu (x) \rightarrow A_\mu (x) + \partial_\mu \Omega(x) \), is ensured by current conservation, \( \partial_\mu j^\mu = 0 \), provided \( \Omega(z(+\infty)) = \Omega(z(-\infty)) \).

Not quite gauge invariant, but quasi-invariant.
This expression is invariant under

- Lorentz transformations, $(A^\mu_\nu)$
- Gauge transformations $A \rightarrow A + d\Omega(x)$
- Gen. coordinate transf. $z^\mu \rightarrow z'^\mu(z)$

This coupling is consistent with the minimal derivative substitution

$$ p_\mu \rightarrow p_\mu - eA_\mu(z), \quad \partial_\mu \rightarrow \partial_\mu + ieA_\mu(z) $$

Good for quantization:

$$ \partial_\mu \Psi \rightarrow (\partial_\mu + ieA_\mu(z))\Psi $$
The simplest Chern-Simons action

Take

\[ I[A] = \kappa \int_{M^{2n+1}} \langle A \wedge (dA)^n + c_1 A^3 \wedge (dA)^{n-1} + \cdots + c_n A^{2n+1} \rangle \]

for an abelian connection and set \( n=0 \):

\[ I[A] = e \int_{\Gamma} A(z) \]

Is this a sensible action?

\[ 1 = 0 \]

???

not exactly…
Varying the action,

$$\delta I[A] = e \int_\Gamma \delta A(z) = 0$$

This only means $\delta A = d\Omega$ with $\Omega(-\infty) = \Omega(\infty)$, or $\partial \Gamma = 0$.

The classical configurations are arbitrary $U(1)$ connections with PBC or living in a periodic 1d spacetime

Alternatively, $I$ can also be viewed as an action for the embedding coordinates $z^\mu$, 

$$I[z] = e \int_\Gamma A(z)$$
Varying the action,

\[
\delta I[z] = e \int_\Gamma \delta z^\mu (\partial_\mu A_\nu - \partial_\nu A_\mu) \dot{z}^\nu + e A_\mu \delta z^\mu \bigg|_{\partial \Gamma}
\]

\[\delta I = 0 \Rightarrow F_{\mu\nu}(z) \dot{z}^\nu = 0\]

The classical orbits are those with zero Lorentz force:

\[E + \frac{\partial}{\partial t} \times B = 0\]

the electric and magnetic forces cancel each other out.

\textit{N.B.:} In order to obtain the equation of motion, \(z\) must satisfy periodic boundary conditions.

\[A_\mu(z) \delta z^\mu \bigg|_{\partial \Gamma} = 0\]
A bit about the quantum theory

\[ Z[z] = \int [\mu(z)] \exp\left(\frac{i}{\eta} I[z]\right) = \int [\mu(z)] \exp\left(\frac{ie}{\eta} \int A(z)\right) \]

Thus, the integral is dominated by those orbits for which the holonomies are quantized:

\[ \frac{e}{\eta} \oint_{S^1} A(z) = 2n\pi \Rightarrow e \int_{D^2} F = 2n\pi\eta \]

Flux quantization

Does this describe a physically sensible system? What are the degrees of freedom?
This action describes a mechanical system:

\[ I[z] = e \int_{\Gamma} A_{\mu}(z) \mathcal{A} d\tau \]

Let, \( z^\mu = (z^0 = t, z^i), \ i=1,2,\cdots,2s \).

\[ I[z] = e \int A(z) = e \int_{\Gamma} [A_0(z) + A_i(z) \mathcal{A}] dt, \]

This action describes a mechanical system:

\[ I[z] = \int_{\Gamma} [p_i \mathcal{A} - H(z)] dt , \]

where \( eA_i(z) = p_i \) \quad (2^{\text{nd}} \text{ class constraints}),

and \( eA_0(z) = -H \)

The equations of motion are Hamilton’s

\[ F_{ij}(z) \mathcal{A}^j = E_i(z) \implies \varepsilon_{ij} \mathcal{A}^j = \partial_i H(z) \]
What is the meaning of the flux quantization?

\[ e \oint_{S^1} A(z) = 2n \pi \eta \]

Substituting \( eA_0(z) = -H \) and \( eA_i(z) = p_i \),

\[ \oint_{S^1} [p_i dq^i - H dt] = 2n \pi \eta \]

Bohr-Sommerfeld quantization rule

• Any mechanical system with \( s \) degrees of freedom can be described by a 0+1 C-S action in a \((2s+1)\)-dimensional target space. (Jackiw-Percacci ’87)
Invariance under canonical transf.

Gauge invariance

Invariance under time reparametrizations

Vanishing Lorentz force

Hamilton’s equations

Gen. coordinate transformations

Bohr-Sommerfeld quantization

Flux/holonomy quantization
A CS action in 2+1 dimensions can be viewed as a coupling between a brane and an external gauge field.

\[ I[z] = \int j^{\mu\nu\lambda} (A_\mu \partial_\nu A_\lambda) d^n x \]

\[ = \kappa \int_A A(z) dA(z) \]

Invariant under:
- General coordinate transformations on the worldvolume \( z^\mu \rightarrow z'^\mu(z) \)
- Lorentz transformations on the target space \((\Lambda^\mu_\nu(z))\)
- Gauge transformations \(A \rightarrow A + d\Omega\) [quasi-invariant]
For $D=3, 5, 7,...$ New possibilities arise:

- Nonabelian algebras

- $A(z)$ can be dynamical (propagating in the worldvolume)

- Worldvolume dynamics (gravity)

- Degeneracy (for $D \geq 5$)

- Quantization? (open problem)
2. CS action in $2n+1$ dimensions
Non abelian CS action in $2n+1$ dimensions

$$I[A] = \kappa \int_{M^{2n+1}} A \wedge (dA)^n + c_1 A^3 \wedge (dA)^{n-1} + \cdots + c_n A^{2n+1}$$

The coefficients $c_1, \ldots, c_n$ are fixed rational numbers,

$$L_{2n+1}^{\text{CS}}(A) = (n+1)\kappa \int_0^1 dt \langle A \wedge F_t^{n+1} \rangle,$$

where

$$F_t = tdA + t^2 A^2$$

Invariant under gauge transformations (up to boundary terms)

$$A' = g^{-1}Ag + g^{-1}dg, \quad g(x) \in G$$
# Classical CS dynamics

\[ I[A] = \int_{\Gamma} L(A) \]

<table>
<thead>
<tr>
<th>( L )</th>
<th>( \delta L = 0 )</th>
<th>( A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0+1</td>
<td>0</td>
<td>Arbitrary connection</td>
</tr>
<tr>
<td>( eA )</td>
<td>0</td>
<td>( \delta A = d\Omega )</td>
</tr>
</tbody>
</table>

| 2+1 | 0 | Pure gauge, nonpropagating, nondegenerate |
| \( \kappa \langle A dA + \frac{2}{3} A \wedge A \wedge A \rangle \) | 0 | \( F = 0 \) |

| 2n+1 | 0 | Nontrivial, propagating, degenerate |
| \( \kappa \langle A (dA)^n + c_1 A^3 (dA)^{n-1} + \cdots + c_n A^{2n+1} \rangle \) | 0 | \( F^{n+1} = 0 \) |
Degeneracy of CS theories ($D=2n+1 \geq 5$)

The problem arises from the fact that for $D=2n+1$ with $n \geq 2$, the field equations are nonlinear in the curvature,

$$\left\langle G_k F^n \right\rangle = 0$$

where $G_k$ are the generators of the Lie algebra.

The linearized perturbations around a given classical configuration $F_0$, obey

$$\left\langle G_k F_0^{n-1} \delta F \right\rangle = 0$$

The dynamics depends on the form of $F_0$. 
Consequences of the degeneracy

• Unpredictability of evolution
• Freezing out of degrees of freedom
• Loss of information about the initial data
• Irreversibility of evolution
<table>
<thead>
<tr>
<th>$D/source$</th>
<th>$Quantization$</th>
<th>$Comment$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0+1$ point particle (0-brane)</td>
<td>$e \oint A = e \iint F = 2k \pi \eta$</td>
<td>Holonomy/Flux quant. Dirac quant. condition Bohr-Sommerfeld rule</td>
</tr>
<tr>
<td>$2+1$ membrane</td>
<td>$A = g_k^{-1} dg_k, \ g \in \text{Top. class}$</td>
<td>Finite, power- counting renormalizable</td>
</tr>
<tr>
<td>$2n+1$ $2n$-brane</td>
<td>Holonomies + local deg. of f.</td>
<td>Anybody’s guess</td>
</tr>
</tbody>
</table>

How?
3. CS Gravity actions in $2n+1$ dimensions
1. **Equivalence Principle:**
   - Spacetime is locally approximated by Minkowski space and has the same (local) Lorentz symmetry.
   - GR is the oldest known nonabelian gauge theory; gauge group $SO(3,1)$.

2. Gravitation should be a theory whose output is the spacetime geometry. Therefore, it is best to start with a theory that makes no assumptions about the local geometry.

1&2 ➔ Chern-Simons theory is probably a better choice
There are two characteristic classes associated to the rotation groups $SO(s,t)$: the Euler and the Pontryagin classes. Associated with each of them there are the corresponding CS actions

**Action (first order formalism):**

\[ I[e, \omega] = \kappa \int_{M^D} L_D \ (e, de, \omega, d\omega) \]

- **Vielbein** (metricity)
- **Connection** (parallelism)

\[ R^{ab} = d\omega^{ab} + \omega^a_c \wedge \omega^{cb} = \text{Curvature 2-form} \]

\[ T^a = de^a + \omega^a_b \wedge e^b = \text{Torsion 2-form} \]
Steps for constructing CS gravities:

1. Combine the vielbein and spin connection into a connection for the $dS$, $AdS$, or Poincaré group:

$$A = l^{-1} e^a J_a + \frac{1}{2} \omega^{ab} J_{ab},$$

2. Select the bracket that corresponds to the invariant characteristic class to be used (Euler or Pontryagin), e.g.,

$$\langle J_{a_1} J_{a_2 a_3} \ldots J_{a_{D-1} a_D} \rangle = \epsilon_{a_1 \ldots a_D}$$

3. Write down the lagrangian

$$L(e, \omega) = (n + 1) \int_0^1 dt \langle A \wedge F_t^n \rangle$$
CS gravities ($D=2n-1$) (summary)

- No dimensionful constants (scale invariant)
- No arbitrary adjustable/renormalizable constants
- Possess black hole solutions $\Lambda = -\# l^{-D/2} < 0$
- Admit $\Lambda \neq 0$ and $\Lambda \to 0$ ($l \to \infty$) limit
- Admit SUSY extensions for $\Lambda \leq 0$ and any odd $D$, and yield field theories with spins $\leq 2$ only
- Give rise to acceptable $D=4$ effective theories
4. Coupling to matter sources
It is a beautiful feature of CS theories the fact that they possess no free adjustable coupling constants.

...and it is also one of the difficulties when trying to make sense of them

\[ I[A] = \kappa \int \left[ A \wedge (dA)^n + c_1 A^3 \wedge (dA)^{n-1} + \cdots + c_n A^{2n+1} \right] \]

Quantized

Fixed rational numbers \( \sim O(1) \)

But doesn’t mean one cannot have interactions
Nothing prevents putting together CS actions of different dimensions,

\[ I[A] = \sum_{r=0}^{\infty} \kappa_r \int_{M^{2r+1}} \left( A \wedge (dA)^r + c_1 A^3 \wedge (dA)^{r-1} + \cdots + c_n A^{2r+1} \right) \]

But, what would this mean?

Consider the simplest case:

\[ I[A] = \kappa_3 \int_{M^{2+1}} \left( A \wedge dA + \frac{2}{3} A^3 \right) + \kappa_1 \int_{M^{0+1}} \bar{A} \]

where \( \bar{A} \) is the restriction of \( A \) to an abelian subalgebra

\[ \bar{A} \in G_0 \subset G \]
The effects of coupling to this 0-brane are

1. Breaking the symmetry: \( \mathcal{G} \rightarrow \mathcal{G}_0 \)

2. Introducing a topological defect.

For example, if \( \mathcal{G} = \text{SO}(2,2) \) the 3d part describes 2+1 gravity, and the defect is a sort of ‘conical’ singularity…
Under closer scrutiny, it can be seen that the defect is the result of an identification by an abelian subgroup of the $AdS_3$ group: a 2+1 black hole!

The mass and angular momentum are related to the strength of the coupling constant ($\kappa_1$) and the particular subgroup of $SO(2,2)$ that is used.

Coupling more $2n$-branes in this way, more complex structures can be produced (black holes, branes, ...?)
5. Summary
• CS actions have been used in physics much longer than one usually thinks: e-m coupling, all of classical mechanics!
• CS theories can be viewed as boundary theories coming from topological field theories in even $D=2n$ manifolds.
• They have no free adjustable parameters and require no metric structure.
• Degeneracy for $D \geq 5$: limited predictability, irreversible loss of degrees of freedom, dynamical dimensional reduction.
• There exist CS (super-) gravities with dimensionless couplings, all fields have spins $\leq 2$ and the metric is not a fundamental field but a condensate…
CS theories are so exceptional, it’s not only worth studying them. It is also understandable if one looses his mind because of them…

- The natural way to couple CS theories is to $2n$-branes.
- The different branes produce topological defects of co-dimension 2, 4,…
- They break supersymmetry down to $\frac{1}{2}$, $\frac{1}{4}$, … of the one in the highest dimensional CS form.
- …..
Thanks!