Worldline Path Integral Formalism
new results and applications

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Outline of the talk

Mainly based on 0205182, 0211134, 0312064, 0503155, 0510010, 0612236, 0701055
Bastianelli, Benincasa, OC, Giombi, Latini, Pisani, Zriottı

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2. Worldline formalism in flat space
   - the case of scalar QED
     → 1D path integral in flat space
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2. Worldline formalism in flat space
   - the case of scalar QED
     → 1D path integral in flat space

3. Worldline formalism in curved space
   - 1-loop effective action for a scalar field
     → 1D path integral in curved space
   - UV regularization of the path integral
   - IR aspects: zero modes on the circle $S^1$
Outline of the talk

4. Worldline formalism with local SUSY’s
   - Spinning particle w/ N=1 $\Rightarrow$ spin-$\frac{1}{2}$ field
   - Spinning particle w/ N=2 $\Rightarrow$ spin-1 field
     $\rightarrow$ coupling to gravity OK
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   - Spinning particle w/ N=1 ⇒ spin-$\frac{1}{2}$ field
   - Spinning particle w/ N=2 ⇒ spin-1 field
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5. $SO(N)$ Spinning particle w/ N>2 ⇒ spin-$\frac{N}{2}$ field
   - one-loop qzn in flat space
   - Dof’s from orthogonal polynomials method $\forall D, N$
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   - Spinning particle w/ N=2 \Rightarrow \text{spin-1 field}
     \Rightarrow \text{coupling to gravity OK}

5. $SO(N)$ Spinning particle w/ N>2 \Rightarrow \text{spin-}\frac{N}{2} \text{ field}
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6. Manifolds with boundary
   - Method of the “image charge”
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7. Outlook
Introduction

Worldline method
QFT results from QM path integrals
⇒ no need to compute momentum integrals and Dirac traces

Alternative way to compute correlation functions
Introduction

Worldline method
QFT results from QM path integrals
⇒ no need to compute momentum integrals and Dirac traces

Alternative way to compute correlation functions
Effective actions of quantum fields coupled to external fields (gravity, vector), chiral and conformal anomalies
Worldline formalism in flat space

- Case of scalar contribution to QED at 1-loop
Worldline formalism in flat space

- Case of scalar contribution to QED at 1-loop
- Classical action:

\[ S[\phi, \phi^*, A] = \int d^D x \left( |(\partial_\mu + ieA_\mu)\phi|^2 + m^2|\phi|^2 \right) \]
Worldline formalism in flat space

Case of scalar contribution to QED at 1-loop

Classical action:

\[ S[\phi, \phi^*, A] = \int d^Dx \left( |(\partial_\mu + ieA_\mu)\phi|^2 + m^2|\phi|^2 \right) \]

The corresponding 1-loop effective action is

\[ e^{-\Gamma[A]} = \int D\phi D\phi^* e^{-S[\phi, \phi^*, A]} = \text{Det}^{-1}(-\nabla^2_A + m^2) \]
Thus

\[ \Gamma[A] = \text{Tr} \log (-\nabla^2_A + m^2) \]

\[ = - \int_0^\infty \frac{dT}{T} \text{Tr} \ e^{-(-\nabla^2_A+m^2)T} \]

\[ = - \int_0^\infty \frac{dT}{T} \int_{PBC} \ e^{-\int_0^T d\tau \left( \frac{1}{4} \dot{x}^2 + ieA_\mu(x)\dot{x}^\mu + m^2 \right)} \]

\[ = \sum \]
Worldline formalism in flat space

Thus

\[ \Gamma[A] = \text{Tr} \, \log \left( -\nabla_A^2 + m^2 \right) \]

\[ = - \int_0^\infty \frac{dT}{T} \text{Tr} \, e^{(-\nabla_A^2+m^2)T} \]

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quantum mechanical path integrals
Expand in powers of $A_\mu$ (sum of plane waves)

$$A_\mu = \sum_{i=1}^{N} \varepsilon_{i,\mu} e^{i p_i \cdot x}$$
Expand in powers of $A_\mu$ (sum of plane waves)

$$A_\mu = \sum_{i=1}^{N} \varepsilon_{i,\mu} e^{ip_i \cdot x}$$

get averages of “photon vertex operators”

$$\left\langle \varepsilon_{1,\mu_1} \dot{x}^{\mu_1}(\tau_1) e^{ip_1 \cdot x(\tau_1)} \cdots \varepsilon_{N,\mu_N} \dot{x}^{\mu_N}(\tau_N) e^{ip_N \cdot x(\tau_N)} \right\rangle$$
Worldline formalism in flat space

- Expand in powers of $A_\mu$ (sum of plane waves)

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- and obtain the “Bern-Kosower master formula”
Bern-Kosower master formula

\[ \Gamma[p_1, \varepsilon_1; \ldots; p_N, \varepsilon_N] = \]
Bern-Kosower master formula

\[
\Gamma[p_1, \varepsilon_1; \ldots; p_N, \varepsilon_N] = -(-ie)^N (2\pi)^D \delta^D \left( \sum_{i=1}^{N} p_i \right)
\]

\[
\int_0^\infty \frac{dT}{T} \frac{e^{-m^2 T}}{(4\pi T)^{D/2}} \prod_{i=1}^{N} \int_0^T d\tau_i
\]

\[
\exp \sum_{i,j=1}^{N} \left[ \frac{1}{2} \Delta_{ij} p_i \cdot p_j - i \Delta_{ij} \varepsilon_i \cdot p_j + \frac{1}{2} \Delta_{ij} \varepsilon_i \cdot \varepsilon_j \right] \bigg|_{\text{lin } \varepsilon_i}
\]
Bern-Kosower master formula

\[ \Gamma[p_1, \varepsilon_1; \ldots; p_N, \varepsilon_N] = -(-ie)^N (2\pi)^D \delta^D \left( \sum_{i=1}^N p_i \right) \]

\[ \int_0^\infty \frac{dT}{T} \frac{e^{-m^2 T}}{(4\pi T)^{D/2}} \prod_{i=1}^N \int_0^T d\tau_i \]

\[ \exp \sum_{i,j=1}^N \left[ \frac{1}{2} \Delta_{ij} p_i \cdot p_j - i \cdot \Delta_{ij} \varepsilon_i \cdot p_j + \frac{1}{2} \cdot \Delta_{ij} \varepsilon_i \cdot \varepsilon_j \right] \left| \right| \lim_{\varepsilon_i \to 0} \]

integral over the modulus of the circle

one-loop determinant for the free path integral
Worldline formalism in curved space

A real scalar field coupled to gravity

\[ S[\phi, g] = \int d^D x \sqrt{g} \frac{1}{2} \left( g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2 + \xi R \phi^2 \right) \]
A real scalar field coupled to gravity

\[ S[\phi, g] = \int d^D x \sqrt{g} \frac{1}{2} (g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2 + \xi R \phi^2) \]

produces an effective action \( (e^{-\Gamma[g]} = \int D\phi \ e^{-S[\phi, g]} ) \)

\[ \Gamma[g] = \frac{1}{2} \text{Tr} \log(-\nabla^2 + m^2 + \xi R) = \]
which can be represented as

\[ \Gamma[g] = -\frac{1}{2} \int_0^\infty \frac{dT}{T} \int_S Dx \ e^{-S[x^\mu]} \]

with

\[ S[x^\mu] = \int_0^1 d\tau \left( \frac{1}{4T} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + T(m^2 + \xi R(x)) \right) \]
Worldline formalism in curved space

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1d non-linear sigma model
The effective action

\[ \Gamma[g] = \frac{1}{2} \text{Tr} \log(-\nabla^2 + m^2 + \xi R) \]

can be obtained directly from 1\textsuperscript{st} qzn.
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can be obtained directly from 1st qzn.

start from the relativistic point-particle action

\[ S[e, x^\mu] = \int_0^1 d\tau \frac{1}{2} [e^{-1} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + e(m^2 + \xi R(x))] \]
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gauge fix the diffeomorphisms \( e = 2T \)
Worldline formalism in curved space

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gauge fix the diffeomorphisms $e = 2T$

divide out the length of the circle
Worldline formalism in curved space

1. UV regularization of the non-linear $\sigma$ model
   3 regularization schemes have been studied
   - Mode Regularization (Bastianelli, OC, Schalm, van Niuwenhuizen)
   - Time Slicing (de Boer, Peeters, Skenderis, van Niuwenhuizen)
   - Dimensional Regularization (Bastianelli, OC, van Nieuwenhuizen)
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DR allows for covariant counterterms

\[ V_{CT} = -\frac{\hbar}{8}R \]
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   DR allows for covariant counterterms

   $$V_{CT} = -\frac{\hbar}{8} R$$

2. Factorization of zero modes
   - non-covariant total derivatives
   - treated with BRST methods
Effective action from DR worldline

\[ \Gamma[g] = -\frac{1}{2} \int_0^\infty \frac{dT}{T} \int DxDaDbDc \ e^{-S} \]

with

\[ S = \int_0^1 d\tau \left( \frac{1}{4T} g_{\mu\nu}(\dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu) + T(m^2 + \bar{\xi} R) \right) \]

where \( \bar{\xi} = \xi - \frac{1}{4} \) includes the DR counterterm.
Effective action from DR worldline

\[ \Gamma[g] = -\frac{1}{2} \int_0^\infty \frac{dT}{T} \int DxDaDbDc \ e^{-S} \]

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where \( \bar{\xi} = \xi - \frac{1}{4} \) includes the DR counterterm.

- bosonic ghosts \( a \) and fermionic ghosts \( b, c \) provide the non-trivial path integral measure.
Effective action from DR worldline

Expand in \( h_{\mu\nu} = g_{\mu\nu} - \delta_{\mu\nu} \), substitute the \( h^N \) term with

\[
h_{\mu\nu} = \sum_{i=1}^{N} \epsilon_{\mu\nu}^{(i)} e^{ip_i \cdot x}
\]

and pick terms linear in \( \epsilon^{(i)} \Rightarrow \)

\( N \)-graviton amplitude in momentum space \( \Gamma^{\epsilon_1,\ldots,\epsilon_N}_{(p_1,\ldots,p_N)} \).

- Get quantum mechanical correlators of the form

\[
\left\langle (\dot{x}_{1}^{\mu_1} \dot{x}_{1}^{\nu_1} + a_{1}^{\mu_1} a_{1}^{\nu_1} + b_{1}^{\mu_1} c_{1}^{\nu_1}) e^{ip_1 \cdot x_1} \right. \\
\left. \cdots (\dot{x}_{N}^{\mu_N} \dot{x}_{N}^{\nu_N} + a_{N}^{\mu_N} a_{N}^{\nu_N} + b_{N}^{\mu_N} c_{N}^{\nu_N}) e^{ip_N \cdot x_N} \right\rangle
\]

graviton vertex operator  graviton vertex operator
Explicit computation

E.g. Two-graviton amplitude

\[ \ldots + \xi \]
Explicit computation

E.g. Two-graviton amplitude

Case $\bar{\xi} = 0$ (i.e. $\xi = \frac{1}{4}$).

Quadratic part in $h_{\mu\nu}$

$$\tilde{\Gamma}^{\epsilon_1,\epsilon_2}_{(p_1,p_2)} = \frac{1}{2} \int_0^\infty \frac{dT}{T} e^{-m^2T} \frac{1}{(4\pi T)^{D/2}} \int d^D x_0$$

$$\times \left\langle \frac{1}{2} \left[ \int_0^1 d\tau \frac{1}{4T} \left( h_{\mu\nu}(\dot{y}^\mu \dot{y}^\nu + a^\mu a^\nu + b^\mu c^\nu) \right) \right]^2 \right\rangle_{lin \; \epsilon_1,\epsilon_2}$$

where $h_{\mu\nu} = \epsilon^{(1)}_{\mu\nu} e^{ip_1 \cdot x} + \epsilon^{(2)}_{\mu\nu} e^{ip_2 \cdot x} \quad x = x_0 + y$
Explicit computation

Use Wick contractions and get

\[
\Gamma_{(p,-p)}^{\epsilon_1\epsilon_2} = \frac{1}{8} \frac{1}{(4\pi)^{D/2}} \int_0^\infty \frac{dT}{T^{1+D/2}} e^{-m^2 T}
\]

\[
\times (r_1 I_1 + r_2 I_2 + 2T p^2 (r_3 I_3 + r_4 I_4) + 4T^2 p^4 r_5 I_5)
\]

where \( r_i = \epsilon_{\mu\nu}^{(1)} R_i^{\mu\nu\alpha\beta} \epsilon_{\alpha\beta}^{(2)} \) and

\[
R_1^{\mu\nu\alpha\beta} = \delta_{\mu\nu} \delta_{\alpha\beta}, \quad R_2^{\mu\nu\alpha\beta} = \delta_{\mu\alpha} \delta_{\nu\beta} + \delta_{\mu\beta} \delta_{\nu\alpha}
\]

\[
R_3^{\mu\nu\alpha\beta} = \frac{1}{p^2} \left( \delta_{\mu\alpha} p^\nu p^\beta + \delta_{\nu\alpha} p^\mu p^\beta + \delta_{\mu\beta} p^\nu p^\alpha + \delta_{\nu\beta} p^\mu p^\alpha \right)
\]

\[
R_4^{\mu\nu\alpha\beta} = \frac{1}{p^2} \left( \delta_{\mu\nu} p^\alpha p^\beta + \delta_{\alpha\beta} p^\mu p^\nu \right), \quad R_5^{\mu\nu\alpha\beta} = \frac{1}{p^4} p^\mu p^\nu p^\alpha p^\beta
\]
Explicit computation

\[ I_1 = \int_0^1 d\tau \int_0^1 d\sigma \ (\dot{\Delta} \cdot + \Delta_{gh})|_\tau (\dot{\Delta} \cdot + \Delta_{gh})|_\sigma \ e^{-2T\dot{p}^2\Delta_0} \]

\[ I_2 = \int_0^1 d\tau \int_0^1 d\sigma \ (\dot{\Delta} \cdot^2 - \Delta_{gh}^2) \ e^{-2T\dot{p}^2\Delta_0} \]

\[ I_3 = \int_0^1 d\tau \int_0^1 d\sigma \ \dot{\Delta} \cdot \dot{\Delta} \cdot \dot{\Delta} \cdot \ e^{-2T\dot{p}^2\Delta_0} \]

\[ I_4 = \int_0^1 d\tau \int_0^1 d\sigma \ (\dot{\Delta} \cdot + \Delta_{gh})|_\tau (\Delta \cdot)^2 \ e^{-2T\dot{p}^2\Delta_0} \]

\[ I_5 = \int_0^1 d\tau \int_0^1 d\sigma \ (\dot{\Delta} \cdot)^2 (\Delta \cdot)^2 \ e^{-2T\dot{p}^2\Delta_0} \]
Explicit computation

Use (WL) dimensional regularization when necessary
Translation invariance can be used to fix $\sigma = 0$

\[
I_1 = \int_0^1 d\tau \ e^{-Tp^2(\tau - \tau^2)}
\]

\[
I_2 = \frac{1}{4}Tp^2 - 2 + I_1
\]

\[
I_3 = \frac{1}{8} - \frac{1}{2Tp^2}(1 - I_1)
\]

\[
I_4 = \frac{1}{2Tp^2}(1 - I_1)
\]

\[
I_5 = \frac{1}{8Tp^2} - \frac{3}{4T^2p^4}(1 - I_1)
\]
Explicit computation

Proper time integral can be carried out at complex $D$

\[
(4\pi)^{\frac{D}{2}} \Gamma_{(p,-p)} = -\frac{1}{8} \Gamma \left( -\frac{D}{2} \right) \left[ (m^2)^{\frac{D}{2}} (R_1 - R_2) + \left( (P^2)^{\frac{D}{2}} - (m^2)^{\frac{D}{2}} \right) (S_1 + S_2) \right]
\]

\[
- \frac{1}{32} \Gamma \left( 1 - \frac{D}{2} \right) p^2 (m^2)^{\frac{D}{2} - 1} S_2
\]

where

\[
(P^2)^a = \int_0^1 d\tau \left( m^2 + p^2 (\tau - \tau^2) \right)^a, \quad S_i \text{ transverse}
\]
Explicit computation

Additional term for the case $\bar{\xi} \neq 0$ (i.e. $\xi \neq \frac{1}{4}$)

$$(4\pi)^{\frac{D}{2}} \Delta \Gamma_{(p,-p)} = -\frac{\bar{\xi}}{8} \Gamma \left(1 - \frac{D}{2}\right) p^2 \left[(m^2)^{\frac{D}{2} - 1}(2S_1 + S_2)ight]
- 4(P^2)^{\frac{D}{2} - 1} S_1
- \frac{\bar{\xi}^2}{2} \Gamma \left(2 - \frac{D}{2}\right) p^4 (P^2)^{\frac{D}{2} - 2} S_1$$
Explicit computation

Additional term for the case $\bar{\xi} \neq 0$ (i.e. $\xi \neq \frac{1}{4}$)

\[
(4\pi)^{\frac{D}{2}} \Delta \Gamma_{(p, -p)} = -\frac{\bar{\xi}}{8} \Gamma \left(1 - \frac{D}{2}\right) p^{2} \left[(m^{2})^{\frac{D-1}{2}}(2S_{1} + S_{2}) \right.

- 4(P^{2})^{\frac{D-1}{2}} S_{1} \right]

- \frac{\bar{\xi}^{2}}{2} \Gamma \left(2 - \frac{D}{2}\right) p^{4}(P^{2})^{\frac{D-2}{2}} S_{1}
\]

Ward Identity from general coordinate invariance

\[
\nabla_{\mu}^{(x)} \frac{1}{\sqrt{g(x)}} \frac{\delta \Gamma[g]}{\delta g_{\mu\nu}(x)} = 0 \quad \checkmark
\]
Extensions

Effective action for spin 1/2 coupled to gravity

Obtained by considering N=1 supersymmetric extension of previous path integral.
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Effective action for spin 1/2 coupled to gravity

- Obtained by considering N=1 supersymmetric extension of previous path integral.

- The supersymmetric partners $\psi^\mu = e^\mu_a \psi^a$ of the coordinates $x^\mu$ generate the gamma matrices. One can use either $\psi^a$ or $\psi^\mu$. 
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- Using $\psi^\mu$ there is no need of introducing the vielbein $e^\mu_a$: one can work directly with the metric $g_{\mu\nu}$.

- Dimensional regularization can be extended to this model as well: DR is a supersymmetric regularization.
Extensions

Path integral with $\psi^\mu$ has additional bosonic ghosts $\alpha^\mu$

$$\Gamma[g_{\mu\nu}] = \frac{1}{2} \int_0^\infty \frac{dT}{T} \oint_{PBC} DxDaDbDc \oint_{ABC} D\psi D\alpha \ e^{-S}$$

with

$$S = \int_0^1 d\tau \frac{1}{4T} \left[ g_{\mu\nu}(x)(\dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu) \right]$$

$$+ g_{\mu\nu}(x)(\psi^\mu \dot{\psi}^\nu + \alpha^\mu \alpha^\nu) - \partial_\mu g_{\nu\lambda}(x)\psi^\mu \psi^\nu \dot{x}^\lambda \right] + Tm^2$$
Extensions

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with

$$S = \int_0^1 d\tau \frac{1}{4T} \left[ g_{\mu\nu}(x)(\dot{x}^\mu \dot{x}^\nu + a^\mu a^\nu + b^\mu c^\nu) + g_{\mu\nu}(x)(\psi^\mu \dot{\psi}^\nu + \alpha^\mu \alpha^\nu) - \partial_\mu g_{\nu\lambda}(x)\psi^\mu \psi^\nu \dot{x}^\lambda \right] + Tm^2$$

- Linear in $g_{\mu\nu}$ (only vertices with a single graviton emission)!
Effective action for spin 1 (Bastianelli, Benincasa, Giombi, ’05)

Obtained by considering N=2 locally supersymmetric extension of previous path integral.
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- It projects-in $p$–forms, with $p = \frac{D}{2} - q - 1$

$$D = 4, \ q = 0 \ \Rightarrow \ p = 1 \text{ vector field}$$
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- Coupling to ext gravity $\sqrt{\phantom{x}}$
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- Obtained by considering N=2 locally supersymmetric extension of previous path integral.
- $SO(2)$ gauge symmetry: yields a new constraint
- Can introduce a Chern Simons coupling $q \int_0^1 d\tau \alpha$
- It projects-in $p$–forms, with $p = \frac{D}{2} - q - 1$

$D = 4, \ q = 0 \ \Rightarrow \ p = 1$ vector field

- Coupling to ext gravity √
- Massive spin 1 by KK reduction
Higher spin fields

Obtained by considering $N>2$ locally supersymmetric extension of previous models

$SO(N)$ spinning particle models
Higher spin fields

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- $SO(N)$ spinning particle models
- $SO(N)$ gauge symmetry $\rightarrow$ new constraints
  Bargmann-Wigner EoM's
Higher spin fields

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- In 4D projects-in a spin-$\frac{N}{2}$ field
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  $N = 3$ in 4D $\rightarrow$ gravitino
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  - \( N = 4 \) in 4D \( \rightarrow \) graviton
  - \( N = 3 \) in 4D \( \rightarrow \) gravitino
- In generic D, massless rep.'s of the conformal group
  - \( SO(D, 2) \) (Siegel)
Higher spin fields

Starting point

\[ S[X] = \int dt \left( p_{\mu} \dot{x}^{\mu} + \frac{1}{2} \psi_{i,\mu} \dot{\psi}^{i,\mu} - \frac{1}{2} \delta^{\mu\nu} p_{\mu} p_{\nu} \right) \]
Higher spin fields

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Symmetry algebra

\[ H = \frac{1}{2} \delta^{\mu\nu} p_\mu p_\nu \quad Q_i = p \cdot \psi_i \quad J_{ij} = \psi_i \cdot \psi_j \]

\( SO(N) \) generators
Higher spin fields

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Symmetry algebra

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\[ SO(N) \] generators

It can be gauged: add gauge fields \( G = (e, \chi_i, a_{ij}) \)

\[ L = p_\mu \dot{x}^\mu + \frac{1}{2} \psi_{i,\mu} \psi^{i,\mu} - \frac{e}{2} \delta^{\mu\nu} p_\mu p_\nu - \chi_i p \cdot \psi^i - a_{ij} \psi^i \cdot \psi^j \]
Higher spin fields

Canonical qzn (Brink, Di Vecchia, Howe, Penati, Pernici, Townsend,...)

\[ [\psi_{i,\mu}, \psi_{j,\nu}] = \delta^j_i \delta_{\mu}^{\nu} \quad \text{set of N Clifford algebras} \]
Higher spin fields

- Canonical qzn (Brink, Di Vecchia, Howe, Penati, Pernici, Townsend,...)

\[
\left[ \psi_{i,\mu}, \psi^{j,\nu} \right] = \delta_i^j \delta_{\mu}^{\nu} \quad \text{set of } N \text{ Clifford algebras}
\]

- wave function is a multispinor \( \Psi_{\alpha_1...\alpha_N}(x) \)
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\[ \partial_{\alpha_i} \bar{\alpha}_i \Psi ...\bar{\alpha}_i...(x) = 0 \]

- vector constraints select propagating DoF's

\[ \psi_{i,\mu}^{\psi_{j,\mu}} \approx 0 \quad \Longrightarrow \quad (\gamma^\mu)_{\alpha_i} \bar{\alpha}_i (\gamma_\mu)_{\alpha_j} \bar{\alpha}_j \Psi ...\bar{\alpha}_i...\bar{\alpha}_j...(x) = 0 \]
Higher spin fields

- **Canonical qzn** (Brink, Di Vecchia, Howe, Penati, Pernici, Townsend,...)

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\[ (\gamma^{T}_{\mu} \Gamma \gamma^{\mu})^{\alpha_i\alpha_j} \Psi_{\alpha_1...\alpha_j}(x) = 0, \quad \Gamma \in \{ C, C\gamma^* , C\gamma_{\mu}, \ldots \} \]
Higher spin fields

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  \[ \gamma^T_\mu \Gamma^{(n)} \gamma^\mu \sim (n - \frac{D}{2}) \Gamma^{(n)} \Rightarrow n = \frac{D}{2} \text{ acts trivially} \]
Higher spin fields

E.g. $D = 4$, $N = 3$, trivial constraint $\gamma_\mu^T \Gamma^{(2)} \gamma_\mu$

$$\Psi_{\alpha_1 \alpha_2 \alpha} = (\Gamma^{\mu \nu})_{\alpha_1 \alpha_2} \chi_{\mu \nu \alpha}$$

$$\Rightarrow \chi_{\mu \nu \alpha} = -\chi_{\nu \mu \alpha}$$
Higher spin fields

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$$\Rightarrow \chi_{\mu \nu \alpha} = -\chi_{\nu \mu \alpha}$$

- non-trivial constraints

$$\begin{cases}
\partial^\mu \chi_{\mu \nu \alpha} = \partial_{[\sigma} \chi_{\mu \nu]} \chi_{\alpha} = 0 \\
\partial_\alpha \tilde{\alpha} \chi_{\mu \nu} \tilde{\alpha} = (\gamma^\mu)_{\alpha} \tilde{\alpha} \chi_{\mu \nu} \tilde{\alpha} = 0
\end{cases} \Rightarrow \begin{cases}
\chi_{\mu \nu \alpha} = \partial_\mu \phi_{\nu \alpha} - \partial_\nu \phi_{\mu \alpha} \\
\partial_\mu \phi - \partial \phi_{\mu} = 0
\end{cases}$$

Rarita – Schwinger eq. \Rightarrow spin-$\frac{3}{2}$ field
Higher spin fields

- E.g. $D = 4$, $N = 3$, trivial constraint $\gamma^{T}_\mu \Gamma^{(2)} \gamma^\mu$

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- non-trivial constraints

\[
\begin{align*}
\partial^\mu \chi_{\mu \nu \alpha} &= \partial_{[\sigma} \chi_{\mu \nu]} \alpha = 0
\partial\tilde{\alpha} \chi_{\mu \nu \tilde{\alpha}} &= (\gamma^\mu_\alpha)_{\tilde{\alpha}} \chi_{\mu \nu \tilde{\alpha}} = 0
\end{align*}
\]

$$\Rightarrow \begin{cases} 
\chi_{\mu \nu \alpha} = \partial_\mu \phi_{\nu \alpha} - \partial_\nu \phi_\mu \alpha \\
\partial_\mu \tilde{\phi} - \tilde{\partial} \phi_\mu = 0 
\end{cases}$$

Rarita–Schwinger eq. $\Rightarrow$ spin-$\frac{3}{2}$ field

- Similarly $D = 4$, $N = 4 \Rightarrow$ spin-2 field
Higher spin fields

Goal: path integral qzn.
- Start from free fields
Higher spin fields

Goal: path integral qzn.
- Start from free fields
- Configuration space action

\[
S[X, G] = \int_0^1 d\tau \left[ \frac{1}{2} e^{-1} (\dot{x}^\mu - \chi_i \dot{\psi}_i^\mu)^2 + \frac{1}{2} \psi_i^\mu (\delta_{ij} \partial_\tau - a_{ij}) \psi_j^\mu \right]
\]
Higher spin fields

Goal: path integral qzn.
- Start from free fields
- Configuration space action

\[ S[X, G] = \int_0^1 d\tau \left[ \frac{1}{2} e^{-1} (\dot{x}^\mu - \chi_i \psi_i^\mu)^2 + \frac{1}{2} \psi_i^\mu (\delta_{ij} \partial_\tau - a_{ij}) \psi_j \right] \]

- Need to find the correct integration measure
Higher spin fields

Goal: path integral qzn.

- Start from free fields
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\[ S[X, G] = \int_0^1 d\tau \left[ \frac{1}{2} e^{-1}(\dot{x}^\mu - \chi_i \psi_i^\mu)^2 + \frac{1}{2} \psi_i^\mu (\delta_{ij} \partial_\tau - a_{ij}) \psi_j^\mu \right] \]

- Need to find the correct integration measure
- Gauge transf. on sugra multiplet

\[
\begin{align*}
\delta e &= \dot{\xi} + 2\chi_i \epsilon_i \\
\delta \chi_i &= \dot{\epsilon}_i - a_{ij} \epsilon_j + \alpha_{ij} \chi_j \\
\delta a_{ij} &= \dot{\alpha}_{ij} + \alpha_{im} a_{mj} + \alpha_{jm} a_{im}
\end{align*}
\]
Higher spin fields

One-loop partition function on $S_1$

$$Z \sim \int_{T^1} \frac{\mathcal{D}X \mathcal{D}G}{\text{Vol (Gauge)}} e^{-S[X,G]}$$

PBC (ABC) for bosonic (fermionic) fields
Higher spin fields

- One-loop partition function on $S_1$

$$Z \sim \int_{T^1} \frac{DXDG}{\text{Vol (Gauge)}} e^{-S[x,G]}$$

PBC (ABC) for bosonic (fermionic) fields

- gauge fix local symmetries

$$e = \beta, \quad \text{modulus of the circle}$$

$$\chi_i = 0, \quad \text{no zero \textendash mode for } \epsilon_i \quad \text{b/c ABC}$$

$$a_{ij} = \hat{a}_{ij}(\theta_k), \quad k = 1, \ldots, r = \text{rank of } SO(N)$$
Higher spin fields

- One-loop partition function on $S_1$

$$Z \sim \int_{T^1} \frac{D\mathcal{X}D\mathcal{G}}{\text{Vol (Gauge)}} \ e^{-S[\mathcal{X},\mathcal{G}]}$$

PBC (ABC) for bosonic (fermionic) fields

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$$a_{ij} = \hat{a}_{ij} (\theta_k), \ k = 1, \ldots, r = \text{rank of } SO(N)$$

- Obtain the correct measure via FP trick
Higher spin fields

\[ Z = -\frac{1}{2} \int_0^\infty \frac{d\beta}{\beta} \int \frac{d^D x}{(2\pi/\beta)^{D/2}} K_N \left[ \prod_{k=1}^r \int_0^{2\pi} \frac{d\theta_k}{2\pi} \right] \]

\[ \times \left( \text{Det} \left( \partial_\tau - \hat{a}_{vec} \right)_{ABC} \right)^{D/2-1} \text{Det}' \left( \partial_\tau - \hat{a}_{adj} \right)_{PBC} \]
Higher spin fields

\[ Z = -\frac{1}{2} \int_0^\infty \frac{d\beta}{\beta} \int \frac{d^D x}{(2\pi\beta)^{D/2}} K_N \left[ \prod_{k=1}^r \int_0^{2\pi} \frac{d\theta_k}{2\pi} \right] \]

\[ \times \left( \text{Det} (\partial_\tau - \hat{a}_{vec})_{ABC} \right)^{D/2 - 1} \left( \text{Det}' (\partial_\tau - \hat{a}_{adj})_{PBC} \right) \]

one-loop fermionic determinant + susy ghosts

gauge symmetry ghosts
Higher spin fields

\[ Z = -\frac{1}{2} \int_0^\infty \frac{d\beta}{\beta} \int \frac{d^Dx}{(2\pi/\beta)^D/2} K_N \left[ \prod_{k=1}^r \int_0^{2\pi} \frac{d\theta_k}{2\pi} \right] \]

\[ \times \left( \text{Det} \left( \partial_\tau - \hat{a}_{vec} \right)_{ABC} \right)^{D/2 - 1} \text{Det}' \left( \partial_\tau - \hat{a}_{adj} \right)_{PBC} \]

\[ Dof(D, N) = K_N \left[ \prod_{k=1}^r \int_0^{2\pi} \frac{d\theta_k}{2\pi} \right] \]

\[ \times \left( \text{Det} \left( \partial_\tau - \hat{a}_{vec} \right)_{ABC} \right)^{D/2 - 1} \text{Det}' \left( \partial_\tau - \hat{a}_{adj} \right)_{PBC} \]

Computes the # of degrees of freedom. \( Dof(D, 0) = 1 \)
Higher spin fields

$N = 2r, \ r = \text{rank of the group}$

$\hat{a}_{ij} = 
\begin{pmatrix}
0 & \theta_1 & 0 & 0 & \cdot & 0 & 0 \\
-\theta_1 & 0 & 0 & 0 & \cdot & 0 & 0 \\
0 & 0 & 0 & \theta_2 & \cdot & 0 & 0 \\
0 & 0 & -\theta_2 & 0 & \cdot & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & 0 & 0 & 0 & \cdot & 0 & \theta_r \\
0 & 0 & 0 & 0 & \cdot & -\theta_r & 0
\end{pmatrix}$
Higher spin fields

\[ N = 2r, \ r = \text{rank of the group} \]

\[ \hat{a}_{ij} = \begin{pmatrix}
0 & \theta_1 & 0 & 0 & . & 0 & 0 \\
-\theta_1 & 0 & 0 & 0 & . & 0 & 0 \\
0 & 0 & 0 & \theta_2 & . & 0 & 0 \\
0 & 0 & -\theta_2 & 0 & . & 0 & 0 \\
. & . & . & . & . & . & . \\
0 & 0 & 0 & 0 & . & 0 & \theta_r \\
0 & 0 & 0 & 0 & . & -\theta_r & 0
\end{pmatrix} \]

\[ \theta' \text{'s are angles: large gauge transf.'s} \Rightarrow \theta_i \cong \theta_i + 2\pi n \]
Higher spin fields

\[ \frac{1}{K_{2r}} = \frac{2^r r!}{2}, \] # copies of fundamental domain

different regions identified up to constant gauge transf.'s

- \( r! \), permutation of \( r \) \( \theta \)'s
- \( 2^r \), \( Z_2 \)-symmetry on \( O(N) \)
Higher spin fields

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\frac{1}{K_{2r}} = \frac{2^r r!}{2}, \quad \# \text{ copies of fundamental domain}
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 r!, \text{ permutation of } r \theta \text{'s}
\]

\[
2^r, \text{ } Z_2\text{-symmetry on } O(N)
\]

\[
\text{Det} \left( \partial_\tau - \hat{a}_{vec} \right)_{PBC} = \prod_{k=1}^{r} \text{Det} \left( \partial_\tau + i\theta_r \right) \text{Det} \left( \partial_\tau - i\theta_r \right)
\]

\[
= \prod_{k=1}^{r} \left( 2 \cos \frac{\theta_k}{2} \right)^2
\]
Higher spin fields

\[
\text{Det}' (\partial_\tau - \hat{a}_{adj})_{PBC} = \prod_{k=1}^{r} \text{Det}' (\partial_\tau)
\]
\[
\times \prod_{k<l} \text{Det} (\partial_\tau + i(\theta_k + \theta_l)) \text{Det} (\partial_\tau - i(\theta_k + \theta_l))
\]
\[
\times \prod_{k<l} \text{Det} (\partial_\tau + i(\theta_k - \theta_l)) \text{Det} (\partial_\tau - i(\theta_k - \theta_l))
\]
\[
= \prod_{k<l} \left(2 \sin \frac{\theta_k + \theta_l}{2}\right)^2 \left(2 \sin \frac{\theta_k - \theta_l}{2}\right)^2
\]
Higher spin fields

\[
Dof(D, N) = \frac{2}{2^r r!} \left[ \prod_{k=1}^{r} \int_{0}^{2\pi} \frac{d\theta_k}{2\pi} \left( 2 \cos \frac{\theta_k}{2} \right)^{D-2} \right] \\
\times \prod_{k<l} \left( 2 \sin \frac{\theta_k + \theta_l}{2} \right)^2 \left( 2 \sin \frac{\theta_k - \theta_l}{2} \right)^2 \\
= \frac{2}{2^r r!} \prod_{k=1}^{r} \int_{0}^{2\pi} \frac{d\theta_k}{2\pi} \left( 2 \cos \frac{\theta_k}{2} \right)^{D-2} \\
\times \prod_{k<l} \left[ \left( 2 \cos \frac{\theta_k}{2} \right)^2 - \left( 2 \cos \frac{\theta_l}{2} \right)^2 \right]^2 \\
Dof(2d + 1, N) = 0
\]
Higher spin fields

• Change of variables \( x_k = \sin^2 \frac{\theta_k}{2} \)

\[
Dof(2d, 2r) = \frac{2^{2(d-1)r+(r-1)(2r-1)}}{\pi^r r!} \times \prod_{k=1}^{r} \int_{0}^{1} dx_k \ x_k^{-1/2} (1 - x_k)^{d-3/2} \prod_{k<l} (x_l - x_k)^2
\]
Higher spin fields

- Change of variables \( x_k = \sin^2 \frac{\theta_k}{2} \)

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Dof(2d, 2r) = \frac{2^{2(d-1)r+(r-1)(2r-1)}}{\pi^r r!} \times \prod_{k=1}^r \int_0^1 dx_k \, x_k^{-1/2} (1 - x_k)^{d-3/2} \prod_{k<l} (x_l - x_k)^2
\]

(Van der Monde determinant)^2: matrix models
Higher spin fields

- Change of variables $x_k = \sin^2 \frac{\theta_k}{2}$

$$Dof(2d, 2r) = \frac{2^{2(d-1)r+(r-1)(2r-1)}}{\pi^r r!}$$

$$\times \prod_{k=1}^{r} \int_0^1 dx_k \ x_k^{-1/2} (1 - x_k)^{d-3/2} \prod_{k<l} (x_l - x_k)^2$$

$$\Delta^2(x_i) = \det \begin{pmatrix} p_0(x_1) & \cdots & p_{r-1}(x_1) \\ p_0(x_2) & \cdots & p_{r-1}(x_2) \\ \vdots & \ddots & \vdots \\ p_0(x_r) & \cdots & p_{r-1}(x_r) \end{pmatrix} \begin{pmatrix} p_0(x_1) & \cdots & p_0(x_r) \\ p_1(x_1) & \cdots & p_1(x_r) \\ \vdots & \ddots & \vdots \\ p_{r-1}(x_1) & \cdots & p_{r-1}(x_r) \end{pmatrix}$$

$$= \det K(x_i, x_j), \quad p_k(x) = x^k + a_{k-1}x^{k-1} + \cdots, \quad \forall a_i$$
Higher spin fields

- Change of variables $x_k = \sin^2 \frac{\theta_k}{2}$

$$\text{Dof}(2d, 2r) = \frac{2^{2(d-1)} r + (r-1)(2r-1)}{\pi^r r!}$$

$$\times \prod_{k=1}^{r} \int_0^1 dx_k \ x_k^{-1/2} (1 - x_k)^{d-3/2} \prod_{k<l} (x_l - x_k)^2$$

$$K(x_i, x_j) = \sum_{k=0}^{r-1} p_k(x_i) p_k(x_j) \int_0^1 dx \ w(x) p_n(x) p_m(x) = h_n \delta_{nm}$$

$$w(x) = x^{-\frac{1}{2}} (1 - x)^{d-\frac{3}{2}} \quad \text{Jacobi polynomials}$$
Higher spin fields

Change of variables \( x_k = \sin^2 \frac{\theta_k}{2} \)

\[
Dof(2d, 2r) = \frac{2^{2(d-1)r + (r-1)(2r-1)}}{\pi^r r!} \prod_{k=1}^{r} \int_0^1 dx_k \frac{x_k^{-1/2}(1-x_k)^{d-3/2}}{2} \frac{1}{x_k} \times \prod_{k<l} (x_l - x_k)^2
\]

\[
\frac{1}{r!} \int_0^1 dx_r \ w(x_r) \cdots \int_0^1 dx_1 \ w(x_1) \Delta^2(x_i) = \prod_{k=0}^{r-1} h_k
\]
Final results

Even dimension

$$Dof(2d, 2r) = 2^{r-1} \frac{(2d - 2)!}{[(d - 1)!]^2} \prod_{k=1}^{r-1} \frac{k (2k - 1)! (2k + 2d - 3)!}{(2k + d - 2)! (2k + d - 1)!}$$
Final results

- **Even dimension**

\[
Dof(2d, 2r) = 2^{r-1} \frac{(2d - 2)!}{[(d - 1)!]^2} \prod_{k=1}^{r-1} \frac{k (2k - 1)! (2k + 2d - 3)!}{(2k + d - 2)! (2k + d - 1)!}
\]

- **Similarly** \(Dof(2d, 2r + 1)\)
Final results

- Even dimension

\[ Dof(2d, 2r) = 2^{r-1} \frac{(2d - 2)!}{[(d - 1)!]^2} \prod_{k=1}^{r-1} \frac{k (2k - 1)! (2k + 2d - 3)!}{(2k + d - 2)! (2k + d - 1)!} \]

- Similarly \( Dof(2d, 2r + 1) \)

Few interesting special cases

- \( Dof(2, N) = 1, \ \forall N \ \checkmark \)
- \( Dof(4, N) = 2, \ \forall N \ \checkmark \)
Final results

- Obtained correct path integral measure for $SO(N)$ spinning ptc, for generic $N$ and $D$
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**Final results**

- Obtained correct path integral measure for $SO(N)$ spinning ptc, for generic $N$ and $D$
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  - Covariantization for $N > 2$ problematic for a generic background; algebra doesn’t seem to close
Final results

- Obtained correct path integral measure for $SO(N)$ spinning ptc, for generic $N$ and $D$
- In 4d, higher spin fields propagate into the loop
- Couple it to maximally symmetric backgrounds
- More general couplings?
  - covariantization for $N > 2$ problematic for a generic background; algebra doesn’t seem to close
- More general symmetry group (Hallowell, Waldron ’07)
Manifolds with boundary

Given the differential operator

\[ H_x = -\frac{1}{2} \nabla^2_x + V(x) \]
Given the differential operator

\[ H_x = -\frac{1}{2} \nabla_x^2 + V(x) \]

Heat kernel is the solution of the heat equation (Schroedinger eq. in euclidean time)

\[ -\frac{\partial}{\partial \beta} K(x, y; \beta) = H_x K(x, y; \beta) \]

with b.c.'s

\[ K(x, y; 0) = \delta^D(x - y) \]
Manifolds with boundary

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with b.c.’s

\[ K(x, y; 0) = \delta^D(x - y) \]

- it can be perturbatively solved via DeWitt ansatz
Manifolds with boundary

- DeWitt ansatz

\[
K(x, y; \beta) = \frac{1}{(2\pi \beta)^{D/2}} e^{-S_0[\bar{x}]} \Omega(x, y; \beta),
\]

\[
\Omega(x, y; \beta) \sim \sum_{n=0}^{\infty} a_n(x, y) \beta^n
\]
Manifolds with boundary

- DeWitt ansatz

\[ K(x, y; \beta) = \frac{1}{(2\pi\beta)^{D/2}} e^{-S_0[\bar{x}]} \Omega(x, y; \beta), \]

\[ \Omega(x, y; \beta) \sim \sum_{n=0}^{\infty} a_n(x, y) \beta^n \]
Manifolds with boundary

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Feynman measure

Action for the classical configuration \( \bar{x}(\tau) \)
Manifolds with boundary

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Feynman measure

Action for the classical configuration \( \bar{x}(\tau) \)

Seeley-DeWitt coefficients
DeWitt ansatz

\[ K(x, y; \beta) = \frac{1}{(2\pi \beta)^{D/2}} e^{-S_0[\bar{x}]} \Omega(x, y; \beta), \]

Feynman measure

Action for the classical configuration \( \bar{x}(\tau) \)

Seeley-DeWitt coefficients

It’s a short-time perturbative expansion
Manifolds with boundary

- Path integral representation of the Heat kernel

\[ K(x, y; \beta) = \int_{x(0)=x}^{x(\beta)=y} Dx \ e^{-S[x]} = \langle y | e^{-\beta \hat{H}} | x \rangle \]
Manifolds with boundary

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\[ S[x] = S_0[x] + S_{int}[x] \]

\[ S_0[x] = \frac{1}{\beta} \int_0^1 d\tau \ \frac{1}{2} \delta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu, \quad S_{int}[x] = \frac{1}{\beta} \int_0^1 d\tau \ \beta^2 V(x) \]
Manifolds with boundary

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Flat target-space metric
Manifolds with boundary

- Path integral representation of the Heat kernel

\[ K(x, y; \beta) = \int_{x(0)=x}^{x(\beta)=y} Dx \, e^{-S[x]} = \langle y | e^{-\beta \hat{H}} | x \rangle \]

\[ S[x] = S_0[x] + S_{int}[x] \]

\[ S_0[x] = \frac{1}{\beta} \int_0^1 d\tau \, \frac{1}{2} \delta_{\mu\nu} \dot{x}^\mu \dot{x}^\nu, \quad S_{int}[x] = \frac{1}{\beta} \int_0^1 d\tau \, \beta^2 V(x) \]

Flat target-space metric

- Path integral for generic \( V \) not known

- Treat \( V \) as perturbation → vertices from Taylor expanding about initial/final point
Path integral representation of the Heat kernel

\[ K(x, y; \beta) = \int_{x(0) = x}^{x(\beta) = y} Dx \ e^{-S[x]} = \langle y | e^{-\beta \hat{H}} | x \rangle \]

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Flat target-space metric

- Path integral for generic \( V \) not known
- Treat \( V \) as perturbation → vertices from Taylor expanding about initial/final point
Manifolds with boundary

- Boundaryless flat space
Manifolds with boundary

- Boundaryless flat space

Split \( x(\tau) = \bar{x}(\tau) + q(\tau) \)
Manifolds with boundary

- Boundaryless flat space
  
  Split \( x(\tau) = \bar{x}(\tau) + q(\tau) \)
  
  Measure

\[
Dx = \prod_{0<\tau<1} d^D q(\tau), \quad q^\mu(0) = q^\mu(1) = 0 \quad q^\mu(\tau) \in \mathbb{R}^D \quad \forall \tau
\]
Manifolds with boundary

- Boundaryless flat space
  - Split $x(\tau) = \bar{x}(\tau) + q(\tau)$
  - Measure
    $$Dx = \prod_{0<\tau<1} d^D q(\tau), \quad q^\mu(0) = q^\mu(1) = 0 \quad q^\mu(\tau) \in \mathbb{R}^D \quad \forall \tau$$

- Pictorially
**Manifolds with boundary**

- Boundaryless flat space

  - **Split** \( x(\tau) = \bar{x}(\tau) + q(\tau) \)

  - **Measure**

    \[
    Dx = \prod_{0<\tau<1} d^D q(\tau), \quad q^\mu(0) = q^\mu(1) = 0 \quad q^\mu(\tau) \in \mathbb{R}^D \quad \forall \tau
    \]

- \( \frac{1}{(2\pi\beta)^{D/2}} \rightarrow \) Semiclassical (one-loop) contribution
Manifolds with boundary

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Dx = \prod_{0<\tau<1} d^D q(\tau), \quad q^\mu(0) = q^\mu(1) = 0 \quad q^\mu(\tau) \in \mathbb{R}^D \quad \forall \tau
\]

- Semiclassical (one-loop) contribution

\[
\frac{1}{(2\pi\beta)^{D/2}} \quad \text{Semiclassical (one-loop) contribution}
\]

- \( a_n(x, y; \beta) \rightarrow (n + 1) \)-loop contribution in the worldline path-integral via Wick’s theorem
Manifolds with boundary

- Flat 1D space with boundary: particle on the half line
Manifolds with boundary

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\[ M = \mathbb{R}_+ \quad \partial M = \{0\} \]
Manifolds with boundary

- Flat 1D space with boundary: particle on the half line

\[ M = \mathbb{R}_+ \quad \partial M = \{0\} \]

- Dirichlet - Neumann b.c.’s

\[ K(x, x_2; \beta) = 0, \quad \partial_n K(x, x_2; \beta) = 0 \quad \text{for} \quad x \in \partial M \]
Manifolds with boundary

- Flat 1D space with boundary: particle on the half line
  \[ M = \mathbb{R}_+ \quad \partial M = \{0\} \]

- Dirichlet - Neumann b.c.’s
  \[ K(x, x_2; \beta) = 0, \quad \partial_n K(x, x_2; \beta) = 0 \quad \text{for} \quad x \in \partial M \]

- Two classical paths (straight lines) of \( S_0 \) now
Manifolds with boundary

- Flat 1D space with boundary: particle on the half line
  
  \[ M = \mathbb{R}_+ \quad \partial M = \{0\} \]

- Dirichlet - Neumann b.c.’s
  
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- Write the ansatz as McAvity-Osborn '91
  
  \[
  K(x_1, x_2; \beta) = \frac{1}{(2\pi\beta)^{1/2}} \left( e^{-S_0[\bar{x}_1]} \Omega_1(x_1, x_2; \beta) + \gamma e^{-S_0[\bar{x}_2]} \Omega_2(x_1, x_2; \beta) \right), \quad \gamma = -1/1
  \]
Manifolds with boundary

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  \[ M = \mathbb{R}_+ \quad \partial M = \{0\} \]

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  \[ + \gamma e^{-S_0[x_2]} \Omega_2(x_1, x_2; \beta), \quad \gamma = -1/1 \]

- How to implement it in the path integral?
Manifolds with boundary

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- How to implement it in the path integral? i.e. How to implement the constraint \( x(\tau) \geq 0 \) in the path integral?
Two classes of paths

1. Paths bounded to the interior of $M$
2. Paths that hit the boundary

Image charge method: paths that bounce at the boundary are equivalent to paths that are reflected off the boundary.
Manifolds with boundary

Two classes of paths

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Image charge method: paths that bounce at the boundary are equivalent to paths that are reflected off the boundary

1. Extend the potential to the whole $\mathbb{R}$ as an even function

$$V(x(\tau)) \rightarrow \tilde{V}(x(\tau)) = \theta(x(\tau))V(x(\tau)) + \theta(-x(\tau))V(-x(\tau))$$
Manifolds with boundary

Two classes of paths
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Image charge method: paths that bounce at the boundary are equivalent to paths that are reflected off the boundary

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   $$V(x(\tau)) \mapsto \tilde{V}(x(\tau)) = \theta(x(\tau))V(x(\tau)) + \theta(-x(\tau))V(-x(\tau))$$

2. For Dirichlet/Neumann bc’s write the kernel as
   $$K_M(x_1, x_2; \beta) = K_{\mathbb{R}}(x_1, x_2; \beta) \mp K_{\mathbb{R}}(x_1, -x_2; \beta) \quad x_{1,2} \in \mathbb{M}$$
Manifolds with boundary

For Dbc the heuristics is quite clear: the second term

$$K_{\mathbb{R}}(x_1, -x_2; \beta) = \sum \text{paths that hit the boundary}$$
Manifolds with boundary

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\[ K_{\mathbb{R}}(x_1, -x_2; \beta) = \sum \text{paths that hit the boundary} \]

\[ \Rightarrow K_M(x_1, x_2; \beta) = \sum \text{paths that do NOT hit the boundary} \]

\[ K_M(x_1, 0; \beta) = 0 \]
Manifolds with boundary

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For Nbc: by direct inspection

\[ \partial_2 K_M(x_1, 0; \beta) = 0 \]
Manifolds with boundary

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Left w/ computation of whole-line path integrals w/ \( \tilde{V} \)
Manifolds with boundary

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Left w/ computation of whole-line path integrals w/ \( \tilde{V} \)

Not yet straightforward: naive perturbation theory w/ \( \theta(x(\tau)) \) is problematic
Manifolds with boundary

Task: computation of
Manifolds with boundary

Task: computation of

\[ \langle x_2, \beta | x_1, 0 \rangle_R = e^{-S_0[x_{cl}]} \int_{q(0)=0}^{q(1)=0} Dq \ e^{-S_2[q]} e^{-\beta \int d\tau \tilde{V}(x_{cl}(\tau)+q(\tau))} \]
Manifolds with boundary

Task: computation of

\[ \langle x_2, \beta | x_1, 0 \rangle_\mathbb{R} = e^{-S_0[x_{cl}]} \int_{q(0)=0}^{q^{(1)}=0} Dq \ e^{-S_2[q]} e^{-\beta \int d\tau \ \tilde{V}(x_{cl}(\tau)+q(\tau))} \]

\[ = e^{-\frac{1}{2\beta} (x_2-x_1)^2} \int_{q(0)=0}^{q^{(1)}=0} Dq \ e^{-S_2[q]} \left( 1 - \beta \int_0^1 d\tau \ \tilde{V}(x_{cl}(\tau) + q(\tau)) \right) \]

\[ + \frac{\beta^2}{2!} \int_0^1 d\tau_1 \int_0^1 d\tau_2 \ \tilde{V}(x_{cl}(\tau_1) + q(\tau_1)) \tilde{V}(x_{cl}(\tau_2) + q(\tau_2)) + \cdots \]
Manifolds with boundary

Task: computation of

\[ \langle x_2, \beta | x_1, 0 \rangle_{\mathbb{R}} = e^{-S_0[x_{cl}]} \int_{q(0)=0}^{q(1)=0} Dq \ e^{-S_2[q]} e^{-\beta \int d\tau \ \tilde{V}(x_{cl}(\tau)+q(\tau))} \]

Single–$V$ insertion: extract the worldline integral out

\[ \int_0^1 d\sigma \int_{q(0)=0}^{q(1)=0} Dq \ e^{-S_2[q]} \left[ \theta(x_{cl}(\sigma) + q(\sigma))V(x_{cl}(\sigma) + q(\sigma)) + \theta(-x_{cl}(\sigma) - q(\sigma))V(-x_{cl}(\sigma) - q(\sigma)) \right] \]
Manifolds with boundary

Task: computation of

$$\langle x_2, \beta | x_1, 0 \rangle_\mathbb{R} = e^{-S_0[x_{cl}]} \int_{q(0)=0}^{q(1)=0} Dq \ e^{-S_2[q]} e^{-\beta \int d\tau \ \tilde{V}(x_{cl}(\tau)+q(\tau))}$$

Single-$V$ insertion: extract the worldline integral out

$$\int_0^1 d\sigma \int_{q(0)=0}^{q(1)=0} Dq \ e^{-S_2[q]} \left[ \theta(x_{cl}(\sigma)+q(\sigma))V(x_{cl}(\sigma)+q(\sigma)) \right.$$ 

$$+ \theta(-x_{cl}(\sigma)-q(\sigma))V(-x_{cl}(\sigma)-q(\sigma)) \right]$$

For fixed $\sigma$ constraints act only on $q(\sigma)$
Manifolds with boundary

- Task: computation of

\[
\langle x_2, \beta | x_1, 0 \rangle_{\mathbb{R}} = e^{-S_0[x_{cl}]} \int_{q(0)=0}^{q(1)=0} Dq \ e^{-S_2[q]} e^{-\beta \int d\tau \ \tilde{V}(x_{cl}(\tau)+q(\tau))}
\]

- Single-\(V\) insertion: extract the worldline integral out

\[
\int_0^1 d\sigma \int_{q(0)=0}^{q(1)=0} Dq \ e^{-S_2[q]} \left[ \theta(x_{cl}(\sigma) + q(\sigma))V(x_{cl}(\sigma) + q(\sigma)) \right. \\
\left. + \theta(-x_{cl}(\sigma) - q(\sigma))V(-x_{cl}(\sigma) - q(\sigma)) \right] \\
q(\sigma) \geq -x_{cl}(\sigma)
\]
Manifolds with boundary

Pictorially

Worldline Path Integral Formalism new results and applications – p.44/48
Pictorially

Split the integral in two pieces $[0, \sigma], [\sigma, 1]$; then, for fixed $\sigma$
Pictorially

Split the integral in two pieces $[0, \sigma], [\sigma, 1]$; then, for fixed $\sigma$

$$
\int_{q(0)=0}^{q(1)=0} Dq \ e^{-S_2[q]} \theta(x_{cl}(\sigma) + q(\sigma)) V(x_{cl}(\sigma) + q(\sigma))
$$

$$
= \int_{-x_{cl}(\sigma)}^{\infty} dy \int_{q(0)=0}^{q(\sigma)=y} Dq \ e^{-S_2[q]} \int_{q(\sigma)=y}^{q(1)=0} Dq \ e^{-S_2[q]} V(x_{cl}(\sigma) + y)
$$
Manifolds with boundary

- Pictorially

- Split the integral in two pieces $[0, \sigma], [\sigma, 1]$; then, for fixed $\sigma$

\[
\int_{q(0)=0}^{q(1)=0} Dq \ e^{-S_2[q]} \theta(x_{cl}(\sigma) + q(\sigma))V(x_{cl}(\sigma) + q(\sigma))
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= \int_{-x_{cl}(\sigma)}^{\infty} dy \int_{q(0)=0}^{q(\sigma)=y} Dq \ e^{-S_2[q]} \int_{q(\sigma)=y}^{q(1)=0} Dq \ e^{-S_2[q]} V(x_{cl}(\sigma) + y)
\]
Manifolds with boundary

Pictorially

Split the integral in two pieces \([0, \sigma], [\sigma, 1]\); then, for fixed \(\sigma\)

\[
\int_{-x_{cl}(\sigma)}^{\infty} dy \cdot \frac{1}{2\pi \beta \sqrt{\sigma(1-\sigma)}} e^{-\frac{1}{2\beta\sigma(1-\sigma)}y^2} V(x_{cl}(\sigma) + y)
\]
Manifolds with boundary

- Pictorially

Split the integral in two pieces $[0, \sigma]$, $[\sigma, 1]$; then, for fixed $\sigma$

$$\int_{-x_{cl}(\sigma)}^{\infty} dy \frac{1}{2\pi \beta \sqrt{\sigma(1-\sigma)}} e^{-\frac{1}{2\beta \sigma (1-\sigma)} y^2} V(x_{cl}(\sigma) + y)$$

- Similarly

$$\int_{-\infty}^{-x_{cl}(\sigma)} dy \frac{1}{2\pi \beta \sqrt{\sigma(1-\sigma)}} e^{-\frac{1}{2\beta \sigma (1-\sigma)} y^2} V(-x_{cl}(\sigma) - y)$$
Results

- Single-V whole-line heat kernel

\[
\langle x_2, \beta | x_1, 0 \rangle_R = \frac{e^{-\frac{1}{2\beta} (x_2-x_1)^2}}{(2\pi\beta)^{1/2}} \left( 1 - \int_0^1 d\sigma \ l_\sigma \int_{-\infty}^{+\infty} dy \ e^{-\frac{y^2}{2\beta\sigma(1-\sigma)}} V(x_{cl}(\sigma) + y) \right. \\
+ \left. \int_0^1 d\sigma \ l_\sigma \int_{-\infty}^{-x_{cl}(\sigma)} dy \ e^{-\frac{y^2}{2\beta\sigma(1-\sigma)}} \left[ V(x_{cl}(\sigma) + y) - V(-x_{cl}(\sigma) - y) \right] \right)
\]
Results

Even potential $V$ ($\Rightarrow \tilde{V} = V$): third line vanishes

$$\langle x_2, \beta | x_1, 0 \rangle_{\mathbb{R}_+} = \frac{e^{-\frac{1}{2\beta}(x_2-x_1)^2}}{(2\pi\beta)^{1/2}} \left[ 1 - \beta \tilde{V} - \frac{\beta^2}{2 \cdot 3!} \tilde{V}'' \left( 1 - \frac{(x_2-x_1)^2}{\beta} \right) + O(\beta^3) \right] + \frac{e^{-\frac{1}{2\beta}(x_2+x_1)^2}}{(2\pi\beta)^{1/2}} \left[ 1 - \beta \tilde{V} - \frac{\beta^2}{2 \cdot 3!} \tilde{V}'' \left( 1 - \frac{(x_2+x_1)^2}{\beta} \right) + O(\beta^3) \right]$$
Results

- Even potential $V$ ($\Rightarrow \tilde{V} = V$): third line vanishes

\[
\langle x_2, \beta| x_1, 0 \rangle_{\mathbb{R}_+} = \frac{e^{-\frac{1}{2\beta}(x_2-x_1)^2}}{(2\pi\beta)^{1/2}} \left[ 1 - \beta \tilde{V} - \frac{\beta^2}{2 \cdot 3!} \tilde{V}'' \left( 1 - \frac{(x_2-x_1)^2}{\beta} \right) + O(\beta^3) \right] + \frac{e^{-\frac{1}{2\beta}(x_2+x_1)^2}}{(2\pi\beta)^{1/2}} \left[ 1 - \beta \tilde{V} - \frac{\beta^2}{2 \cdot 3!} \tilde{V}'' \left( 1 - \frac{(x_2+x_1)^2}{\beta} \right) + O(\beta^3) \right]
\]

- McAvity-Osborn ansatz, w/ $\Omega_i$'s integer power series in $\beta$ and $|x_2 - x_1| \sqrt{\ }$
Results

Even potential $V$ ($\Rightarrow \tilde{V} = V$): third line vanishes

$$\langle x_2, \beta | x_1, 0 \rangle_{\mathbb{R}_+} = \frac{e^{-\frac{1}{2\beta}(x_2-x_1)^2}}{(2\pi\beta)^{1/2}} \left[ 1 - \beta \tilde{V} - \frac{\beta^2}{2 \cdot 3!} \tilde{V}'' \left( 1 - \frac{(x_2-x_1)^2}{\beta} \right) + O(\beta^3) \right]$$

$$+ \frac{e^{-\frac{1}{2\beta}(x_2+x_1)^2}}{(2\pi\beta)^{1/2}} \left[ 1 - \beta \tilde{V} - \frac{\beta^2}{2 \cdot 3!} \tilde{V}'' \left( 1 - \frac{(x_2+x_1)^2}{\beta} \right) + O(\beta^3) \right]$$

McAvity-Osborn ansatz, w/ $\Omega^i_s$ integer power series in $\beta$ and $|x_2 - x_1| \sqrt{\cdot}$

Result coincides w/ conventional expansion (Wick’s theorem): no $\theta$’s involved $\sqrt{\cdot}$
Results

- Generic potential: additional boundary contributions
Results

- Generic potential: additional boundary contributions
- Can obtain the perturbative expansion (in $\beta$) of the partition function $\text{Tr}_{\mathbb{R}^+} e^{-\beta \hat{H}}$

\[
\text{Tr}_{\mathbb{R}^+} e^{-\beta \hat{H}} = \int_0^\infty dx \langle x, \beta|x, 0 \rangle_{\mathbb{R}^+}
\]
\[
= \frac{1}{(2\pi\beta)^{1/2}} \left[ \int_0^\infty dx \left( 1 - \beta V(x) + \beta^2 \left( \frac{1}{2} V^2(x) - \frac{1}{12} V''(x) \right) \right) \right]
\]
\[
\mp \sqrt{\frac{\pi \beta}{8}} \left( 1 - \beta V(0) + \beta^2 \left( \frac{1}{2} V^2(0) - \frac{1}{8} V''(0) \right) \right)
\]
\[
- \frac{\beta^2}{6} (2^+, -1^-) V'(0) + O(\beta^3)
\]
Results

- Generic potential: additional boundary contributions
- Can obtain the perturbative expansion (in $\beta$) of the partition function $\text{Tr}_{\mathbb{R}^+} e^{-\beta \hat{H}}$

\[
\text{Tr}_{\mathbb{R}^+} e^{-\beta \hat{H}} = \int_0^\infty dx \langle x, \beta | x, 0 \rangle_{\mathbb{R}^+}
\]

\[
= \frac{1}{(2\pi \beta)^{1/2}} \left[ \int_0^\infty dx \left( 1 - \beta V(x) + \beta^2 \left( \frac{1}{2} V^2(x) - \frac{1}{12} V'''(x) \right) \right) \right]
\]

\[
\mp \sqrt{\frac{\pi \beta}{8}} \left( 1 - \beta V(0) + \beta^2 \left( \frac{1}{2} V^2(0) - \frac{1}{8} V''(0) \right) \right)
\]

\[- \frac{\beta^2}{6} (2_+,-1_-) V'(0) + O(\beta^3) \]

- It’s the needed object in anomaly computations and worldline formalism
Outlook

- Developed worldline path integral method in (flat) manifolds with boundary
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- Carried on expansion up to order $\beta^{7/2}$
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- Simplify the implementation of the constraints into the path integral
Outlook

- Developed worldline path integral method in (flat) manifolds with boundary
- Carried on expansion up to order $\beta^{7/2}$
- Higher-order increasingly difficult

- Simplify the implementation of the constraints into the path integral
- Mixed (Robin) boundary conditions: need to add $\delta$—function potential