Holomorphy
Non-renormalization theorems

Consider

\[ W_{\text{tree}} = \frac{m}{2} \phi^2 + \frac{\lambda}{3} \phi^3 \]

\( \phi \) is a chiral superfield; scalar component \( \phi \), fermion component by \( \psi \).

\( R \)-charge

\[ [R, Q_\alpha] = -Q_\alpha \]

\( R[\psi] = R[\phi] - 1, \ R[\theta] = 1 \)

Lagrangian in toy model has Yukawa coupling

\[ \mathcal{L} \supset \frac{\lambda}{3} \phi \psi \psi \]

which must have zero \( R \)-charge, so

\[ 3R[\phi] - 2 = 0 \]

therefore \( R[W] = 2 \), or \( \mathcal{L}_{\text{int}} = \int d^2 \theta W \)
Toy Model

\[
\begin{array}{c|c|c}
U(1) & \times & U(1)_R \\
\phi & 1 & 1 \\
m & -2 & 0 \\
\lambda & -3 & -1 \\
\end{array}
\]

treat the mass and coupling as background spurion fields
integrate out modes from \( \Lambda \) down to \( \mu \), then the symmetries and holomorphy of the effective superpotential restrict it to be of the form

\[
W_{\text{eff}} = m\phi^2 h\left(\frac{\lambda\phi}{m}\right) = \sum_n a_n \lambda^n m^{1-n} \phi^{n+2},
\]

weak coupling limit \( \lambda \to 0 \) restricts \( n \geq 0 \)
the massless limit \( m \to 0 \) restricts \( n \leq 1 \) so

\[
W_{\text{eff}} = \frac{m}{2} \phi^2 + \frac{\lambda}{3} \phi^3 = W_{\text{tree}}
\]

superpotential is not renormalized
Wavefunction renormalization

\[ \mathcal{L}_{\text{kin.}} = Z \partial_\mu \phi^* \partial^\mu \phi + i Z \bar{\psi} \sigma^\mu \partial_\mu \psi \]

Z is a non-holomorphic function

\[ Z = Z(m, \lambda, m^\dagger, \lambda^\dagger, \mu, \Lambda) \]

If we integrate out modes down to \( \mu > m \) at one-loop order

\[ Z = 1 + c \lambda \lambda^\dagger \ln \left( \frac{\Lambda^2}{\mu^2} \right) \]

where \( c \) is a constant determined by the perturbative calculation. If we integrate out modes down to scales below \( m \) we have

\[ Z = 1 + c \lambda \lambda^\dagger \ln \left( \frac{\Lambda^2}{mm^\dagger} \right) \]

Wavefunction renormalization means that couplings of canonically normalized fields run

running mass and running coupling are given by

\[ \frac{m}{Z}, \frac{\lambda}{Z^{3/2}} \]
Integrating out

Consider a model with two different chiral superfields:

\[ W = \frac{1}{2} M \phi_H^2 + \frac{\lambda}{2} \phi_H \phi^2 \]

three global \( U(1) \) symmetries:

\[
\begin{array}{c|ccc}
 & U(1)_A & U(1)_B & U(1)_R \\
\phi_H & 1 & 0 & 1 \\
\phi & 0 & 1 & \frac{1}{2} \\
M & -2 & 0 & 0 \\
\lambda & -1 & -2 & 0 \\
\end{array}
\]

where \( U(1)_A \) and \( U(1)_B \) are spurious symmetries for \( M, \lambda \neq 0 \)
If we want to integrate out modes down to $\mu < M$, we must integrate out $\phi_H$. An arbitrary term in the effective superpotential has the form

$$\phi^j M^k \lambda^p$$

To preserve the symmetries we must have $j = 4$, $k = -1$, and $p = 2$. By comparing with tree-level perturbation theory we can determine the coefficient:

$$W_{\text{eff}} = -\frac{\lambda^2 \phi^4}{8M}$$

algebraic equation of motion:

$$\frac{\partial W}{\phi_H} = M \phi_H + \frac{\lambda}{2} \phi^2 = 0$$

solve this equation for $\phi_H$ and plug the result back into the superpotential.
Another Example

\[ W = \frac{1}{2} M \phi_H^2 + \frac{\lambda}{2} \phi_H \phi^2 + \frac{y}{6} \phi^3 \]

\( \phi_H \) equation of motion:

\[ \phi_H = -\frac{M}{y} \left( 1 \pm \sqrt{1 - \frac{\lambda y \phi^2}{M^2}} \right) \]

as \( y \to 0 \), the two vacua approach \( \phi_H = -\lambda \phi/(2M) \) (as in previous example) and \( \phi_H = \infty \). Integrating out \( \phi_H \) yields

\[ W_{\text{eff}} = \frac{M^3}{3y^2} \left( 1 - \frac{3\lambda y \phi^2}{2M^2} \pm \left( 1 - \frac{\lambda y \phi^2}{M^2} \right) \sqrt{1 - \frac{\lambda y \phi^2}{M^2}} \right) \]

singularities in \( W_{\text{eff}} \) indicate points in the parameter space and the space of \( \phi \) VEVs where \( \phi_H \) becomes massless and we should not have integrated it out.
Singularities

The mass of $\phi_H$ can be found by calculating

$$\frac{\partial^2 W}{\partial \phi_H^2} = M + y\phi_H$$

and substituting in the solution for $\phi_H$:

$$\frac{\partial^2 W}{\partial \phi_H^2} = \mp M \sqrt{1 - \frac{\lambda y\phi^2}{M^2}}$$

Using holomorphy assign $y$ charges (-3,0,-1) under $U(1)_A \times U(1)_B \times U(1)_R$ then

$$W_{\text{eff}} = \frac{M^3}{y^2} f \left( \frac{\lambda y\phi^2}{M^2} \right)$$

for some function $f$, just as we found from explicitly integrating out $\phi_H$
The holomorphic gauge coupling

chiral superfield for an $SU(N)$ gauge supermultiplet:

$$W^a_\alpha = -i\lambda^a_\alpha(y) + \theta_\alpha D^a(y) - (\sigma^{\mu\nu}\theta)_\alpha F^a_{\mu\nu}(y) - (\theta\theta)\sigma^\mu D_\mu \lambda^{a\dagger}(y),$$

$a = 1, \ldots, N^2 - 1$

$$\tau \equiv \frac{\theta_{YM}}{2\pi} + \frac{4\pi i}{g^2},$$

SUSY Yang–Mills action as a superpotential term

$$\frac{1}{16\pi i} \int d^4x \int d^2\theta \, \tau \, W^a_\alpha W^a_\alpha + h.c. = \int d^4x \left[ -\frac{1}{4g^2} F^{a\mu\nu} F^a_{\mu\nu} - \frac{\theta_{YM}}{32\pi^2} F^{a\mu\nu} \tilde{F}^a_{\mu\nu} + \frac{i}{g^2} \lambda^{a\dagger} \bar{\sigma}^\mu D_\mu \lambda^a + \frac{1}{2g^2} D^a D^a \right]$$

g only in $\tau$ which is a holomorphic parameter, but gauge fields are not canonically normalized
Running coupling

one-loop running $g$ is given by the RG equation:

$$\mu \frac{dg}{d\mu} = -\frac{b}{16\pi^2} g^3$$

where for an $SU(N)$ gauge theory with $F$ flavors and $\mathcal{N} = 1$ SUSY,

$$b = 3N - F$$

The solution for the running coupling is

$$\frac{1}{g^2(\mu)} = -\frac{b}{8\pi^2} \ln \left( \frac{|\Lambda|}{\mu} \right)$$

where $|\Lambda|$ is the intrinsic scale of the non-Abelian gauge theory.
Holomorphic Intrinsic Scale

\[ \tau_{1-\text{loop}} = \frac{\theta_{YM}}{2\pi} + \frac{4\pi i}{g^2(\mu)} \]
\[ = \frac{1}{2\pi i} \ln \left[ \left( \frac{|\Lambda|}{\mu} \right)^b e^{i\theta_{YM}} \right] \]

\[ \Lambda \equiv |\Lambda| e^{i\theta_{YM}/b} \]
\[ = \mu e^{2\pi i \tau/b} \]

\[ \tau_{1-\text{loop}} = \frac{b}{2\pi i} \ln \left( \frac{\Lambda}{\mu} \right) \]
CP Violating Term

\[ F^{a\mu\nu} \tilde{F}^a_{\mu\nu} = 4\epsilon^{\mu\nu\rho\sigma} \partial_\mu \text{Tr} \left( A_\nu \partial_\rho A_\sigma + \frac{2}{3} A_\nu A_\rho A_\sigma \right) \]

total derivative: no effect in perturbation theory

nonperturbative effects: instantons have a nontrivial, topological winding number, \( n \)

\[ \frac{\theta_{YM}}{32\pi^2} \int d^4 x \ F^{a\mu\nu} \tilde{F}^a_{\mu\nu} = n \theta_{YM} . \]

Since the path integral has the form

\[ \int \mathcal{D} A^a \mathcal{D} \lambda^a \mathcal{D} D^a \ e^{iS} \]

\[ \theta_{YM} \rightarrow \theta_{YM} + 2\pi \]

is a symmetry of the theory
**Instanton Action**

The Euclidean action of an instanton configuration can be bounded

\[
0 \leq \int d^4x \text{Tr} \left( F_{\mu\nu} \pm \tilde{F}_{\mu\nu} \right)^2 = \int d^4x \text{Tr} \left( 2F^2 \pm 2F\tilde{F} \right)
\]

\[
\int d^4x \text{Tr} F^2 \geq |\int d^4x \text{Tr} F\tilde{F}| = 16\pi^2|n|
\]

One instanton effects are suppressed by

\[
e^{-S_{\text{int}}} = e^{-\left(8\pi^2/g^2(\mu)\right) + i\theta_{\text{YM}}} = \left( \frac{A}{\mu} \right)^b
\]
Effective Superpotential

integrate down to the scale $\mu$

$$W_{\text{eff}} = \frac{\tau(\Lambda; \mu)}{16\pi i} W_\alpha W_\alpha$$

physics periodic in $\theta_{\text{YM}}$ equivalent to

$$\Lambda \rightarrow e^{2\pi i/\Lambda} \Lambda$$

in general:

$$\tau(\Lambda; \mu) = \frac{b}{2\pi i} \ln \left( \frac{\Lambda}{\mu} \right) + f(\Lambda; \mu),$$

where $f$ has Taylor series representation in positive powers of $\Lambda$.

$$\Lambda \rightarrow e^{2\pi i/\Lambda} \Lambda$$

in perturbative term shifts $\theta_{\text{YM}}$ by $2\pi$, $f$ must be invariant under this transformation, so the Taylor series must be in positive powers of $\Lambda^b$
Effective Superpotential

in general, we can write:

\[ \tau(\Lambda; \mu) = \frac{b}{2\pi i} \ln \left( \frac{\Lambda}{\mu} \right) + \sum_{n=1}^{\infty} a_n \left( \frac{\Lambda}{\mu} \right)^{bn}. \]

holomorphic gauge coupling only receives one-loop corrections and non-perturbative \( n \)-instanton corrections, no perturbative running beyond one-loop
Symmetry of $SU(N)$ SUSY YM

$U(1)_R$ symmetry is broken by instantons
anomaly index: gaugino $R$-charge times $T(Ad)$

Because of the anomaly, the chiral rotation

$$\lambda^a \rightarrow e^{i\alpha} \lambda^a$$

is equivalent to shift

$$\theta_{YM} \rightarrow \theta_{YM} - 2N\alpha$$

$2N$ because the gaugino $\lambda^a$ is in the adjoint representation, $2N$ zero modes in instanton background
chiral rotation is only a symmetry when

$$\alpha = \frac{k\pi}{N}$$

$U(1)_R$ explicitly broken to discrete $Z_{2N}$ subgroup
Spurion Analysis

Treat $\tau$ as a spurion chiral superfield, define spurious symmetry

$$\lambda^a \rightarrow e^{i\alpha} \lambda^a, \quad \tau \rightarrow \tau + \frac{N\alpha}{\pi}$$

Assuming that SUSY YM has no massless particles, then holomorphy and symmetries determine the effective superpotential to be:

$$W_{\text{eff}} = a\mu^3 e^{2\pi i\tau/N}$$

This is the unique form because under the spurious $U(1)_R$ rotation the superpotential (which has $R$-charge 2) transforms as

$$W_{\text{eff}} \rightarrow e^{2i\alpha} W_{\text{eff}}$$
Gaugino condensation

treat \( \tau \) as a background chiral superfield, the \( \mathcal{F} \) component of \( \tau \) \( (\mathcal{F}_\tau) \) acts as a source for \( \lambda^a \lambda^a \) gaugino condensate given by

\[
\langle \lambda^a \lambda^a \rangle = 16\pi i \frac{\partial}{\partial F_\tau} \ln Z = 16\pi i \frac{\partial}{\partial F_\tau} \int d^2 \theta W_{\text{eff}} \\
= 16\pi i \frac{\partial}{\partial \tau} W_{\text{eff}} = 16\pi i \frac{2\pi i}{N} a\mu^3 e^{2\pi i \tau/N}
\]

Drop nonperturbative corrections to running, plug in \( b = 3N \):

\[
\langle \lambda^a \lambda^a \rangle = -\frac{32\pi^2}{N} a\Lambda^3
\]

vacuum does not respect the discrete \( Z_N \) symmetry since

\[
\langle \lambda^a \lambda^a \rangle \rightarrow e^{2i\alpha} \langle \lambda^a \lambda^a \rangle
\]

only invariant for \( k = 0 \) or \( k = N \)

\( Z_{2N} \rightarrow Z_2 \), implies \( N \) degenerate vacua

\( \theta_{\text{YM}} \rightarrow \theta_{\text{YM}} + 2\pi \) sweeps out \( N \) different values for \( \langle \lambda^a \lambda^a \rangle \)
NSVZ revisited

Three seemingly contradictory statements:

- the SUSY gauge coupling runs only at one-loop

\[ \beta(g) = -\frac{g^3}{16\pi^2} \left( 3T(Ad) - \sum_j T(r_j) \right) \]

with matter chiral superfields \( Q_j \) in representations \( r_j \)

- the “exact” \( \beta \) function is

\[ \beta(g) = -\frac{g^3}{16\pi^2} \left( 3T(Ad) - \sum_j T(r_j)(1-\gamma_j) \right) \]

- one- and two-loop terms in \( \beta \) function are scheme independent
Changing renormalization schemes

\[ g' = g + ag^3 + \mathcal{O}(g^5) \]

If \( \beta \) function is given by

\[ \beta(g) = b_1 g^3 + b_2 g^5 + \mathcal{O}(g^7) \]

then

\[ \beta'(g') = \beta(g) \frac{\partial g}{\partial g'} = b_1 g'^3 + b_2 g'^5 + \mathcal{O}(g'^7) \]

dependence on \( a \) only appears at higher order
Holomorphic vs Canonical Coupling

\[ \mathcal{L}_h = \frac{1}{4} \int d^2 \theta \frac{1}{g_h^2} W^a(V_h) W^a(V_h) + h.c., \]

\[ \frac{1}{g_h^2} = \frac{1}{g^2} - i \frac{\theta_{YM}}{8\pi^2} = \frac{\tau}{4\pi i}, \]

\[ V_h = (A^a_\mu, \chi^a, D^a). \]

canonical gauge coupling for canonically normalized fields:

\[ \mathcal{L}_c = \frac{1}{4} \int d^2 \theta \left( \frac{1}{g_c^2} - i \frac{\theta_{YM}}{8\pi^2} \right) W^a(g_c V_c) W^a(g_c V_c) + h.c. \]

not equivalent under \( V_h = g_c V_c \) because of rescaling anomaly with matter fields \( Q_j \), additional rescaling anomaly from:

\[ Q'_j = Z_j(\mu, \mu')^{1/2} Q_j \]

rescaling anomaly completely determined by the axial anomaly
Rescaling Anomaly

for fermions with a rescaling $Z = e^{2i\alpha}$. We can rewrite the axial anomaly in a manifestly supersymmetric form using the path integral measure as

$$D(e^{i\alpha}Q)D(e^{-i\alpha}\bar{Q}) = DQ\bar{D}\bar{Q}$$

$$\times \exp\left(-\frac{i}{4} \int d^2\theta \left(\frac{T(r_j)}{8\pi^2}2i\alpha\right) W^a W^a + h.c.\right)$$

take $\alpha$ to be complex gives general case:

$$D(Z_j^{1/2}Q_j)D(Z_j^{1/2}\bar{Q}_j) = DQ_j\bar{D}\bar{Q}_j$$

$$\times \exp\left(-\frac{i}{4} \int d^2\theta \left(\frac{T(r_j)}{8\pi^2} \ln Z_j\right) W^a W^a + h.c.\right)$$
Rescaling Anomaly

for the gauge fields (gauginos) taking $Z_\lambda = g_c^2$

$$\mathcal{D}(g_c V_c) = \mathcal{D}V_c \times \exp \left( -\frac{i}{4} \int d^2 \theta \left( \frac{2T(Ad)}{8\pi^2} \ln(g_c) \right) W^a(g_c V_c) W^a(g_c V_c) + h.c. \right)$$

Thus, for pure SUSY Yang–Mills we have

$$Z = \int \mathcal{D}V_h \exp \left( \frac{i}{4} \int d^2 \theta \frac{1}{g_h^2} W^a(V_h) W^a(V_h) + h.c. \right)$$

$$= \int \mathcal{D}V_c \exp \left( \frac{i}{4} \int d^2 \theta \left( \frac{1}{g_h^2} - \frac{2T(Ad)}{8\pi^2} \ln(g_c) \right) W^a(g_c V_c) W^a(g_c V_c) + h.c. \right)$$

So

$$\frac{1}{g_c^2} = \text{Re} \left( \frac{1}{g_h^2} \right) - \frac{2T(Ad)}{8\pi^2} \ln(g_c)$$
Rescaling Anomaly

including the matter fields:

\[
\frac{1}{g_c^2} = \text{Re} \left( \frac{1}{g_h^2} \right) - \frac{2T(Ad)}{8\pi^2} \ln(g_c) - \sum_j \frac{T(r_j)}{8\pi^2} \ln(Z_j)
\]

Differentiating with respect to \( \ln \mu \), this leads precisely to

\[
\beta(g) = -\frac{g^3}{16\pi^2} \left( \frac{3T(Ad) - \sum_j T(r_j)(1-\gamma_j)}{1-T(Ad)g^2/8\pi^2} \right)
\]

relation between the two couplings is logarithmic, one cannot be expanded in a Taylor series around zero in the other