Holomorphy

Non-renormalization theorems

Consider

$$W_{\text{tree}} = \frac{m}{2}\phi^2 + \frac{\lambda}{3}\phi^3$$

ϕ is a chiral superfield; scalar component ϕ, fermion component by ψ.

R-charge

$$[R, Q_\alpha] = -Q_\alpha$$

$R[\psi] = R[\phi] - 1$, $R[\theta] = 1$
Lagrangian in toy model has Yukawa coupling

$$\mathcal{L} \supset \frac{\lambda}{3}\phi\psi\psi$$

which must have zero R-charge, so

$$3R[\phi] - 2 = 0$$

therefore $R[W] = 2$, or $\mathcal{L}_{\text{int}} = \int d^2\theta\, W$

Toy Model

	$U(1)$	$\times$	$U(1)_R$
ϕ	1		1
m	-2		0
λ	-3		-1

treat the mass and coupling as background spurion fields

integrate out modes from Λ down to μ, then the symmetries and holomorphy of the effective superpotential restrict it to be of the form

$$W_{\rm eff} = m\phi^2\, h\left(\frac{\lambda\phi}{m}\right) = \sum_n a_n \lambda^n m^{1-n} \phi^{n+2} \ ,$$

weak coupling limit $\lambda \to 0$ restricts $n \geq 0$
the massless limit $m \to 0$ restricts $n \leq 1$ so

$$W_{\rm eff} = \frac{m}{2}\phi^2 + \frac{\lambda}{3}\phi^3 = W_{\rm tree}$$

superpotential is not renormalized

Wavefunction renormalization

$$\mathcal{L}_{\text{kin.}} = Z\partial_\mu \phi^* \partial^\mu \phi + iZ\overline{\psi}\overline{\sigma}^\mu \partial_\mu \psi$$

Z is a non-holomorphic function

$$Z = Z(m, \lambda, m^\dagger, \lambda^\dagger, \mu, \Lambda)$$

If we integrate out modes down to $\mu > m$ at one-loop order

$$Z = 1 + c\lambda\lambda^\dagger \ln\left(\frac{\Lambda^2}{\mu^2}\right)$$

where c is a constant determined by the perturbative calculation. If we integrate out modes down to scales below m we have

$$Z = 1 + c\lambda\lambda^\dagger \ln\left(\frac{\Lambda^2}{mm^\dagger}\right)$$

Wavefunction renormalization means that couplings of canonically normalized fields run

running mass and running coupling are given by

$$\frac{m}{Z}, \frac{\lambda}{Z^{\frac{3}{2}}}$$

Integrating out

Consider a model with two different chiral superfields:

$$W = \frac{1}{2} M \phi_H^2 + \frac{\lambda}{2} \phi_H \phi^2$$

three global $U(1)$ symmetries:

	$U(1)_A$	$U(1)_B$	$U(1)_R$
ϕ_H	1	0	1
ϕ	0	1	$\frac{1}{2}$
M	-2	0	0
λ	-1	-2	0

where $U(1)_A$ and $U(1)_B$ are spurious symmetries for $M, \lambda \neq 0$

Integrating out

If we want to integrate out modes down to $\mu < M$, we must integrate out ϕ_H. An arbitrary term in the effective superpotential has the form

$$\phi^j M^k \lambda^p$$

To preserve the symmetries we must have $j = 4$, $k = -1$, and $p = 2$. By comparing with tree-level perturbation theory we can determine the coefficient:

$$W_{\mathrm{eff}} = -\frac{\lambda^2 \phi^4}{8M}$$

algebraic equation of motion:

$$\frac{\partial W}{\phi_H} = M\phi_H + \frac{\lambda}{2}\phi^2 = 0$$

solve this equation for ϕ_H and plug the result back into the superpotential

Another Example

$$W = \frac{1}{2} M \phi_H^2 + \frac{\lambda}{2} \phi_H \phi^2 + \frac{y}{6} \phi_H^3$$

ϕ_H equation of motion:

$$\phi_H = -\frac{M}{y} \left(1 \pm \sqrt{1 - \frac{\lambda y \phi^2}{M^2}} \right)$$

as $y \to 0$, the two vacua approach $\phi_H = -\lambda\phi/(2M)$ (as in previous example) and $\phi_H = \infty$. Integrating out ϕ_H yields

$$W_{\text{eff}} = \frac{M^3}{3y^2} \left(1 - \frac{3\lambda y \phi^2}{2M^2} \pm \left(1 - \frac{\lambda y \phi^2}{M^2} \right) \sqrt{1 - \frac{\lambda y \phi^2}{M^2}} \right)$$

singularities in W_{eff} indicate points in the parameter space and the space of ϕ VEVs where ϕ_H becomes massless and we should not have integrated it out

Singularities

The mass of ϕ_H can be found by calculating

$$\frac{\partial^2 W}{\partial \phi_H^2} = M + y\phi_H$$

and substituting in the solution forϕ_H:

$$\frac{\partial^2 W}{\partial \phi_H^2} = \mp M\sqrt{1 - \frac{\lambda y \phi^2}{M^2}}$$

Using holomorphy assign y charges (-3,0,-1) under $U(1)_A \times U(1)_B \times U(1)_R$ then

$$W_{\rm eff} = \frac{M^3}{y^2} f\left(\frac{\lambda y \phi^2}{M^2}\right)$$

for some function f, just as we found from explicitly integrating out ϕ_H

The holomorphic gauge coupling

chiral superfield for an $SU(N)$ gauge supermultiplet:

$$W_\alpha^a = -i\lambda_\alpha^a(y) + \theta_\alpha D^a(y) - (\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}^a(y) - (\theta\theta)\sigma^\mu D_\mu \lambda^{a\dagger}(y) \ ,$$

$a = 1, \ldots, N^2 - 1$

$$\tau \equiv \frac{\theta_{\rm YM}}{2\pi} + \frac{4\pi i}{g^2} \ ,$$

SUSY Yang–Mills action as a superpotential term

$$\frac{1}{16\pi i}\int d^4x \int d^2\theta \, \tau \, W_\alpha^a W_\alpha^a + h.c. =$$
$$\int d^4x \left[-\frac{1}{4g^2} F^{a\mu\nu} F_{\mu\nu}^a - \frac{\theta_{\rm YM}}{32\pi^2} F^{a\mu\nu} \widetilde{F}_{\mu\nu}^a + \frac{i}{g^2} \lambda^{a\dagger} \overline{\sigma}^\mu D_\mu \lambda^a + \frac{1}{2g^2} D^a D^a \right]$$

g only in τ which is a holomorphic parameter, but gauge fields are not canonically normalized

Running coupling

one-loop running g is given by the RG equation:

$$\mu \frac{dg}{d\mu} = -\frac{b}{16\pi^2} g^3$$

where for an $SU(N)$ gauge theory with F flavors and $\mathcal{N} = 1$ SUSY,

$$b = 3N - F$$

The solution for the running coupling is

$$\frac{1}{g^2(\mu)} = -\frac{b}{8\pi^2} \ln\left(\frac{|\Lambda|}{\mu}\right)$$

where $|\Lambda|$ is the intrinsic scale of the non-Abelian gauge theory

Holomorphic Intrinsic Scale

$$\tau_{1-\text{loop}} = \frac{\theta_{\text{YM}}}{2\pi} + \frac{4\pi i}{g^2(\mu)}$$
$$= \frac{1}{2\pi i} \ln \left[\left(\frac{|\Lambda|}{\mu} \right)^b e^{i\theta_{\text{YM}}} \right]$$

$$\Lambda \equiv |\Lambda| e^{i\theta_{\text{YM}}/b}$$
$$= \mu e^{2\pi i \tau / b}$$

$$\tau_{1-\text{loop}} = \frac{b}{2\pi i} \ln \left(\frac{\Lambda}{\mu} \right)$$

CP Violating Term

$$F^{a\mu\nu}\widetilde{F}^{a}_{\mu\nu} = 4\epsilon^{\mu\nu\rho\sigma}\partial_\mu \mathrm{Tr}\left(A_\nu\partial_\rho A_\sigma + \tfrac{2}{3}A_\nu A_\rho A_\sigma\right)$$

total derivative: no effect in perturbation theory
nonperturbative effects: instantons have a nontrivial, topological winding number, n

$$\tfrac{\theta_{\mathrm{YM}}}{32\pi^2}\int d^4x\, F^{a\mu\nu}\widetilde{F}^{a}_{\mu\nu} = n\,\theta_{\mathrm{YM}}\ .$$

Since the path integral has the form

$$\int \mathcal{D}A^a\mathcal{D}\lambda^a\mathcal{D}D^a\, e^{iS}$$

$$\theta_{\mathrm{YM}} \to \theta_{\mathrm{YM}} + 2\pi$$

is a symmetry of the theory

Instanton Action

The Euclidean action of an instanton configuration can be bounded

$$0 \quad \leq \quad \int d^4x Tr \left(F_{\mu\nu} \pm \widetilde{F}_{\mu\nu} \right)^2 = \int d^4x Tr \left(2F^2 \pm 2F\widetilde{F} \right)$$

$$\int d^4x Tr F^2 \geq | \int d^4x Tr F \widetilde{F} | = 16\pi^2 |n|$$

one instanton effects are suppressed by

$$e^{-S_{\mathrm{int}}} = e^{-(8\pi^2/g^2(\mu))+i\theta_{\mathrm{YM}}} = \left(\frac{\Lambda}{\mu} \right)^b$$

Effective Superpotential

integrate down to the scale μ

$$W_{\rm eff} = \frac{\tau(\Lambda;\mu)}{16\pi i} W_\alpha^a W_\alpha^a$$

physics periodic in $\theta_{\rm YM}$ equivalent to

$$\Lambda \to e^{2\pi i/b}\Lambda$$

in general:

$$\tau(\Lambda;\mu) = \frac{b}{2\pi i} \ln\left(\frac{\Lambda}{\mu}\right) + f(\Lambda;\mu)\ ,$$

where f has Taylor series representation in positive powers of Λ.

$$\Lambda \to e^{2\pi i/b}\Lambda$$

in perturbative term shifts $\theta_{\rm YM}$ by 2π, f must be invariant under this transformation, so the Taylor series must be in positive powers of Λ^b

Effective Superpotential

in general, we can write:

$$\tau(\Lambda;\mu) = \tfrac{b}{2\pi i}\ln\left(\tfrac{\Lambda}{\mu}\right) + \textstyle\sum_{n=1}^{\infty} a_n \left(\tfrac{\Lambda}{\mu}\right)^{bn} .$$

holomorphic gauge coupling only receives one-loop corrections and non-perturbative n-instanton corrections, no perturbative running beyond one-loop

Symmetry of $SU(N)$ SUSY YM

$U(1)_R$ symmetry is broken by instantons
anomaly index: gaugino R-charge times $T(Ad)$

Because of the anomaly, the chiral rotation

$$\lambda^a \to e^{i\alpha}\lambda^a$$

is equivalent to shift

$$\theta_{\rm YM} \to \theta_{\rm YM} - 2N\alpha$$

$2N$ because the gaugino λ^a is in the adjoint representation, $2N$ zero modes in instanton background
chiral rotation is only a symmetry when

$$\alpha = \frac{k\pi}{N}$$

$U(1)_R$ explicitly broken to discrete Z_{2N} subgroup

Spurion Analysis

Treat τ as a spurion chiral superfield, define spurious symmetry

$$\lambda^a \to e^{i\alpha}\lambda^a \ , \ \ \tau \to \tau + \tfrac{N\alpha}{\pi}$$

Assuming that SUSY YM has no massless particles, then holomorphy and symmetries determine the effective superpotential to be:

$$W_{\text{eff}} \quad = \quad a\mu^3 e^{2\pi i\tau/N}$$

This is the unique form because under the spurious $U(1)_R$ rotation the superpotential (which has R-charge 2) transforms as

$$W_{\text{eff}} \to e^{2i\alpha} W_{\text{eff}}$$

Gaugino condensation

treat τ as a background chiral superfield, the $\mathcal{F}$ component of τ ($\mathcal{F}_\tau$) acts as a source for $\lambda^a\lambda^a$

gaugino condensate given by

$$\begin{aligned}\langle\lambda^a\lambda^a\rangle &= 16\pi i \frac{\partial}{\partial F_\tau}\ln Z = 16\pi i \frac{\partial}{\partial F_\tau}\int d^2\theta W_{\text{eff}} \\ &= 16\pi i \frac{\partial}{\partial \tau} W_{\text{eff}} = 16\pi i \frac{2\pi i}{N} a\mu^3 e^{2\pi i\tau/N}\end{aligned}$$

Drop nonperturbative corrections to running, plug in $b = 3N$:

$$\langle\lambda^a\lambda^a\rangle = -\frac{32\pi^2}{N} a\Lambda^3$$

vacuum does not respect the discrete Z_N symmetry since

$$\langle\lambda^a\lambda^a\rangle \to e^{2i\alpha}\langle\lambda^a\lambda^a\rangle$$

only invariant for $k = 0$ or $k = N$

$Z_{2N} \to Z_2$, implies N degenerate vacua

$\theta_{\text{YM}} \to \theta_{\text{YM}} + 2\pi$ sweeps out N different values for $\langle\lambda^a\lambda^a\rangle$

NSVZ revisited

Three seemingly contradictory statements:

- the SUSY gauge coupling runs only at one-loop

$$\beta(g) = -\frac{g^3}{16\pi^2}\left(3T(Ad) - \sum_j T(r_j)\right)$$

  with matter chiral superfields Q_j in representations r_j

- the "exact" β function is

$$\beta(g) = -\frac{g^3}{16\pi^2}\frac{\left(3T(Ad) - \sum_j T(r_j)(1-\gamma_j)\right)}{1 - T(Ad)g^2/8\pi^2}$$

- one- and two-loop terms in β function are scheme independent

Changing renormalization schemes

$$g' = g + ag^3 + \mathcal{O}(g^5)$$

If β function is given by

$$\beta(g) = b_1 g^3 + b_2 g^5 + \mathcal{O}(g^7)$$

then

$$\beta'(g') = \beta(g)\frac{\partial g}{\partial g'} = b_1 g'^3 + b_2 g'^5 + \mathcal{O}(g'^7)$$

dependence on a only appears at higher order

Holomorphic vs Canonical Coupling

$$\mathcal{L}_h = \tfrac{1}{4}\int d^2\theta \tfrac{1}{g_h^2} W^a(V_h)W^a(V_h) + h.c.,$$

$$\begin{aligned} \tfrac{1}{g_h^2} &= \tfrac{1}{g^2} - i\tfrac{\theta_{\rm YM}}{8\pi^2} = \tfrac{\tau}{4\pi i} \ , \\ V_h &= (A_\mu^a, \lambda^a, D^a). \end{aligned}$$

canonical gauge coupling for canonically normalized fields:

$$\mathcal{L}_c = \tfrac{1}{4}\int d^2\theta \left(\tfrac{1}{g_c^2} - i\tfrac{\theta_{\rm YM}}{8\pi^2}\right) W^a(g_c V_c)W^a(g_c V_c) + h.c.$$

not equivalent under $V_h = g_c V_c$ because of rescaling anomaly
with matter fields Q_j, additional rescaling anomaly from:

$$Q_j' = Z_j(\mu, \mu')^{1/2} Q_j$$

rescaling anomaly completely determined by the axial anomaly

Rescaling Anomaly

for fermions with a rescaling $Z = e^{2i\alpha}$. We can rewrite the axial anomaly in a manifestly supersymmetric form using the path integral measure as

$$\begin{aligned}\mathcal{D}(e^{i\alpha}Q)\mathcal{D}(e^{-i\alpha}\overline{Q}) &= \mathcal{D}Q\mathcal{D}\overline{Q} \\ &\times \exp\left(\frac{-i}{4}\int d^2\theta \left(\frac{T(r_j)}{8\pi^2}2i\alpha\right) W^a W^a + h.c.\right)\end{aligned}$$

take α to be complex gives general case:

$$\begin{aligned}\mathcal{D}(Z_j^{1/2}Q_j)\mathcal{D}(Z_j^{1/2}\overline{Q}_j) &= \mathcal{D}Q_j\mathcal{D}\overline{Q}_j \\ &\times \exp\left(\frac{-i}{4}\int d^2\theta \left(\frac{T(r_j)}{8\pi^2}\ln Z_j\right) W^a W^a + h.c.\right)\end{aligned}$$

Rescaling Anomaly

for the gauge fields (gauginos) taking $Z_\lambda = g_c^2$

$$
\begin{aligned}
& \mathcal{D}(g_c V_c) \\
= \quad & \mathcal{D}V_c \times \exp\left(\frac{-i}{4}\int d^2\theta \left(\frac{2T(Ad)}{8\pi^2}\ln(g_c)\right) W^a(g_c V_c) W^a(g_c V_c) + h.c.\right)
\end{aligned}
$$

Thus, for pure SUSY Yang–Mills we have

$$
\begin{aligned}
Z &= \int \mathcal{D}V_h \exp\left(\frac{i}{4}\int d^2\theta \frac{1}{g_h^2} W^a(V_h) W^a(V_h) + h.c.\right) \\
&= \int \mathcal{D}V_c \exp\left(\frac{i}{4}\int d^2\theta \left(\frac{1}{g_h^2} - \frac{2T(Ad)}{8\pi^2}\ln(g_c)\right) W^a(g_c V_c) W^a(g_c V_c) + h.c.\right)
\end{aligned}
$$

So

$$
\frac{1}{g_c^2} = \mathrm{Re}\left(\frac{1}{g_h^2}\right) - \frac{2T(Ad)}{8\pi^2}\ln(g_c)
$$

Rescaling Anomaly

including the matter fields:

$$\frac{1}{g_c^2} = \mathrm{Re}\left(\frac{1}{g_h^2}\right) - \frac{2T(Ad)}{8\pi^2}\ln(g_c) - \sum_j \frac{T(r_j)}{8\pi^2}\ln(Z_j)$$

Differentiating with respect to $\ln\mu$, this leads precisely to

$$\beta(g) = -\frac{g^3}{16\pi^2}\frac{\left(3T(Ad) - \sum_j T(r_j)(1-\gamma_j)\right)}{1 - T(Ad)g^2/8\pi^2}$$

relation between the two couplings is logarithmic, one cannot be expanded in a Taylor series around zero in the other