

Holomorphy

Non-renormalization theorems

Consider

$$W_{\text{tree}} = \frac{m}{2} \phi^2 + \frac{\lambda}{3} \phi^3$$

ϕ is a chiral superfield; scalar component ϕ , fermion component by ψ .

R -charge

$$[R, Q_\alpha] = -Q_\alpha$$

$$R[\psi] = R[\phi] - 1, \quad R[\theta] = 1$$

Lagrangian in toy model has Yukawa coupling

$$\mathcal{L} \supset \frac{\lambda}{3} \phi \psi \psi$$

which must have zero R -charge, so

$$3R[\phi] - 2 = 0$$

therefore $R[W] = 2$, or $\mathcal{L}_{\text{int}} = \int d^2\theta W$

Toy Model

$$\begin{array}{ccc} & U(1) & \times & U(1)_R \\ \phi & 1 & & 1 \\ m & -2 & & 0 \\ \lambda & -3 & & -1 \end{array}$$

treat the mass and coupling as background spurion fields

integrate out modes from Λ down to μ , then the symmetries and holomorphy of the effective superpotential restrict it to be of the form

$$W_{\text{eff}} = m\phi^2 h\left(\frac{\lambda\phi}{m}\right) = \sum_n a_n \lambda^n m^{1-n} \phi^{n+2} ,$$

weak coupling limit $\lambda \rightarrow 0$ restricts $n \geq 0$

the massless limit $m \rightarrow 0$ restricts $n \leq 1$ so

$$W_{\text{eff}} = \frac{m}{2}\phi^2 + \frac{\lambda}{3}\phi^3 = W_{\text{tree}}$$

superpotential is not renormalized

Wavefunction renormalization

$$\mathcal{L}_{\text{kin.}} = Z \partial_\mu \phi^* \partial^\mu \phi + i Z \bar{\psi} \bar{\sigma}^\mu \partial_\mu \psi$$

Z is a non-holomorphic function

$$Z = Z(m, \lambda, m^\dagger, \lambda^\dagger, \mu, \Lambda)$$

If we integrate out modes down to $\mu > m$ at one-loop order

$$Z = 1 + c \lambda \lambda^\dagger \ln \left(\frac{\Lambda^2}{\mu^2} \right)$$

where c is a constant determined by the perturbative calculation. If we integrate out modes down to scales below m we have

$$Z = 1 + c \lambda \lambda^\dagger \ln \left(\frac{\Lambda^2}{m m^\dagger} \right)$$

Wavefunction renormalization means that couplings of canonically normalized fields run
running mass and running coupling are given by

$$\frac{m}{Z}, \frac{\lambda}{Z^{\frac{3}{2}}}$$

Integrating out

Consider a model with two different chiral superfields:

$$W = \frac{1}{2}M\phi_H^2 + \frac{\lambda}{2}\phi_H\phi^2$$

three global $U(1)$ symmetries:

	$U(1)_A$	$U(1)_B$	$U(1)_R$
ϕ_H	1	0	1
ϕ	0	1	$\frac{1}{2}$
M	-2	0	0
λ	-1	-2	0

where $U(1)_A$ and $U(1)_B$ are spurious symmetries for $M, \lambda \neq 0$

Integrating out

If we want to integrate out modes down to $\mu < M$, we must integrate out ϕ_H . An arbitrary term in the effective superpotential has the form

$$\phi^j M^k \lambda^p$$

To preserve the symmetries we must have $j = 4$, $k = -1$, and $p = 2$. By comparing with tree-level perturbation theory we can determine the coefficient:

$$W_{\text{eff}} = -\frac{\lambda^2 \phi^4}{8M}$$

algebraic equation of motion:

$$\frac{\partial W}{\partial \phi_H} = M \phi_H + \frac{\lambda}{2} \phi^2 = 0$$

solve this equation for ϕ_H and plug the result back into the superpotential

Another Example

$$W = \frac{1}{2}M\phi_H^2 + \frac{\lambda}{2}\phi_H\phi^2 + \frac{y}{6}\phi_H^3$$

ϕ_H equation of motion:

$$\phi_H = -\frac{M}{y} \left(1 \pm \sqrt{1 - \frac{\lambda y \phi^2}{M^2}} \right)$$

as $y \rightarrow 0$, the two vacua approach $\phi_H = -\lambda\phi/(2M)$ (as in previous example) and $\phi_H = \infty$. Integrating out ϕ_H yields

$$W_{\text{eff}} = \frac{M^3}{3y^2} \left(1 - \frac{3\lambda y \phi^2}{2M^2} \pm \left(1 - \frac{\lambda y \phi^2}{M^2} \right) \sqrt{1 - \frac{\lambda y \phi^2}{M^2}} \right)$$

singularities in W_{eff} indicate points in the parameter space and the space of ϕ VEVs where ϕ_H becomes massless and we should not have integrated it out

Singularities

The mass of ϕ_H can be found by calculating

$$\frac{\partial^2 W}{\partial \phi_H^2} = M + y\phi_H$$

and substituting in the solution for ϕ_H :

$$\frac{\partial^2 W}{\partial \phi_H^2} = \mp M \sqrt{1 - \frac{\lambda y \phi^2}{M^2}}$$

Using holomorphy assign y charges $(-3,0,-1)$ under $U(1)_A \times U(1)_B \times U(1)_R$ then

$$W_{\text{eff}} = \frac{M^3}{y^2} f\left(\frac{\lambda y \phi^2}{M^2}\right)$$

for some function f , just as we found from explicitly integrating out ϕ_H

The holomorphic gauge coupling

chiral superfield for an $SU(N)$ gauge supermultiplet:

$$W_\alpha^a = -i\lambda_\alpha^a(y) + \theta_\alpha D^a(y) - (\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}^a(y) - (\theta\theta)\sigma^\mu D_\mu \lambda^{a\dagger}(y) ,$$

$$a = 1, \dots, N^2 - 1$$

$$\tau \equiv \frac{\theta_{\text{YM}}}{2\pi} + \frac{4\pi i}{g^2} ,$$

SUSY Yang–Mills action as a superpotential term

$$\frac{1}{16\pi i} \int d^4x \int d^2\theta \tau W_\alpha^a W_\alpha^a + h.c. = \int d^4x \left[-\frac{1}{4g^2} F^{a\mu\nu} F_{\mu\nu}^a - \frac{\theta_{\text{YM}}}{32\pi^2} F^{a\mu\nu} \tilde{F}_{\mu\nu}^a + \frac{i}{g^2} \lambda^{a\dagger} \bar{\sigma}^\mu D_\mu \lambda^a + \frac{1}{2g^2} D^a D^a \right]$$

g only in τ which is a holomorphic parameter, but gauge fields are not canonically normalized

Running coupling

one-loop running g is given by the RG equation:

$$\mu \frac{dg}{d\mu} = -\frac{b}{16\pi^2} g^3$$

where for an $SU(N)$ gauge theory with F flavors and $\mathcal{N} = 1$ SUSY,

$$b = 3N - F$$

The solution for the running coupling is

$$\frac{1}{g^2(\mu)} = -\frac{b}{8\pi^2} \ln \left(\frac{|\Lambda|}{\mu} \right)$$

where $|\Lambda|$ is the intrinsic scale of the non-Abelian gauge theory

Holomorphic Intrinsic Scale

$$\begin{aligned}\tau_{1\text{-loop}} &= \frac{\theta_{\text{YM}}}{2\pi} + \frac{4\pi i}{g^2(\mu)} \\ &= \frac{1}{2\pi i} \ln \left[\left(\frac{|\Lambda|}{\mu} \right)^b e^{i\theta_{\text{YM}}} \right]\end{aligned}$$

$$\begin{aligned}\Lambda &\equiv |\Lambda| e^{i\theta_{\text{YM}}/b} \\ &= \mu e^{2\pi i\tau/b}\end{aligned}$$

$$\tau_{1\text{-loop}} = \frac{b}{2\pi i} \ln \left(\frac{\Lambda}{\mu} \right)$$

CP Violating Term

$$F^{a\mu\nu} \tilde{F}_{\mu\nu}^a = 4\epsilon^{\mu\nu\rho\sigma} \partial_\mu \text{Tr} \left(A_\nu \partial_\rho A_\sigma + \frac{2}{3} A_\nu A_\rho A_\sigma \right)$$

total derivative: no effect in perturbation theory

nonperturbative effects: instantons have a nontrivial, topological winding number, n

$$\frac{\theta_{\text{YM}}}{32\pi^2} \int d^4x F^{a\mu\nu} \tilde{F}_{\mu\nu}^a = n \theta_{\text{YM}} .$$

Since the path integral has the form

$$\int \mathcal{D}A^a \mathcal{D}\lambda^a \mathcal{D}D^a e^{iS}$$

$$\theta_{\text{YM}} \rightarrow \theta_{\text{YM}} + 2\pi$$

is a symmetry of the theory

Instanton Action

The Euclidean action of an instanton configuration can be bounded

$$0 \leq \int d^4x \text{Tr} \left(F_{\mu\nu} \pm \tilde{F}_{\mu\nu} \right)^2 = \int d^4x \text{Tr} \left(2F^2 \pm 2F\tilde{F} \right)$$

$$\int d^4x \text{Tr} F^2 \geq \left| \int d^4x \text{Tr} F\tilde{F} \right| = 16\pi^2 |n|$$

one instanton effects are suppressed by

$$e^{-S_{\text{int}}} = e^{-(8\pi^2/g^2(\mu)) + i\theta_{\text{YM}}} = \left(\frac{\Lambda}{\mu} \right)^b$$

Effective Superpotential

integrate down to the scale μ

$$W_{\text{eff}} = \frac{\tau(\Lambda; \mu)}{16\pi i} W_{\alpha}^a W_{\alpha}^a$$

physics periodic in θ_{YM} equivalent to

$$\Lambda \rightarrow e^{2\pi i/b} \Lambda$$

in general:

$$\tau(\Lambda; \mu) = \frac{b}{2\pi i} \ln \left(\frac{\Lambda}{\mu} \right) + f(\Lambda; \mu) ,$$

where f has Taylor series representation in positive powers of Λ .

$$\Lambda \rightarrow e^{2\pi i/b} \Lambda$$

in perturbative term shifts θ_{YM} by 2π , f must be invariant under this transformation, so the Taylor series must be in positive powers of Λ^b

Effective Superpotential

in general, we can write:

$$\tau(\Lambda; \mu) = \frac{b}{2\pi i} \ln \left(\frac{\Lambda}{\mu} \right) + \sum_{n=1}^{\infty} a_n \left(\frac{\Lambda}{\mu} \right)^{bn}.$$

holomorphic gauge coupling only receives one-loop corrections and non-perturbative n -instanton corrections, no perturbative running beyond one-loop

Symmetry of $SU(N)$ SUSY YM

$U(1)_R$ symmetry is broken by instantons
anomaly index: gaugino R -charge times $T(Ad)$
Because of the anomaly, the chiral rotation

$$\lambda^a \rightarrow e^{i\alpha} \lambda^a$$

is equivalent to shift

$$\theta_{\text{YM}} \rightarrow \theta_{\text{YM}} - 2N\alpha$$

$2N$ because the gaugino λ^a is in the adjoint representation, $2N$ zero modes in instanton background
chiral rotation is only a symmetry when

$$\alpha = \frac{k\pi}{N}$$

$U(1)_R$ explicitly broken to discrete Z_{2N} subgroup

Spurion Analysis

Treat τ as a spurion chiral superfield, define spurious symmetry

$$\lambda^a \rightarrow e^{i\alpha} \lambda^a, \quad \tau \rightarrow \tau + \frac{N\alpha}{\pi}$$

Assuming that SUSY YM has no massless particles, then holomorphy and symmetries determine the effective superpotential to be:

$$W_{\text{eff}} = a\mu^3 e^{2\pi i\tau/N}$$

This is the unique form because under the spurious $U(1)_R$ rotation the superpotential (which has R -charge 2) transforms as

$$W_{\text{eff}} \rightarrow e^{2i\alpha} W_{\text{eff}}$$

Gaugino condensation

treat τ as a background chiral superfield, the \mathcal{F} component of τ (\mathcal{F}_τ) acts as a source for $\lambda^a \lambda^a$
gaugino condensate given by

$$\begin{aligned}\langle \lambda^a \lambda^a \rangle &= 16\pi i \frac{\partial}{\partial F_\tau} \ln Z = 16\pi i \frac{\partial}{\partial F_\tau} \int d^2\theta W_{\text{eff}} \\ &= 16\pi i \frac{\partial}{\partial \tau} W_{\text{eff}} = 16\pi i \frac{2\pi i}{N} a \mu^3 e^{2\pi i \tau / N}\end{aligned}$$

Drop nonperturbative corrections to running, plug in $b = 3N$:

$$\langle \lambda^a \lambda^a \rangle = -\frac{32\pi^2}{N} a \Lambda^3$$

vacuum does not respect the discrete Z_N symmetry since

$$\langle \lambda^a \lambda^a \rangle \rightarrow e^{2i\alpha} \langle \lambda^a \lambda^a \rangle$$

only invariant for $k = 0$ or $k = N$

$Z_{2N} \rightarrow Z_2$, implies N degenerate vacua

$\theta_{\text{YM}} \rightarrow \theta_{\text{YM}} + 2\pi$ sweeps out N different values for $\langle \lambda^a \lambda^a \rangle$

NSVZ revisited

Three seemingly contradictory statements:

- the SUSY gauge coupling runs only at one-loop

$$\beta(g) = -\frac{g^3}{16\pi^2} \left(3T(Ad) - \sum_j T(r_j) \right)$$

with matter chiral superfields Q_j in representations r_j

- the “exact” β function is

$$\beta(g) = -\frac{g^3}{16\pi^2} \frac{\left(3T(Ad) - \sum_j T(r_j)(1-\gamma_j) \right)}{1 - T(Ad)g^2/8\pi^2}$$

- one- and two-loop terms in β function are scheme independent

Changing renormalization schemes

$$g' = g + ag^3 + \mathcal{O}(g^5)$$

If β function is given by

$$\beta(g) = b_1g^3 + b_2g^5 + \mathcal{O}(g^7)$$

then

$$\beta'(g') = \beta(g) \frac{\partial g}{\partial g'} = b_1g'^3 + b_2g'^5 + \mathcal{O}(g'^7)$$

dependence on a only appears at higher order

Holomorphic vs Canonical Coupling

$$\mathcal{L}_h = \frac{1}{4} \int d^2\theta \frac{1}{g_h^2} W^a(V_h) W^a(V_h) + h.c.,$$

$$\begin{aligned} \frac{1}{g_h^2} &= \frac{1}{g^2} - i \frac{\theta_{\text{YM}}}{8\pi^2} = \frac{\tau}{4\pi i}, \\ V_h &= (A_\mu^a, \lambda^a, D^a). \end{aligned}$$

canonical gauge coupling for canonically normalized fields:

$$\mathcal{L}_c = \frac{1}{4} \int d^2\theta \left(\frac{1}{g_c^2} - i \frac{\theta_{\text{YM}}}{8\pi^2} \right) W^a(g_c V_c) W^a(g_c V_c) + h.c.$$

not equivalent under $V_h = g_c V_c$ because of rescaling anomaly with matter fields Q_j , additional rescaling anomaly from:

$$Q'_j = Z_j(\mu, \mu')^{1/2} Q_j$$

rescaling anomaly completely determined by the axial anomaly

Rescaling Anomaly

for fermions with a rescaling $Z = e^{2i\alpha}$. We can rewrite the axial anomaly in a manifestly supersymmetric form using the path integral measure as

$$\mathcal{D}(e^{i\alpha}Q)\mathcal{D}(e^{-i\alpha}\bar{Q}) = \mathcal{D}Q\mathcal{D}\bar{Q} \times \exp\left(\frac{-i}{4} \int d^2\theta \left(\frac{T(r_j)}{8\pi^2} 2i\alpha\right) W^a W^a + h.c.\right)$$

take α to be complex gives general case:

$$\mathcal{D}(Z_j^{1/2}Q_j)\mathcal{D}(Z_j^{1/2}\bar{Q}_j) = \mathcal{D}Q_j\mathcal{D}\bar{Q}_j \times \exp\left(\frac{-i}{4} \int d^2\theta \left(\frac{T(r_j)}{8\pi^2} \ln Z_j\right) W^a W^a + h.c.\right)$$

Rescaling Anomaly

for the gauge fields (gauginos) taking $Z_\lambda = g_c^2$

$$\begin{aligned} & \mathcal{D}(g_c V_c) \\ = & \mathcal{D}V_c \times \exp\left(\frac{-i}{4} \int d^2\theta \left(\frac{2T(Ad)}{8\pi^2} \ln(g_c)\right) W^a(g_c V_c) W^a(g_c V_c) + h.c.\right) \end{aligned}$$

Thus, for pure SUSY Yang–Mills we have

$$\begin{aligned} Z &= \int \mathcal{D}V_h \exp\left(\frac{i}{4} \int d^2\theta \frac{1}{g_h^2} W^a(V_h) W^a(V_h) + h.c.\right) \\ &= \int \mathcal{D}V_c \exp\left(\frac{i}{4} \int d^2\theta \left(\frac{1}{g_h^2} - \frac{2T(Ad)}{8\pi^2} \ln(g_c)\right) W^a(g_c V_c) W^a(g_c V_c) + h.c.\right) \end{aligned}$$

So

$$\frac{1}{g_c^2} = \text{Re}\left(\frac{1}{g_h^2}\right) - \frac{2T(Ad)}{8\pi^2} \ln(g_c)$$

Rescaling Anomaly

including the matter fields:

$$\frac{1}{g_c^2} = \text{Re} \left(\frac{1}{g_h^2} \right) - \frac{2T(Ad)}{8\pi^2} \ln(g_c) - \sum_j \frac{T(r_j)}{8\pi^2} \ln(Z_j)$$

Differentiating with respect to $\ln \mu$, this leads precisely to

$$\beta(g) = -\frac{g^3}{16\pi^2} \frac{\left(3T(Ad) - \sum_j T(r_j)(1-\gamma_j) \right)}{1 - T(Ad)g^2/8\pi^2}$$

relation between the two couplings is logarithmic, one cannot be expanded in a Taylor series around zero in the other