Extending Bubbling AdS: Going Beyond the ½ BPS Sector

Sera Cremonini
University of Michigan

UC Davis, February 1st, 2007
Outline

- Basics of AdS/CFT
- Problem with Interactions in AdS/CFT
  - Interactions on the boundary vs. in the bulk
- AdS/CFT beyond small perturbations
  - The ½ BPS sector and the one matrix model (bubbling)
  - The role of collective field theory
- Going beyond the ½ BPS sector
  - Multi-matrix models: interactions revisited
  - Bubbling with less SUSY? The ¼ and ⅛ BPS sectors
- Towards Dynamics? (directions for the future)
The AdS/CFT Conjecture

“Holographic” Duality:

Theory of Gravity \rightleftharpoons Conformal field theory on boundary

IIB string theory on AdS$_5 \times$ S$^5 \rightleftharpoons$ $\mathcal{N}=4$ U(N) SYM in 3+1 d

Strong / Weak Coupling Duality:

$\lambda = g_{YM}^2 N \sim g_s N \sim R^4/l_s^4$

Trust perturbative analysis in YM theory when

Classical gravity description reliable when

$\lambda = g_{YM}^2 N \sim g_s N \sim \frac{R^4}{l_s^4} \ll 1$

$\frac{R^4}{l_s^4} \sim g_s N \sim g_{YM}^2 N \gg 1$

(and $N$ large)
**Type IIB SUGRA on AdS$_5 \times S^5$**

- SO($2,d$) symmetry of d-dim conformal group
- SO($6$) R-symmetry
- SO($2,d$) isometry group of AdS$_{d+1}$
- SO($6$) isometry of S$^5$

**Hyperboloid**

$$X_0^2 + X_{d+1}^2 - \sum_{i=1}^{d} X_i^2 = R^2$$

**Simple example: AdS$_3$ in global coordinates**

$$ds^2 = \frac{R^2}{\cos^2 \theta} (-d\tau^2 + d\theta^2 + \sin^2 \theta \, d\Omega^2)$$

Boundary is at $\theta = \pi/2$: $R \times S^1$
My favorite AdS picture

$S^5$ at each point

Boundary

Interior or “bulk”
AdS/CFT beyond SUGRA: pp-wave background

So far correspondence only between SUGRA and SYM (no strings)

**Progress:** Extending AdS/CFT to string theory

*Why plane-wave background?*

String propagation on **pp-wave background** can be solved exactly

Green-Scharwz light-cone action becomes **quadratic** can be quantized

**STRING THEORY on pp-waves**

AdS/CFT

Sector of CFT (large R-charge)
Focus on “particle” moving very rapidly (large J) along $\psi$ and sitting near $\rho = \theta = 0$

Systematically:

\[
\begin{align*}
\tilde{x}^\pm &= \frac{t^\pm \psi}{2} \\
 x^+ &= \tilde{x}^+, \quad x^- = R^2 \tilde{x}^- , \quad \rho = \frac{r}{R} , \quad \theta = \frac{y}{R} , \quad R \to \infty
\end{align*}
\]

\[
ds^2 = -4dx^+ dx^- - (\vec{r}^2 + \vec{y}^2)(dx^+)^2 + d\vec{y}^2 + d\vec{r}^2
\]

Main result of BMN: matching of SPECTRUM in large J limit (large R charge)

What about interactions?
Cubic Interactions

Simple model of interactions:

\[ S = \int d^5x \sqrt{g} \left[ \sum_i \frac{1}{2}(\partial \phi_i)^2 + \frac{1}{2}m_i^2\phi_i^2 + \lambda \phi_1 \phi_2 \phi_3 \right] \]

For fields on boundary of AdS, well-defined prescription (GKP-W prescription):

\[ \langle \mathcal{O}_1 \mathcal{O}_2 \mathcal{O}_3 \rangle = \int d^5x \sqrt{g} K_{\Delta_1}(x; \bar{x}_1) K_{\Delta_2}(x; \bar{x}_2) K_{\Delta_3}(x; \bar{x}_3) \]

\[ \langle \mathcal{O} \mathcal{O} \rangle = \frac{1}{|x_1 - x_2|^{2\Delta}} \]

Position of vertex is integrated over AdS
Interactions inside AdS?

**Prescription** for calculating interactions **INSIDE** AdS?

In pp-limit **boundary is lost**

There should be a dictionary **BUT** **bulk-boundary** prescription may **not be fundamental or complete**

Approach: construct and study the Hamiltonian
So far...

- To understand AdS/CFT need to set up a **precise dictionary** between states of two theories.

- In “original” AdS/CFT **perturbations** on AdS$_5 \times S^5$

Can we go **beyond the perturbative description**?

We may consider SUGRA **solutions** that are **asymptotically AdS$_5 \times S^5$** as **GOOD CANDIDATES for dual states** in the CFT.

**Hope:** carry out this program in the **FULL BPS sector** of the respective theories.

First step in this direction:

**dictionary for $\frac{1}{2}$ BPS sector** of Type IIB string theory (LLM, hep-th/0409174)
What about the problem with AdS interactions?

**Natural question**: what is the appropriate Hamiltonian?

We will construct the Hamiltonian for:
- $\frac{1}{2}$ BPS sector of the theory (well-known)

Motivate $H$ for **more general geometries** (work in progress)
LLM:
- constructed **exact $1/2$ BPS solutions** in type IIB SUGRA
- identified them with the **free fermion picture** of $1/2$ BPS sector of $N = 4$ SYM

Explicit, **regular** solutions with $SO(4) \times SO(4)$ isometry

**Chiral primaries** in $\mathcal{N} = 4$ SYM ($\Delta = J$)

**fermionic droplets**

\[ \text{geometry} \leftrightarrow \text{Droplet shape} \]
General idea:

10D Spacetime of form

\[ ds_{10}^2 = g_{\mu\nu} dx_\mu dx_\nu + ye^G d\Omega_3^2 + ye^{-G} d\tilde{\Omega}_3^2 \]

Time-like Killing vector \( \implies \) 3 dimensions really “matter”

Crucial feature: all solutions describable in terms of a single function \( z(x_1, x_2, y) \):

\[
\left( \partial_1^2 + \partial_2^2 + y \partial_y \frac{1}{y} \partial_y \right) z(x_1, x_2, y) = 0
\]

Regularity of solutions demands certain boundary conditions on \( y=0 \) plane:

\[ z(x_1, x_2, y = 0) = \pm \frac{1}{2} \]

meaning of \( y \)
(one sphere shrinking smoothly)
Smoothness of solutions: on $y=0$ plane

black and white color coding of solutions

Boundary conditions on $y = 0$ plane specify geometry:

$$z(x_1, x_2, y) = \frac{y^2}{\pi} \int_D \frac{z(x_1', x_2', 0)\,dx'_1\,dx'_2}{[(x - x')^2 + y^2]^2}$$

$$\partial_i \partial_i z + y \partial_y \left( \frac{\partial_y z}{y} \right) = 0$$

Bubbling!

↑ LINEAR!

Where is the fermion description?
We will see in detail:

“Special” 2D plane \((y=0)\) identified with **phase space of fermions**

Fermion droplets (\(=\) geometries):

\[
\text{AdS}_5 \times S^5 \quad \text{(ground state)}
\]
Relation between LLM ansatz and matrix model

Solutions are BPS, thus \( \Delta = U(1)_R \) charge \( = J \)

Angular momentum and flux for all LLM solutions given by:

\[
\Delta = J = \frac{1}{16 \pi^3 l_p^8} \left[ \int d^2 x (x_1^2 + x_2^2) - \frac{1}{2\pi} \left( \int d^2 x \right)^2 \right]
\]

\[
N = \int d^2 x
\]

\( \frac{1}{2} \) BPS Sector described by matrix model with harmonic oscillator potential

Will show this via connection with collective field theory
Matrix Model: Reduction to ½ BPS Sector

Why only 1 matrix for ½ BPS sector?

Start from two-matrix model:

\[ H = \frac{1}{2} \text{Tr}(P_1^2 + P_2^2 + X_1^2 + X_2^2) \quad J = \text{Tr}(P_1 X_2 - P_2 X_1) \]

Rewrite in different way: introduce complex matrices

\[ Z = \frac{1}{\sqrt{2}} (X_1 + iX_2) \quad \Pi = \frac{1}{\sqrt{2}} (P_1 + iP_2) = -i \frac{\partial}{\partial Z^\dagger} \]
\[ Z^\dagger = \frac{1}{\sqrt{2}} (X_1 - iX_2) \quad \Pi^\dagger = \frac{1}{\sqrt{2}} (P_1 - iP_2) = -i \frac{\partial}{\partial Z} \]

With \[ A = \frac{1}{2} (Z + i\Pi), \quad B = \frac{1}{2} (Z - i\Pi) \]

\[ H = \text{Tr}(A^\dagger A + B^\dagger B), \quad J = \text{Tr}(A^\dagger A - B^\dagger B). \]
Constructing states:

\[ \text{Tr}(A^\dagger)^n \ket{0}, \quad E = J = n, \]
\[ \text{Tr}(B^\dagger)^n \ket{0}, \quad E = -J = n, \]
\[ \text{Tr}(A^\dagger)^n \text{Tr}(B^\dagger)^m \ket{0}, \quad E = n + m, \quad J = n - m. \]

\( H = \text{Tr}(A^\dagger A + B^\dagger B), \)
\( J = \text{Tr}(A^\dagger A - B^\dagger B). \)

\( E \) is not \( J \)!

\( \frac{1}{2} \) BPS Sector

(E=J)

\( \quad \leftrightarrow \quad \)

Truncation to A oscillators

(no B oscillators) so SINGLE MATRIX

These states correspond to:

chiral primary operators: \( e.g. \quad \text{Tr} Z^{k_1} \text{Tr} Z^{k_2} \ldots \text{Tr} Z^{k_n} \)

BUT a theory of a single matrix can be described by a collective field theory

Fermion description starts emerging
Collective Field Theory Description

**Simple matrix model**

\[ L = \frac{1}{2} \text{Tr}(\dot{M}^2 - M^2) \]

**diagonalize** \( M(t) \) → \( \lambda_i(t) \)  

**Collective field** \( \phi(x, t) = \sum_{i=1}^{N} \delta(x - \lambda_i(t)) \)

In large \( N \) limit appearance of new spatial dimension

Gravity is “EMERGING” (collective phenomenon)

**dynamics of** \( M(t) \)

\[ H_{col} = \int dx \left( \frac{1}{2} \partial_x \Pi \phi \partial_x \Pi + \frac{\pi^2}{6} \phi^3 + \frac{1}{2} (x^2 - \mu) \phi \right) \]

(normalization) \[ \int dx \phi(x) = N \]

Das-Jevicki
We found 

\[ H_{coll} = \int dx \left( \frac{1}{2} \partial_x \pi \phi \partial_x \Pi + \frac{\pi^2}{6} \phi^3 + \frac{1}{2} (x^2 - \mu) \phi \right) \]

Introduce new fields:

\[ \alpha_{\pm}(x, t) = \partial_x \Pi \pm \pi \phi(x, t) \]

\[ H = \int dx \int_{\alpha_-}^{\alpha_+} d\alpha \frac{1}{2} (\alpha^2 + x^2 - \mu) \]

\[ \Delta = J = \frac{1}{16 \pi^3 l_p^8} \left[ \int_D d^2 x (x_1^2 + x_2^2) - \frac{1}{2\pi} \left( \int_D d^2 x \right)^2 \right] \]

\[ N = \int d^2 x \]

With \( \mu \sim N \) we have matching!

Collective field theory description for ½ BPS SUGRA states

Appropriate Hamiltonian for ½ BPS states!
Back to interactions: the collective field theory cubic vertex

**Fluctuations**

\[ \phi(x, t) = \phi_0(x) + \frac{1}{\sqrt{\pi}} \partial_x \eta(x, t) \]

“Time of flight” coordinate \( \tau = \int \frac{dx}{\phi_0}, \quad 0 < \tau < \pi \)

\[
H = \int d\tau \left[ \frac{1}{2} \Pi^2 + \frac{1}{2} (\partial_\tau \eta)^2 + \frac{1}{6\pi^2 \phi_0^2} (\partial_\tau \eta)^3 + 3\Pi \partial_\tau \eta \Pi \right]
\]

**Static ground state**

\[ \pi \phi_0 = \sqrt{\mu - x^2} \]

**This is what we are interested in**

**Some manipulations:**

\[
\begin{align*}
\alpha &= \alpha_+ - \pi \phi_0, \quad \tau > 0 \\
&= -\alpha_- - \pi \phi_0, \quad \tau < 0 \\
&\quad \quad \quad -\pi < \tau < \pi
\end{align*}
\]

\[ H^{(3)} = \int_{-\pi}^{\pi} \frac{d\tau}{\phi_0^2} \alpha^3(\tau) \]

\[ H^{(3)} = -4\pi \sqrt{n_1 n_2 n_3} (n_1 + n_2 - n_3) a_1 a_2 a_3^\dagger + \ldots \]

This matches the corresponding GR calculation! (see next)
Derivative couplings can be removed by a field redefinition:

\[
(n_\mu n_\mu - m_1^2) s^I = \sum_{J,K} \left( D_{IJK} s^J s^K + E_{IJK} n_\mu s^J n_\nu s^K + F_{IJK} n_\mu n_\nu s^J n_\mu n_\nu s^K \right)
\]

AdS$_5$

Chiral primary $s^I$ with mass $m^2 = j(j-4)$.

Highest-weight states

\[ s = \frac{\sqrt{\Delta (\Delta - 1)}}{\pi (\cosh \mu)^\Delta} \]

\[ < 3 | H_3 | 12 > \sim (\Delta_3 - \Delta_1 - \Delta_2) \sqrt{\Delta_1 \Delta_2 \Delta_3} \delta(j_3 - j_1 - j_2) \]

matches Coll. F.T. cubic vertex

\[ H^{(3)} = -4\pi \sqrt{n_1 n_2 n_3} \left( n_1 + n_2 - n_3 \right) a_1 a_2 a_3^\dagger \]
Going Beyond the $\frac{1}{2}$ BPS Sector

What have we seen so far?

$\frac{1}{2}$ BPS states $\leftrightarrow$ Fermions in harmonic oscillator (collective field theory description)

single-matrix states

Next?

Study **more general states** (outside of $\frac{1}{2}$ BPS Sector)

multi-matrix states

“Brute force” approach to interactions challenging

Alternative approach?
Interactions for two-matrix states

General strategy for reconstructing full AdS interaction:

- **Start from collective field theory vertex**  \( V_3 \neq 0 \)
  (assume correct description for multi-matrix states)

- Use SL(2,R) symmetry of underlying theory to **generate interactions**
  (find **useful identities** using generators that relate vertices that we know to vertices we don’t know)

**Feature of SUGRA:**

\[
V_3 \sim (\Delta_3 - \Delta_1 - \Delta_2) \delta(j_1 + j_2 - j_3)
\]

**vanishes on shell for highest-weight state**

**Meaning** of acting with generators?

**Rewrite the vertex** so that it does not vanish
(cannotical transformation and non-linear field redefinition)

S.C., A. Jevicki, R. de Mello Koch
hep-th/0702???
Warm up: a toy model

Consider **particle in two dimensions** as toy model for two matrix states

\[ H = -\frac{1}{2} \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{\omega^2}{2} (x^2 + y^2) \]

\[ H = a^\dagger a + b^\dagger b \]

\[ J = a^\dagger a - b^\dagger b \]

**States**

\[ |J, n > = \frac{(a^\dagger)^J (b^\dagger)^n}{\sqrt{(J + n)! n!}} |0 > \]

**Recall**

\[ |J, n > = \frac{1}{\sqrt{(J + n)n}} L_+ |J, n - 1 > \]

**Acting with** \( L_0, L_+, L_- \) **on**

\[ \int d^2x \frac{1}{\sqrt{\phi_0(\vec{x})}} \bar{\Psi} J_{1,n_1} \Psi J_{2,n_2} \Psi J_{3,n_3} \]

\[ \int d^2x \frac{1}{\sqrt{\phi_0(\vec{x})}} \left[ L_+^{(3)} - L_-^{(1)} - L_-^{(2)} + \frac{1}{2} (L_0^{(3)} - L_0^{(1)} - L_0^{(2)}) \right] \bar{\Psi} J_{1,n_1} \Psi J_{2,n_2} \Psi J_{3,n_3} = 0 \]
Two matrix interactions in $\text{AdS}_5 \times S^5$

Repeat same procedure but for Hamiltonian

$$H = -\frac{1}{2} \left( \frac{\partial^2}{\partial M^2} + \frac{\partial^2}{\partial N^2} \right) + \frac{1}{2} (M^2 + N^2)$$

**Eigenfunctions** of two-matrix model found by A. Donos, A. Jevicki, J. Rodrigues (hep-th/0507124):

$$H = -\frac{1}{2} \frac{\partial^2}{\partial M^2} + \frac{1}{2} M^2 + B \frac{\partial}{\partial B}$$

**AdS Result**:

Build vertex, act with $\text{SL}(2,\mathbb{R})$ generators, and find **analog of toy model identity**

**Bottom line**: Symmetries may help
Comment on “emergent geometry”

Recall eigenvalues of one matrix yielded a new dimension:

\[ M(t) \rightarrow \Phi(x, t) = \sum_i \delta(x - \lambda_i) \]

Add another matrix \( N(t) \) \( \Rightarrow \) Probe additional direction \( y \)

LLM used ONE MATRIX to describe AdS\(_5 \times S^5\)

With two matrices, hope to eventually probe radial direction of AdS

**BUT** still challenge
Detailed bubbling picture of $1/2$, $1/4$, $1/8$ BPS states?

Work in progress by

Natural questions:

- Can we extend AdS/CFT dictionary to full BPS spectrum?
- Is there bubbling if you have LESS SUSY ($1/4$ BPS and $1/8$ BPS sectors)?
- What is the GAUGE THEORY picture? Fermions?

General SUGRA ansatz for $1/4$ BPS and $1/8$ BPS geometries worked out by

A. Donos
hep-th/0606199, hep-th/0610259

N. Kim,
hep-th/0511029

But here boundary conditions (possible bubbling picture) missing!
(hard to get explicit solutions)
Gravity picture that has emerged:

$\frac{1}{2}$ BPS

\[ ds^{2}_{1/2} = -h^{-2}(dt+V)^2 + h^2(dy^2 + dx_1^2 + dx_2^2) + ye^G d\Omega_3^2 - ye^{-G} d\tilde{\Omega}_3^2 \]

BC's here
(droplets on 2D plane)

$4D \times S^3 \times S^3$

$\frac{1}{4}$ BPS

\[ ds^{2}_{1/4} = -h^{-2}(dt+W)^2 + h^2 dy^2 + \frac{1}{ye^G} ds_4^2 - ye^G d\Omega_3^2 + ye^{-G} d\psi^2 \]

Expect BC's here

$6D \times S^3 \times S^1$

$\frac{1}{8}$ BPS

\[ ds^{2}_{1/8} = -e^{2\alpha}(dt+w)^2 + e^{-2\alpha} ds_6^2 + e^{2\alpha} d\Omega_3^2 \]

$7D \times S^3$

For smoothness look at collapsing spheres!
Difficulties with ¼ BPS construction:
- solving equation for K is challenging (explicit solutions?)
- even hard to reproduce simple ½ BPS states

¼ BPS solution STILL depends on only one function z (as in ½ BPS case):

\[ ds_{1/4}^2 = -h^{-2} (dt+W)^2 + h^2 dy^2 + \frac{1}{ye^G} ds_4^2 + ye^G d\Omega_3^2 + ye^{-G} d\psi^2 \]

\[ z = -2 y \partial_y \left( \frac{1}{y} \partial_y K \right) \]

\[ (g_{m n}^{4D} = \partial_m \partial_n K) \]

\[ \left| \begin{array}{cc}
\partial_z \partial_{\bar{z}} K & \partial_{\bar{w}} \partial_{\bar{z}} K \\
\partial_z \partial_{\bar{w}} K & \partial_{w} \partial_{\bar{w}} K \\
\end{array} \right| = ye^{2 \frac{2}{y} \partial_y K} \left( -2 y \partial_y \left( \frac{1}{y} \partial_y K \right) + 1 \right) \]
Natural Question:
Is there an analog of the “special” 2D plane of the ½ BPS solutions, on which boundary conditions (for regularity) would be defined?

The answer is yes for examples worked out:
- We embedded many known SUGRA solutions into ¼ and 1/8 BPS general geometries
- Found some new solutions for simplifying assumptions on K
- Found relevant boundary conditions

Picture:
Questions still open:

- Can you draw any shape in these 4D and 6D spaces and get a UNIQUE geometry?
- What do we have on gauge theory side? Fermions?

How can this be useful?

- Push forward AdS/CFT duality with less SUSY (and more general geometries)
- Can we understand more realistic gravity/gauge theory dualities starting from the more “controlled” setting of AdS/CFT? (dS/CFT?)
- Can we learn anything about time dependence? Hard question
- Black hole applications
Future Applications?

We saw LLM describes many vacua of the theory:

- **Instanton solutions** interpolating between different LLM vacua?
  (recent work by H. Lin)

- Bubbles merging or separating (**topology change**)?

Can we make **bubbles fluctuate in time**?

Yes, but small fluctuations not necessarily new (just spectrum)

\[ r(\phi, t) = \sum_n a_n(t) e^{in\phi} \]

\[
\text{Known spectrum of } \quad \text{AdS}_5 \times S^5 \quad (\text{Kim et al.})
\]
Topology change: can bubbles split or recombine?

If yes, at transition they would locally look like:

Can we resolve the singularity? Can we describe topology change?
Conclusions…

- Although AdS/CFT is still a conjecture, much progress recently

- **Nice fermion** (bubbling) **picture** for $\frac{1}{2}$ BPS SUGRA solutions

- **Interactions** in bulk are challenging, but **symmetries** may help (they give useful identities for generating **multi-matrix** interactions)

- **Bubbling** picture may survive with less SUSY

Future directions:
- Can we make any progress **beyond static solutions**? Connect with cosmology work?
- Can we understand whether bubbles can merge and separate?
- Can we go **from one vacuum to the other** (possibly relevant for cosmology)?