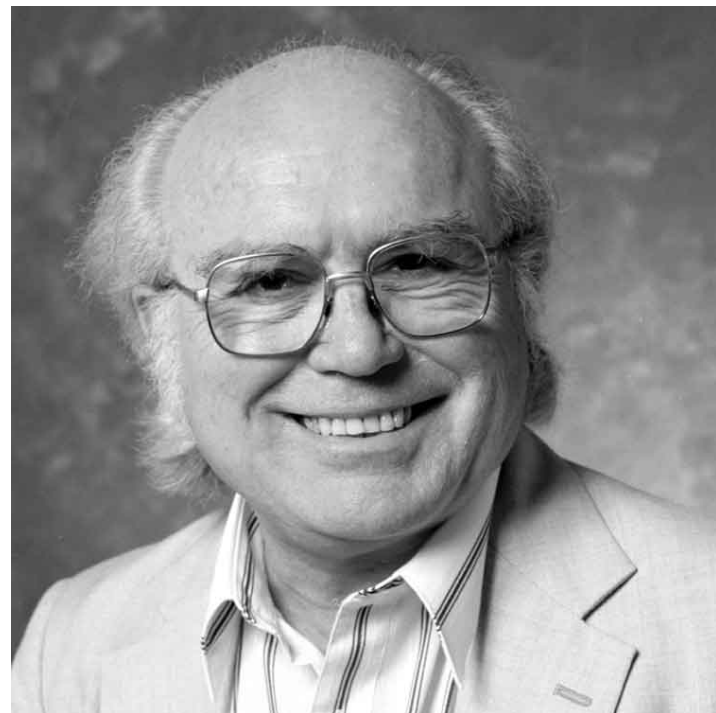


SUSY Lagrangians

Wess-Zumino



The free Wess–Zumino model

$$S = \int d^4x (\mathcal{L}_s + \mathcal{L}_f)$$
$$\mathcal{L}_s = \partial^\mu \phi^* \partial_\mu \phi, \quad \mathcal{L}_f = i\psi^\dagger \bar{\sigma}^\mu \partial_\mu \psi.$$

$$g^{\mu\nu} = \eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$$

$$\phi \rightarrow \phi + \delta\phi$$
$$\psi \rightarrow \psi + \delta\psi$$

$$\delta\phi = \epsilon^\alpha \psi_\alpha$$
$$= \epsilon^\alpha \epsilon_{\alpha\beta} \psi^\beta \equiv \epsilon\psi$$

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\epsilon\psi = -\psi^\beta \epsilon_{\alpha\beta} \epsilon^\alpha = \psi^\beta \epsilon_{\beta\alpha} \epsilon^\alpha = \psi\epsilon$$

$$\delta\phi^* = \epsilon^\dagger_{\dot{\alpha}} \psi^{\dagger\dot{\alpha}} \equiv \epsilon^\dagger \psi^\dagger$$

$$\delta\mathcal{L}_s = \epsilon \partial^\mu \psi \partial_\mu \phi^* + \epsilon^\dagger \partial^\mu \psi^\dagger \partial_\mu \phi$$

$$\delta\psi_\alpha = -i(\sigma^\nu \epsilon^\dagger)_\alpha \partial_\nu \phi \quad \delta\psi^\dagger_{\dot{\alpha}} = i(\epsilon \sigma^\nu)_{\dot{\alpha}} \partial_\nu \phi^*$$

$$\delta\mathcal{L}_f = -\epsilon \sigma^\nu \partial_\nu \phi^* \bar{\sigma}^\mu \partial_\mu \psi + \psi^\dagger \bar{\sigma}^\mu \sigma^\nu \epsilon^\dagger \partial_\mu \partial_\nu \phi$$

Pauli identities:

$$[\sigma^\mu \bar{\sigma}^\nu + \sigma^\nu \bar{\sigma}^\mu]^\beta_\alpha = 2\eta^{\mu\nu} \delta^\beta_\alpha \quad [\bar{\sigma}^\mu \sigma^\nu + \bar{\sigma}^\nu \sigma^\mu]_{\dot{\alpha}}^{\dot{\beta}} = 2\eta^{\mu\nu} \delta_{\dot{\alpha}}^{\dot{\beta}}$$

$$\begin{aligned} \delta\mathcal{L}_f = & -\epsilon \partial^\mu \psi \partial_\mu \phi^* - \epsilon^\dagger \partial^\mu \psi^\dagger \partial_\mu \phi \\ & + \partial_\mu (\epsilon \sigma^\mu \bar{\sigma}^\nu \psi \partial_\nu \phi^* - \epsilon \psi \partial^\mu \phi^* + \epsilon^\dagger \psi^\dagger \partial^\mu \phi). \end{aligned}$$

total derivative so:

$$\delta S = 0$$

Commutators of SUSY transformations

$$(\delta_{\epsilon_2} \delta_{\epsilon_1} - \delta_{\epsilon_1} \delta_{\epsilon_2})\phi = -i(\epsilon_1 \sigma^\mu \epsilon_2^\dagger - \epsilon_2 \sigma^\mu \epsilon_1^\dagger) \partial_\mu \phi$$

$$(\delta_{\epsilon_2} \delta_{\epsilon_1} - \delta_{\epsilon_1} \delta_{\epsilon_2})\psi_\alpha = -i(\sigma^\nu \epsilon_1^\dagger)_\alpha \epsilon_2 \partial_\nu \psi + i(\sigma^\nu \epsilon_2^\dagger)_\alpha \epsilon_1 \partial_\nu \psi$$

Fierz identity:

$$\chi_\alpha (\xi \eta) = -\xi_\alpha (\chi \eta) - (\xi \chi) \eta_\alpha$$

$$\begin{aligned} (\delta_{\epsilon_2} \delta_{\epsilon_1} - \delta_{\epsilon_1} \delta_{\epsilon_2})\psi_\alpha &= -i(\epsilon_1 \sigma^\mu \epsilon_2^\dagger - \epsilon_2 \sigma^\mu \epsilon_1^\dagger) \partial_\mu \psi_\alpha \\ &\quad + i(\epsilon_{1\alpha} \epsilon_2^\dagger \bar{\sigma}^\mu \partial_\mu \psi - \epsilon_{2\alpha} \epsilon_1^\dagger \bar{\sigma}^\mu \partial_\mu \psi). \end{aligned}$$

SUSY algebra closes on-shell.

on-shell the fermion EOM reduces DOF by two

$$p_\mu = (p, 0, 0, p)$$

$$\bar{\sigma}^\mu p_\mu \psi = \begin{pmatrix} 0 & 0 \\ 0 & 2p \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$$

projects out half of DOF

	off-shell	on-shell
ϕ, ϕ^*	2 d.o.f.	2 d.o.f.
$\psi_\alpha, \psi^\dagger_{\dot{\alpha}}$	4 d.o.f.	2 d.o.f.

SUSY is not manifest off-shell

trick: add an auxiliary boson field \mathcal{F}

	off-shell	on-shell
$\mathcal{F}, \mathcal{F}^*$	2 d.o.f.	0 d.o.f.

$$\mathcal{L}_{\text{aux}} = \mathcal{F}^* \mathcal{F}$$

$$\delta\mathcal{F} = -i\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\psi, \quad \delta\mathcal{F}^* = i\partial_\mu\psi^\dagger\bar{\sigma}^\mu\epsilon$$

$$\delta\mathcal{L}_{\text{aux}} = i\partial_\mu\psi^\dagger\bar{\sigma}^\mu\epsilon\mathcal{F} - i\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\psi\mathcal{F}^*$$

modify the transformation of the fermion:

$$\delta\psi_\alpha = -i(\sigma^\nu\epsilon^\dagger)_\alpha\partial_\nu\phi + \epsilon_\alpha\mathcal{F}, \quad \delta\psi^\dagger_{\dot{\alpha}} = +i(\epsilon\sigma^\nu)_{\dot{\alpha}}\partial_\nu\phi^* + \epsilon^\dagger_{\dot{\alpha}}\mathcal{F}^*$$

$$\begin{aligned} \delta^{\text{new}}\mathcal{L}_f &= \delta^{\text{old}}\mathcal{L}_f + i\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\psi\mathcal{F}^* + i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\epsilon\mathcal{F} \\ &= \delta^{\text{old}}\mathcal{L}_f + i\epsilon^\dagger\bar{\sigma}^\mu\partial_\mu\psi\mathcal{F}^* - i\partial_\mu\psi^\dagger\bar{\sigma}^\mu\epsilon\mathcal{F} + \partial_\mu(i\psi^\dagger\bar{\sigma}^\mu\epsilon\mathcal{F}) \end{aligned}$$

last term is a total derivative

$$S^{\text{new}} = \int d^4x \mathcal{L}_{\text{free}} = \int d^4x (\mathcal{L}_s + \mathcal{L}_f + \mathcal{L}_{\text{aux}})$$

is invariant under SUSY transformations:

$$\delta S^{\text{new}} = 0$$

Commutator of two SUSY transformations acting on the fermion

$$\begin{aligned}
 (\delta_{\epsilon_2} \delta_{\epsilon_1} - \delta_{\epsilon_1} \delta_{\epsilon_2}) \psi_\alpha &= -i(\epsilon_1 \sigma^\mu \epsilon_2^\dagger - \epsilon_2 \sigma^\mu \epsilon_1^\dagger) \partial_\mu \psi_\alpha \\
 &\quad + i(\epsilon_{1\alpha} \epsilon_2^\dagger \bar{\sigma}^\mu \partial_\mu \psi - \epsilon_{2\alpha} \epsilon_1^\dagger \bar{\sigma}^\mu \partial_\mu \psi) \\
 &\quad + \delta_{\epsilon_2} \epsilon_{1\alpha} \mathcal{F} - \delta_{\epsilon_1} \epsilon_{2\alpha} \mathcal{F}
 \end{aligned}$$

$$\delta_{\epsilon_2} \epsilon_{1\alpha} \mathcal{F} - \delta_{\epsilon_1} \epsilon_{2\alpha} \mathcal{F} = \epsilon_{1\alpha} (-i \epsilon_2^\dagger \bar{\sigma}^\mu \partial_\mu \psi) - \epsilon_{2\alpha} (-i \epsilon_1^\dagger \bar{\sigma}^\mu \partial_\mu \psi)$$

$$(\delta_{\epsilon_2} \delta_{\epsilon_1} - \delta_{\epsilon_1} \delta_{\epsilon_2}) \psi_\alpha = -i(\epsilon_1 \sigma^\mu \epsilon_2^\dagger - \epsilon_2 \sigma^\mu \epsilon_1^\dagger) \partial_\mu \psi_\alpha$$

SUSY algebra closes for off-shell fermions

Commutator acting on the auxiliary field

$$\begin{aligned}
 (\delta_{\epsilon_2} \delta_{\epsilon_1} - \delta_{\epsilon_1} \delta_{\epsilon_2}) \mathcal{F} &= \delta_{\epsilon_2} (-i \epsilon_1^\dagger \bar{\sigma}^\mu \partial_\mu \psi) - \delta_{\epsilon_1} (-i \epsilon_2^\dagger \bar{\sigma}^\mu \partial_\mu \psi) \\
 &= -i \epsilon_1^\dagger \bar{\sigma}^\mu \partial_\mu (-i \sigma^\nu \epsilon_2^\dagger \partial_\nu \phi + \epsilon_2 \mathcal{F}) \\
 &\quad + i \epsilon_2^\dagger \bar{\sigma}^\mu \partial_\mu (-i \sigma^\nu \epsilon_1^\dagger \partial_\nu \phi + \epsilon_1 \mathcal{F}) \\
 &= -i (\epsilon_1 \sigma^\mu \epsilon_2^\dagger - \epsilon_2 \sigma^\mu \epsilon_1^\dagger) \partial_\mu \mathcal{F} \\
 &\quad - \epsilon_1^\dagger \bar{\sigma}^\mu \sigma^\nu \epsilon_2^\dagger \partial_\mu \partial_\nu \phi + \epsilon_2^\dagger \bar{\sigma}^\mu \sigma^\nu \epsilon_1^\dagger \partial_\mu \partial_\nu \phi
 \end{aligned}$$

Thus for

$$X = \phi, \phi^*, \psi, \psi^\dagger, \mathcal{F}, \mathcal{F}^*$$

$$(\delta_{\epsilon_2} \delta_{\epsilon_1} - \delta_{\epsilon_1} \delta_{\epsilon_2}) X = -i (\epsilon_1 \sigma^\mu \epsilon_2^\dagger - \epsilon_2 \sigma^\mu \epsilon_1^\dagger) \partial_\mu X$$

Noether



Noether's Theorem

Noether theorem:

corresponding to every continuous symmetry is a conserved current.

infinitesimal symmetry $(1 + \epsilon T)X = X + \delta X$

$$\delta\mathcal{L} = \mathcal{L}(X + \delta X) - \mathcal{L}(X) = \partial_\mu V^\mu$$

EOM:

$$\partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu X)} \right) = \frac{\partial\mathcal{L}}{\partial X},$$

$$\begin{aligned} \partial_\mu V^\mu = \delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial X} \delta X + \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu X)} \right) \delta(\partial_\mu X) \\ &= \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu X)} \right) \delta X + \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu X)} \right) \partial_\mu \delta X \\ &= \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu X)} \delta X \right) \end{aligned}$$

$$\epsilon \partial_\mu J^\mu = \partial_\mu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu X)} \delta X - V^\mu \right)$$

Conserved SuperCurrent

conserved supercurrent, J_α^μ :

$$\epsilon J^\mu + \epsilon^\dagger J^{\dagger\mu} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu X)} \delta X - V^\mu$$

$$\epsilon J^\mu + \epsilon^\dagger J^{\dagger\mu} = \delta\phi \partial^\mu \phi^* + \delta\phi^* \partial^\mu \phi + i\psi^\dagger \bar{\sigma}^\mu \delta\psi - V^\mu$$

$$\begin{aligned} \epsilon J^\mu + \epsilon^\dagger J^{\dagger\mu} &= \epsilon\psi \partial^\mu \phi^* + \epsilon^\dagger \psi^\dagger \partial^\mu \phi + i\psi^\dagger \bar{\sigma}^\mu (-i\sigma^\nu \epsilon^\dagger \partial_\nu \phi + \epsilon \mathcal{F}) \\ &\quad - \epsilon \sigma^\mu \bar{\sigma}^\nu \psi \partial_\nu \phi^* + \epsilon\psi \partial^\mu \phi^* - \epsilon^\dagger \psi^\dagger \partial^\mu \phi - i\psi^\dagger \bar{\sigma}^\mu \epsilon \mathcal{F} \\ &= 2\epsilon\psi \partial^\mu \phi^* + \psi^\dagger \bar{\sigma}^\mu \sigma^\nu \epsilon^\dagger \partial_\nu \phi - \epsilon \sigma^\mu \bar{\sigma}^\nu \psi \partial_\nu \phi^* \end{aligned}$$

Using the Pauli identity:

$$J_{\alpha}^{\mu} = (\sigma^{\nu} \bar{\sigma}^{\mu} \psi)_{\alpha} \partial_{\nu} \phi^{*}, \quad J^{\dagger \mu}{}_{\dot{\alpha}} = (\psi^{\dagger} \bar{\sigma}^{\mu} \sigma^{\nu})_{\dot{\alpha}} \partial_{\nu} \phi.$$

conserved supercharges:

$$Q_{\alpha} = \sqrt{2} \int d^3x J_{\alpha}^0, \quad Q^{\dagger}{}_{\dot{\alpha}} = \sqrt{2} \int d^3x J^{\dagger 0}{}_{\dot{\alpha}}$$

generate SUSY transformations

$$[\epsilon Q + \epsilon^{\dagger} Q^{\dagger}, X] = -i\sqrt{2} \delta X$$

Commutators of the supercharges acting on fields give:

$$\begin{aligned} \left[\epsilon_2 Q + \epsilon_2^{\dagger} Q^{\dagger}, [\epsilon_1 Q + \epsilon_1^{\dagger} Q^{\dagger}, X] \right] &- \left[\epsilon_1 Q + \epsilon_1^{\dagger} Q^{\dagger}, [\epsilon_2 Q + \epsilon_2^{\dagger} Q^{\dagger}, X] \right] \\ &= 2(\epsilon_2 \sigma^{\mu} \epsilon_1^{\dagger} - \epsilon_1 \sigma^{\mu} \epsilon_2^{\dagger}) i \partial_{\mu} X \end{aligned}$$

$$\left[[\epsilon_2 Q + \epsilon_2^{\dagger} Q^{\dagger}, \epsilon_1 Q + \epsilon_1^{\dagger} Q^{\dagger}], X \right] = 2(\epsilon_2 \sigma^{\mu} \epsilon_1^{\dagger} - \epsilon_1 \sigma^{\mu} \epsilon_2^{\dagger}) [P_{\mu}, X]$$

Since this is true for any X , we have

$$[\epsilon_2 Q + \epsilon_2^\dagger Q^\dagger, \epsilon_1 Q + \epsilon_1^\dagger Q^\dagger] = 2(\epsilon_2 \sigma^\mu \epsilon_1^\dagger - \epsilon_1 \sigma^\mu \epsilon_2^\dagger) P_\mu$$

Since ϵ_1 and ϵ_2 are arbitrary, we have

$$\begin{aligned} [\epsilon_2 Q, \epsilon_1^\dagger Q^\dagger] &= 2\epsilon_2 \sigma^\mu \epsilon_1^\dagger P_\mu \\ [\epsilon_2^\dagger Q, \epsilon_1 Q^\dagger] &= -2\epsilon_2 \sigma^\mu \epsilon_1^\dagger P_\mu \\ [\epsilon_2 Q, \epsilon_1 Q] &= [\epsilon_2^\dagger Q^\dagger, \epsilon_1^\dagger Q^\dagger] = 0 \end{aligned}$$

Extracting the arbitrary ϵ_1 and ϵ_2 :

$$\begin{aligned} \{Q_\alpha, Q^\dagger_{\dot{\alpha}}\} &= 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu, \\ \{Q_\alpha, Q_\beta\} &= \{Q^\dagger_{\dot{\alpha}}, Q^\dagger_{\dot{\beta}}\} = 0 \end{aligned}$$

which is just the SUSY algebra

The interacting Wess–Zumino model

$$\mathcal{L}_{\text{free}} = \partial^\mu \phi^{*j} \partial_\mu \phi_j + i\psi^{\dagger j} \bar{\sigma}^\mu \partial_\mu \psi_j + \mathcal{F}^{*j} \mathcal{F}_j$$

$$\begin{aligned} \delta\phi_j &= \epsilon\psi_j & \delta\phi^{*j} &= \epsilon^\dagger\psi^{\dagger j} \\ \delta\psi_{j\alpha} &= -i(\sigma^\mu\epsilon^\dagger)_\alpha \partial_\mu \phi_j + \epsilon_\alpha \mathcal{F}_j & \delta\psi_{\dot{\alpha}}^{\dagger j} &= i(\epsilon\sigma^\mu)_{\dot{\alpha}} \partial_\mu \phi^{*j} + \epsilon_{\dot{\alpha}}^\dagger \mathcal{F}^{*j} \\ \delta\mathcal{F}_j &= -i\epsilon^\dagger \bar{\sigma}^\mu \partial_\mu \psi_j & \delta\mathcal{F}^{*j} &= i\partial_\mu \psi^{\dagger j} \bar{\sigma}^\mu \epsilon \end{aligned}$$

most general set of renormalizable interactions:

$$\mathcal{L}_{\text{int}} = -\frac{1}{2}W^{jk}\psi_j\psi_k + W^j\mathcal{F}_j + h.c.,$$

$\psi_j\psi_k = \psi_j^\alpha \epsilon_{\alpha\beta} \psi_k^\beta$ is symmetric under $j \leftrightarrow k$, $\Rightarrow W^{jk}$

potential $U(\phi_j, \phi^{*j})$ breaks SUSY, since a SUSY transformation gives

$$\delta U = \frac{\partial U}{\partial \phi_j} \epsilon\psi_j + \frac{\partial U}{\partial \phi^{*j}} \epsilon^\dagger\psi^{\dagger j}$$

which is linear in ψ_j and $\psi^{\dagger j}$ with no derivatives or \mathcal{F} dependence and cannot be canceled by any other term in $\delta\mathcal{L}_{\text{int}}$

require SUSY

$$\delta\mathcal{L}_{\text{int}}|_{4\text{-spinor}} = -\frac{1}{2}\frac{\partial W^{jk}}{\partial\phi_n}(\epsilon\psi_n)(\psi_j\psi_k) - \frac{1}{2}\frac{\partial W^{jk}}{\partial\phi^{*n}}(\epsilon^\dagger\psi^\dagger n)(\psi_j\psi_k) + h.c.$$

Fierz identity \Rightarrow

$$(\epsilon\psi_j)(\psi_k\psi_n) + (\epsilon\psi_k)(\psi_n\psi_j) + (\epsilon\psi_n)(\psi_j\psi_k) = 0,$$

$\delta\mathcal{L}_{\text{int}}|_{4\text{-spinor}}$ vanishes iff $\partial W^{jk}/\partial\phi_n$ is totally symmetric under the interchange of j, k, n . We also need

$$\frac{\partial W^{jk}}{\partial\phi^{*n}} = 0$$

so W^{jk} is *analytic* (*holomorphic*)

define superpotential W :

$$W^{jk} = \frac{\partial^2}{\partial\phi_j\partial\phi_k} W$$

for renormalizable interactions

$$W = E^j \phi_j + \frac{1}{2} M^{jk} \phi_j \phi_k + \frac{1}{6} y^{jkn} \phi_j \phi_k \phi_n$$

and M^{jk} , y^{jkn} are symmetric under interchange of indices.
take $E^j = 0$ so SUSY is unbroken

$$\delta \mathcal{L}_{\text{int}}|_{\partial} = -i W^{jk} \partial_{\mu} \phi_k \psi_j \sigma^{\mu} \epsilon^{\dagger} - i W^j \partial_{\mu} \psi_j \sigma^{\mu} \epsilon^{\dagger} + h.c.$$

$$W^{jk} \partial_{\mu} \phi_k = \partial_{\mu} \left(\frac{\partial W}{\partial \phi_j} \right)$$

so $\delta \mathcal{L}_{\text{int}}|_{\partial}$ will be a total derivative iff

$$W^j = \frac{\partial W}{\partial \phi_j}$$

remaining terms:

$$\delta \mathcal{L}_{\text{int}}|_{\mathcal{F}, \mathcal{F}^*} = -W^{jk} \mathcal{F}_j \epsilon \psi_k + \frac{\partial W^j}{\partial \phi_k} \epsilon \psi_k \mathcal{F}_j$$

identically cancel if previous conditions are satisfied

proof did not rely on the functional form of W , only that it was holomorphic

integrate out auxiliary fields

action is quadratic in \mathcal{F}

$$\mathcal{L}_{\mathcal{F}} = \mathcal{F}_j \mathcal{F}^{*j} + W^j \mathcal{F}_j + W_j^* \mathcal{F}^{*j}$$

perform the corresponding Gaussian path integral exactly by solving its algebraic equation of motion:

$$\mathcal{F}_j = -W_j^*, \quad \mathcal{F}^{*j} = -W^j$$

without auxiliary fields SUSY transformation ψ would be different for each choice of W

plugging in to \mathcal{L} :

$$\begin{aligned} \mathcal{L} = & \partial^\mu \phi^{*j} \partial_\mu \phi_j + i \psi^{\dagger j} \bar{\sigma}^\mu \partial_\mu \psi_j \\ & - \frac{1}{2} (W^{jk} \psi_j \psi_k + W^{*jk} \psi^{\dagger j} \psi^{\dagger k}) - W^j W_j^* \end{aligned}$$

WZ Lagrangian

$$V(\phi, \phi^*) = W^j W_j^* = \mathcal{F}_j \mathcal{F}^{*j} = M_{jn}^* M^{nk} \phi^{*j} \phi_k + \frac{1}{2} M^{jm} y_{knm}^* \phi_j \phi^{*k} \phi^{*n} + \frac{1}{2} M_{jm}^* y^{knm} \phi^{*j} \phi_k \phi_n + \frac{1}{4} y^{jkm} y_{npm}^* \phi_j \phi_k \phi^{*n} \phi^{*p}$$

as required by SUSY:

$$V(\phi, \phi^*) \geq 0$$

interacting Wess–Zumino model:

$$\begin{aligned} \mathcal{L}_{\text{WZ}} = & \partial^\mu \phi^{*j} \partial_\mu \phi_j + i \psi^{\dagger j} \bar{\sigma}^\mu \partial_\mu \psi_j \\ & - \frac{1}{2} M^{jk} \psi_j \psi_k - \frac{1}{2} M_{jk}^* \psi^{\dagger j} \psi^{\dagger k} - V(\phi, \phi^*) \\ & - \frac{1}{2} y^{jkn} \phi_j \psi_k \psi_n - \frac{1}{2} y_{jkn}^* \phi^{*j} \psi^{\dagger k} \psi^{\dagger n}. \end{aligned}$$

quartic coupling is $|y|^2$ as required to cancel the Λ^2 divergence in ϕ mass

$|\text{cubic coupling}|^2 \propto \text{quartic coupling} \times |M|^2$ as required to cancel the $\log \Lambda$ divergence

linearized equations of motion

$$\begin{aligned}\partial^\mu \partial_\mu \phi_j &= -M_{jn}^* M^{nk} \phi_k + \dots; \\ i\bar{\sigma}^\mu \partial_\mu \psi_j &= M_{jk}^* \psi^{\dagger k} + \dots; \\ i\sigma^\mu \partial_\mu \psi^{\dagger j} &= M^{jk} \psi_k + \dots\end{aligned}$$

Multiplying ψ eqns by $i\sigma^\nu \partial_\nu$, and $i\bar{\sigma}^\nu \partial_\nu$, and using the Pauli identity we obtain

$$\begin{aligned}\partial^\mu \partial_\mu \psi_j &= -M_{jn}^* M^{nk} \psi_k + \dots; \\ \partial^\mu \partial_\mu \psi^{\dagger k} &= -\psi^{\dagger j} M_{jn}^* M^{nk} + \dots\end{aligned}$$

scalars and fermions have the same mass eigenvalues, as required by SUSY

diagonalizing gives a collection of massive chiral supermultiplets.

Yang-Mills



SUSY Yang–Mills

under a gauge transformation gauge field, A_μ^a , and gaugino field, λ^a , transform as:

$$\begin{aligned}\delta_{\text{gauge}} A_\mu^a &= -\partial_\mu \Lambda^a + g f^{abc} A_\mu^b \Lambda^c \\ \delta_{\text{gauge}} \lambda^a &= g f^{abc} \lambda^b \Lambda^c\end{aligned}$$

where Λ^a is an infinitesimal gauge transformation parameter, g is the gauge coupling, and f^{abc} are the antisymmetric structure constants of the gauge group which satisfy

$$[T_{\mathbf{r}}^a, T_{\mathbf{r}}^b] = i f^{abc} T_{\mathbf{r}}^c$$

for the generators T^a for any representation \mathbf{r} . For the adjoint representation :

$$(T_{\mathbf{Ad}}^b)_{ac} = i f^{abc}$$

degrees of freedom

Gauge invariance removes one degree of freedom from the gauge field, while the eqm projects out another, fermion eqm project out half the degrees of freedom, so

	off-shell	on-shell
A_μ^a	3 d.o.f.	2 d.o.f.
$\lambda_\alpha^a, \lambda_{\dot{\alpha}}^{\dagger a}$	4 d.o.f.	2 d.o.f.

for SUSY to be manifest off-shell add a real auxiliary boson field D^a .

	off-shell	on-shell
D^a	1 d.o.f.	0 d.o.f.

SUSY Yang–Mills Lagrangian

$$\mathcal{L}_{\text{SYM}} = -\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a} + i\lambda^{\dagger a}\bar{\sigma}^{\mu}D_{\mu}\lambda^a + \frac{1}{2}D^a D^a$$

where the gauge field strength is given by

$$F_{\mu\nu}^a = \partial_{\mu}A_{\nu}^a - \partial_{\nu}A_{\mu}^a - gf^{abc}A_{\mu}^b A_{\nu}^c$$

and the gauge covariant derivative of the gaugino is

$$D_{\mu}\lambda^a = \partial_{\mu}\lambda^a - gf^{abc}A_{\mu}^b \lambda^c$$

auxiliary field has dimension $[D^a] = 2$

infinitesimal SUSY transformations

- should be linear in ϵ and ϵ^\dagger
- transform A_μ^a and λ^a into each other
- keep A_μ^a real
- maintain the correct dimensions of fields with $[\epsilon] = -\frac{1}{2}$
- infinitesimal change in D^a should vanish when the equations of motion are satisfied
- infinitesimal change in the λ^a should involve the derivative of the A_μ^a so that the infinitesimal changes in the two kinetic terms cancel

but gauge transformation of $\partial_\mu A_\nu^a$ different from λ^a and $F_{\mu\nu}^a$

$$\begin{aligned}\delta A_\mu^a &= -\frac{1}{\sqrt{2}} [\epsilon^\dagger \bar{\sigma}_\mu \lambda^a + \lambda^{\dagger a} \bar{\sigma}_\mu \epsilon] \\ \delta \lambda_\alpha^a &= -\frac{i}{2\sqrt{2}} (\sigma^\mu \bar{\sigma}^\nu \epsilon)_\alpha F_{\mu\nu}^a + \frac{1}{\sqrt{2}} \epsilon_\alpha D^a \\ \delta \lambda^{\dagger a}_{\dot{\alpha}} &= \frac{i}{2\sqrt{2}} (\epsilon^\dagger \bar{\sigma}^\nu \sigma^\mu)_{\dot{\alpha}} F_{\mu\nu}^a + \frac{1}{\sqrt{2}} \epsilon^\dagger_{\dot{\alpha}} D^a \\ \delta D^a &= \frac{-i}{\sqrt{2}} [\epsilon^\dagger \bar{\sigma}^\mu D_\mu \lambda^a - D_\mu \lambda^{\dagger a} \bar{\sigma}^\mu \epsilon]\end{aligned}$$

SUSY gauge theories

add a set of chiral supermultiplets

$$\delta_{\text{gauge}} X_j = ig \Lambda^a T^a X_j$$

for $X_j = \phi_j, \psi_j, \mathcal{F}_j$. gauge covariant derivatives are:

$$\begin{aligned} D_\mu \phi_j &= \partial_\mu \phi_j + ig A_\mu^a T^a \phi_j \\ D_\mu \phi^{*j} &= \partial_\mu \phi^{*j} - ig A_\mu^a \phi^{*j} T^a \\ D_\mu \psi_j &= \partial_\mu \psi_j + ig A_\mu^a T^a \psi_j \end{aligned}$$

new allowed renormalizable interactions:

$$(\phi^* T^a \psi) \lambda^a, \quad \lambda^{\dagger a} (\psi^\dagger T^a \phi) \quad (\phi^* T^a \phi) D^a$$

all are required by SUSY with particular couplings. The first two are required to cancel pieces of the SUSY transformations of the gauge interactions of ϕ and ψ . The third is needed to cancel pieces of the SUSY transformations of the first two terms.

Lagrangian for a SUSY gauge theory

$$\mathcal{L} = \mathcal{L}_{\text{SYM}} + \mathcal{L}_{\text{WZ}} - \sqrt{2}g [(\phi^* T^a \psi)\lambda^a + \lambda^{\dagger a}(\psi^\dagger T^a \phi)] + g(\phi^* T^a \phi)D^a.$$

\mathcal{L}_{WZ} is general Wess-Zumino with gauge-covariant derivatives. superpotential must be gauge invariant:

$$\delta_{\text{gauge}}W = ig\Lambda^a \frac{\partial W}{\partial \phi_i} T^a \phi_i = 0.$$

infinitesimal SUSY transformations of ϕ and ψ are have derivatives promoted to gauge covariant derivatives, \mathcal{F} has an additional term required by the gaugino interactions:

$$\begin{aligned}\delta\phi_j &= \epsilon\psi_j \\ \delta\psi_{j\alpha} &= -i(\sigma^\mu\epsilon^\dagger)_\alpha D_\mu\phi_j + \epsilon_\alpha\mathcal{F}_j \\ \delta\mathcal{F}_j &= -i\epsilon^\dagger\bar{\sigma}^\mu D_\mu\psi_j + \sqrt{2}g(T^a\phi)_j\epsilon^\dagger\lambda^{\dagger a}\end{aligned}$$

Integrating out auxiliary fields

eqm for the auxiliary field D^a :

$$D^a = -g\phi^* T^a \phi$$

scalar potential is given by “ \mathcal{F} -terms” and “ D -terms”:

$$V(\phi, \phi^*) = \mathcal{F}^{*i} \mathcal{F}_i + \frac{1}{2} D^a D^a = W_i^* W^i + \frac{1}{2} g^2 (\phi^* T^a \phi)^2$$

as required by SUSY, is positive definite:

$$V(\phi, \phi^*) \geq 0$$

for the vacuum to preserve SUSY $V = 0 \Rightarrow \mathcal{F}_i = 0$ and $D^a = 0$.

Feynman vertices

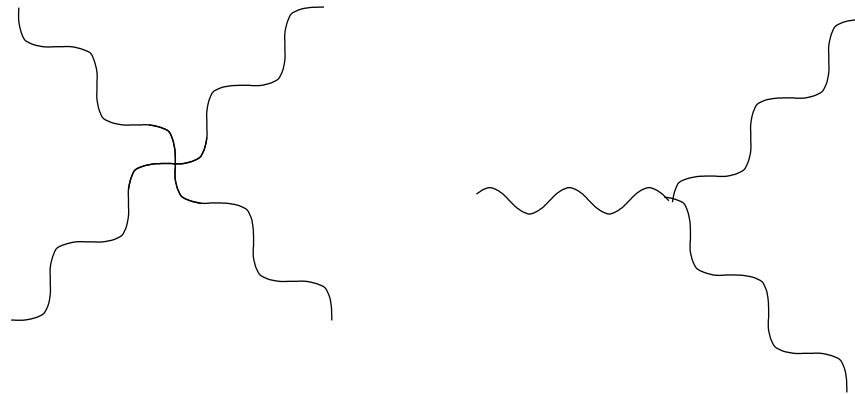


Figure 1:

Cubic and quartic Yang–Mills interactions; wavy lines denote gauge fields.

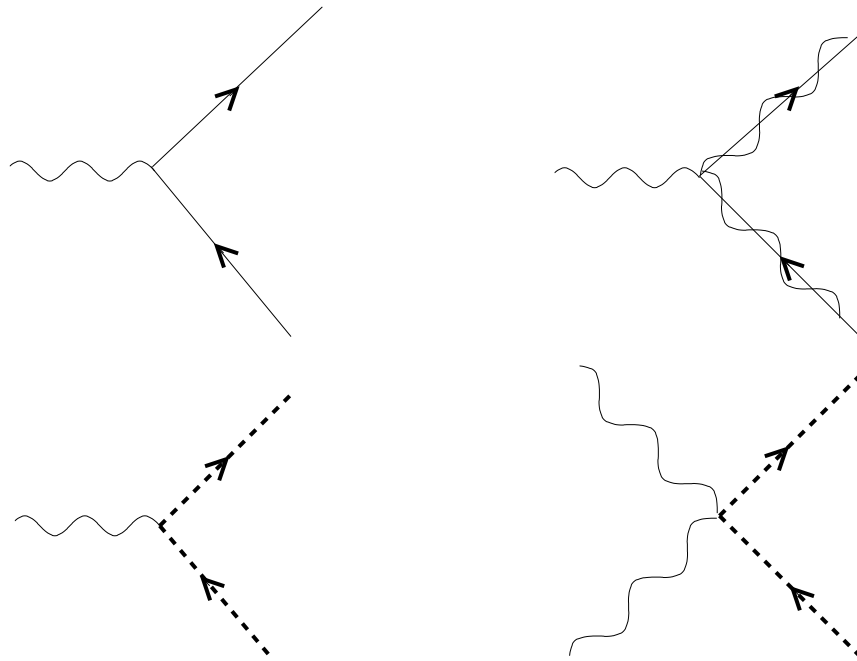


Figure 2:

Interactions required by gauge invariance. Solid lines denote fermions, dashed lines denote scalars, wavy lines denote gauge bosons, wavy/solid lines denote gauginos.

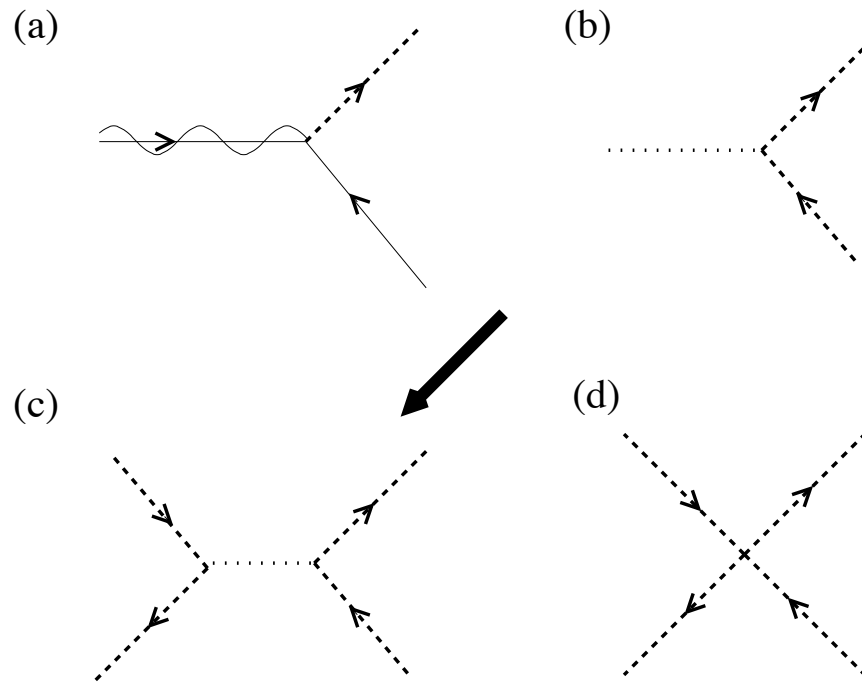


Figure 3:

Additional interactions required by gauge invariance and SUSY: (a) $\phi^* \psi \lambda$, (b) $\phi^* \phi D$ coupling, Note that these three vertices all have the same gauge index structure, being proportional to the gauge generator T^a . Integrating out (c) the auxiliary field gives (d) the quartic scalar coupling proportional to $T^a T^a$.

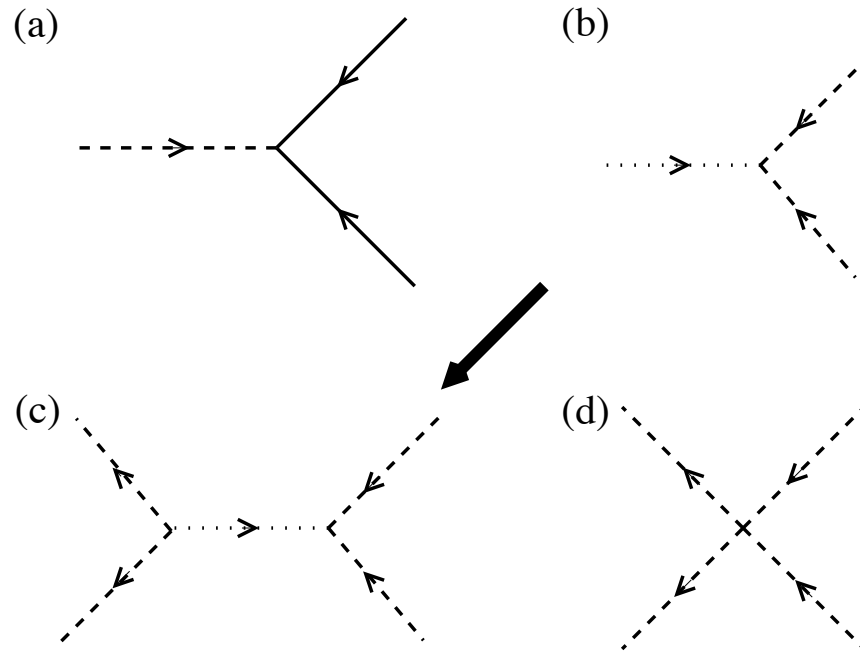


Figure 4:

dimensionless non-gauge interaction vertices in a renormalizable supersymmetric theory: (a) $\phi_i \psi_j \psi_k$ Yukawa interaction vertex $-iy^{ijk}$, (b) $\phi_i \phi_j \mathcal{F}_k$ interaction vertex iy^{ijk} , (c) integrating out the auxiliary field yields, (d) the quartic scalar interaction $-iy^{ijn} y_{kln}^*$ required for cancelling the Λ^2 divergence in the Higgs mass

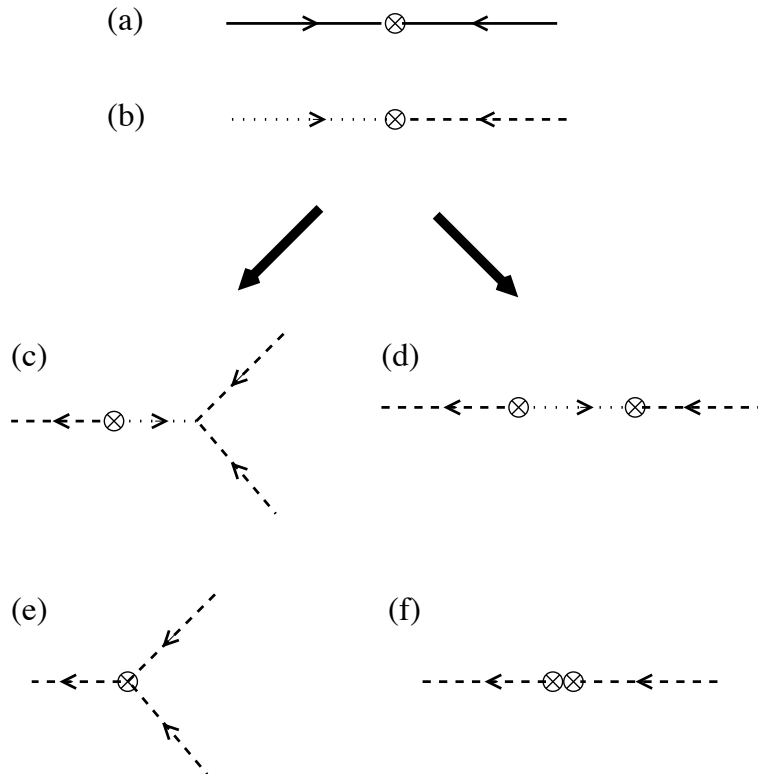


Figure 5:

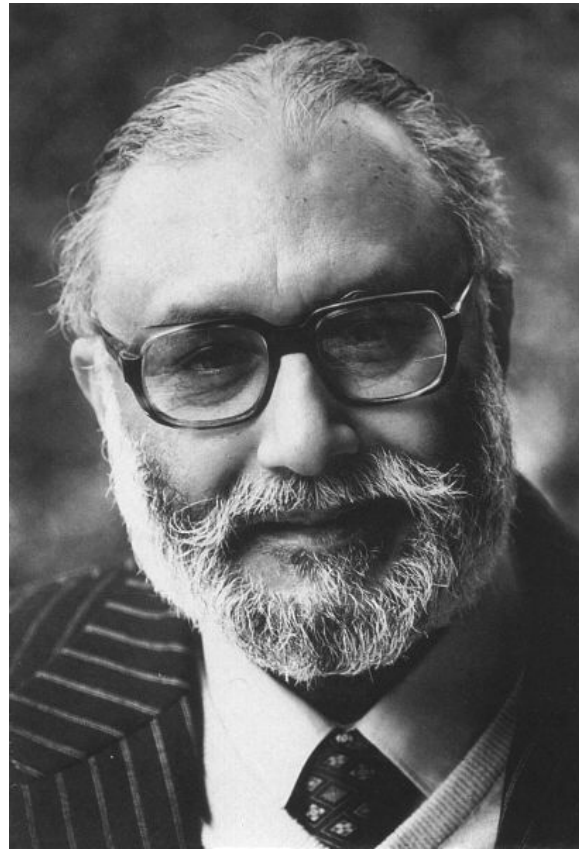
dimensionful couplings: (a) $\psi\psi$ mass insertion $-iM^{ij}$, (b) $\phi\mathcal{F}$ mixing term insertion $+iM^{ij}$, (c) integrating out \mathcal{F} in cubic term, (d) integrating out \mathcal{F} in mass term, (e) $\phi^2\phi^*$ interaction vertex $-iM_{in}^*y^{jkn}$, (f) $\phi^*\phi$ mass insertion $-iM_{ik}^*M^{kj}$ ensuring cancellation of the $\log \Lambda$ in the Higgs mass

Supercurrent

using the Noether theorem one finds that the conserved supercurrent is:

$$J_{\alpha}^{\mu} = \frac{i}{\sqrt{2}} D^a (\sigma^{\mu} \lambda^{\dagger a})_{\alpha} + \mathcal{F}_i i (\sigma^{\mu} \psi^{\dagger i})_{\alpha} \\ + (\sigma^{\nu} \bar{\sigma}^{\mu} \psi_i)_{\alpha} D_{\nu} \phi^{*i} - \frac{1}{2\sqrt{2}} (\sigma^{\nu} \bar{\sigma}^{\rho} \sigma^{\mu} \lambda^{\dagger a})_{\alpha} F_{\nu\rho}^a$$

Salam



Superspace Notation

anticommuting (Grassmann) spinors: $\theta_\alpha, \bar{\theta}_{\dot{\alpha}} = \theta_\alpha^\dagger$. for a single component Grassmann variable η we have:

$$\int d\eta = 0, \int \eta d\eta = 1$$

two-component Grassmann spinor:

$$\{\theta_\alpha, \bar{\theta}_{\dot{\alpha}}\} = 0$$

define:

$$\begin{aligned} d^2\theta &\equiv -\frac{1}{4}d\theta^\alpha d\theta^\beta \epsilon_{\alpha\beta} \\ d^2\bar{\theta} &\equiv -\frac{1}{4}d\bar{\theta}_{\dot{\alpha}} d\bar{\theta}_{\dot{\beta}} \epsilon^{\dot{\alpha}\dot{\beta}} \\ d^4\theta &\equiv d^2\theta d^2\bar{\theta} \end{aligned}$$

Then

$$\begin{aligned} \int d^2\theta \theta^2 &= \int d^2\theta \theta^\sigma \theta_\sigma = -\frac{1}{4} \int d\theta^\alpha d\theta^\beta \epsilon_{\alpha\beta} \theta^\sigma \epsilon_{\sigma\tau} \theta^\tau \\ &= -\frac{1}{4} (\epsilon_{\alpha\beta} \delta^{\beta\sigma} \epsilon_{\sigma\tau} \delta^{\tau\alpha} - \epsilon_{\alpha\beta} \delta^{\alpha\sigma} \epsilon_{\sigma\tau} \delta^{\tau\beta}) \\ &= -\frac{1}{4} (\epsilon_{\alpha\beta} \epsilon_{\beta\alpha} + \epsilon_{\beta\alpha} \epsilon_{\alpha\beta}) = -\frac{1}{2} \epsilon_{\alpha\beta} \epsilon_{\beta\alpha} \\ &= 1 \end{aligned}$$

and for arbitrary spinors χ and ψ we have

$$\int d^2\theta (\chi\theta)(\psi\theta) = -\frac{1}{2}(\chi\psi)$$

define a “superspace coordinate”

$$y^\mu = x^\mu - i\theta\sigma^\mu\bar{\theta}$$

Then we can assemble the fields of a chiral supermultiplet into a *chiral superfield*:

$$\begin{aligned}\Phi(y) &\equiv \phi(y) + \sqrt{2}\theta\psi(y) + \theta^2\mathcal{F}(y) \\ &= \phi(x) - i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi(x) - \frac{1}{4}\theta^2\bar{\theta}^2\partial^2\phi(x) \\ &\quad + \sqrt{2}\theta\psi(x) + \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi(x)\sigma^\mu\bar{\theta} + \theta^2\mathcal{F}(x)\end{aligned}$$

second line follows by Taylor expanding in Grassmann variables θ and $\bar{\theta}$

Superspace SUSY Lagrangians

$$\begin{aligned}
 \int d^4\theta \Phi^\dagger \Phi &= \int d^4\theta \left(\begin{array}{l} \phi^* + i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi^* - \frac{1}{4}\bar{\theta}^2\theta^2\partial^2\phi^* \\ +\sqrt{2}\bar{\theta}\psi^\dagger - \frac{i}{\sqrt{2}}\theta\sigma^\mu\partial_\mu\psi^\dagger\bar{\theta}^2 + \bar{\theta}^2\mathcal{F}^* \end{array} \right) \\
 &\quad \left(\begin{array}{l} \phi - i\theta\sigma^\mu\bar{\theta}\partial_\mu\phi - \frac{1}{4}\theta^2\bar{\theta}^2\partial^2\phi \\ +\sqrt{2}\theta\psi + \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi\sigma^\mu\bar{\theta} + \theta^2\mathcal{F} \end{array} \right) \\
 &= \mathcal{F}^*\mathcal{F} + \partial^\mu\phi^*\partial_\mu\phi + i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi \\
 &\quad -\frac{1}{4}\partial^\mu(\phi^*\partial_\mu\phi + \partial_\mu\phi^*\phi) + \frac{i}{2}\partial_\mu(\psi^\dagger\bar{\sigma}^\mu\psi)
 \end{aligned}$$

so

$$\int d^4x d^4\theta \Phi^\dagger \Phi = \int d^4x \mathcal{L}_{\text{free}}$$

Superpotential

$$\begin{aligned}\int d^2\theta W(\Phi) &= \int d^2\theta (W(\Phi)|_{\theta=0} + \theta W_1 + \theta^2 W_2) = \int d^2\theta \theta^2 W_2 \\ &= W_a \mathcal{F}^a - \frac{1}{2} W^{ab} \psi_a \psi_b - \partial_\mu \left(\frac{1}{4} W^a \bar{\theta}^2 \partial^\mu \phi_a - \frac{i}{\sqrt{2}} W^a \psi_a \sigma^\mu \bar{\theta} \right)\end{aligned}$$

$$\int d^4x d^2\theta W(\Phi) + h.c. = \int d^4x \mathcal{L}_{\text{int}}$$

product of chiral superfields is also a chiral superfield

SUSY variation of the θ^2 component of a chiral superfield is a total derivative

$\int d^4x d^2\theta W$ is always SUSY invariant

Kähler function

$$\int d^4x d^4\theta K(\Phi^\dagger, \Phi)$$

where K is real give kinetic and other (in general non-renormalizable) interactions

SUSY variation of the $\theta^2\bar{\theta}^2$ component of any superfield is a total derivative

Real Superfield

real superfield contains, in addition to the vector supermultiplet, an auxiliary field D^a , plus three real scalar fields and an additional Weyl spinor. In Wess–Zumino gauge the extra scalars and spinor vanish, simplifies to:

$$V^a = \theta\sigma^\mu\bar{\theta}A_\mu^a + \theta^2\bar{\theta}\lambda^{\dagger a} + \bar{\theta}^2\theta\lambda^a + \frac{1}{2}\theta^2\bar{\theta}^2 D^a ,$$

In the Wess–Zumino gauge we have:

$$\begin{aligned} V^a V^b &= \frac{1}{2}\theta^2\bar{\theta}^2 A^{a\mu} A_\mu^b \\ V^a V^b V^c &= 0 \end{aligned}$$

Extended gauge invariance

gauge parameter becomes a chiral superfield, Λ^a

extended gauge transformation can reintroduce (or remove) the extra scalar and spinor fields:

$$\exp(T^a V^a) \rightarrow \exp(T^a \Lambda^{a\dagger}) \exp(T^a V^a) \exp(T^a \Lambda^a)$$

so

$$V^a \rightarrow V^a + \Lambda^a + \Lambda^{a\dagger} + \mathcal{O}(V^a \Lambda^a)$$

chiral superfield Φ transforms under the extended gauge invariance as

$$\Phi \rightarrow e^{-g T^a \Lambda^a} \Phi$$

“field strength” chiral superfield

superspace derivatives defined by

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial\theta^\alpha} - 2i(\sigma^\mu\bar{\theta})_\alpha \frac{\partial}{\partial y^\mu} \\ \bar{D}_{\dot{\alpha}} &= -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} \end{aligned}$$

Then the “field strength” chiral superfield is given by

$$\begin{aligned} T^a W_\alpha^a &= -\frac{1}{4} \bar{D}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}} e^{-T^a V^a} D_\alpha e^{T^a V^a} \\ W_\alpha^a &= -i\lambda_\alpha^a(y) + \theta_\alpha D^a(y) - (\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}^a(y) - (\theta\theta)\sigma^\mu D_\mu \lambda^{\dagger a}(y), \end{aligned}$$

where

$$\sigma^{\mu\nu} = \frac{i}{4} (\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu)$$

SUSY Yang–Mills action

$$\begin{aligned}\int d^4x \mathcal{L}_{SYM} &= \frac{1}{4} \int d^4x d^2\theta W^{a\alpha} W_{\alpha}^a + h.c. \\ &= \frac{1}{4} \int d^4x d^4\theta \text{Tr} T^a W^{a\alpha} e^{-T^a V^a} D_{\alpha} e^{T^a V^a} + h.c.\end{aligned}$$

standard gauge invariant kinetic terms:

$$\int d^4\theta \Phi^{\dagger} e^{gT^a V^a} \Phi$$

or more generally (to include non-renormalizable interactions):

$$\int d^4\theta K(\Phi^{\dagger}, e^{gT^a V^a} \Phi)$$

$\mathcal{N} = 0$ SUSY



Weisskopf

chiral symmetry \Rightarrow multiplicative mass renormalization

$$m_f = m_0 + c_f \frac{\alpha}{16\pi^2} m_0 \ln \left(\frac{\Lambda}{m_0} \right)$$

where Λ is the cutoff

SUSY ensures that the scalar mass is given by the same formula

SUSY dim.less couplings \Rightarrow no Λ^2 divergences

SUSY must be broken in the real world, eg.

$$W = E^a \phi_a$$

gives a scalar potential

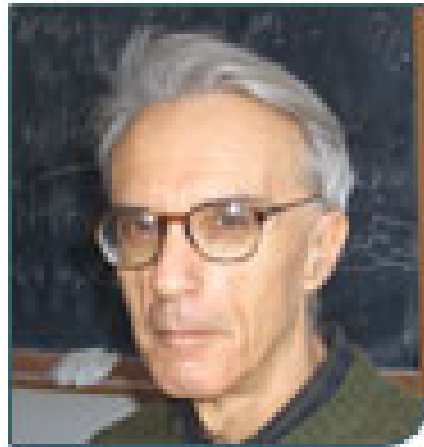
$$V = W_a^* W^a = E^a E_a^* \neq 0$$

which breaks SUSY.

We want to break SUSY such that Higgs – top squark quartic coupling $\lambda = |y_t|^2$. If not we reintroduce a Λ^2 divergence in the Higgs mass:

$$\delta m_h^2 \propto (\lambda - |y_t|^2) \Lambda^2$$

Effective Theory



Grisaru, Girardello

We want an *effective theory* of broken SUSY with only soft breaking terms (operators with dimension < 4). Girardello and Grisaru found:

$$\begin{aligned}\mathcal{L}_{\text{soft}} = & -\frac{1}{2}(M_\lambda \lambda^a \lambda^a + h.c.) - (m^2)_j^i \phi^{*j} \phi_i \\ & -\left(\frac{1}{2} b^{ij} \phi_i \phi_j + \frac{1}{6} a^{ijk} \phi_i \phi_j \phi_k + h.c.\right) \\ & -\frac{1}{2} c_i^{jk} \phi^{i*} \phi_j \phi_k + e^i \phi_i + h.c.\end{aligned}$$

$e^i \phi_i$ is only allowed if ϕ_i is a gauge singlet. The c_i^{jk} term may introduce quadratic divergences if there is a gauge singlet multiplet in the model.

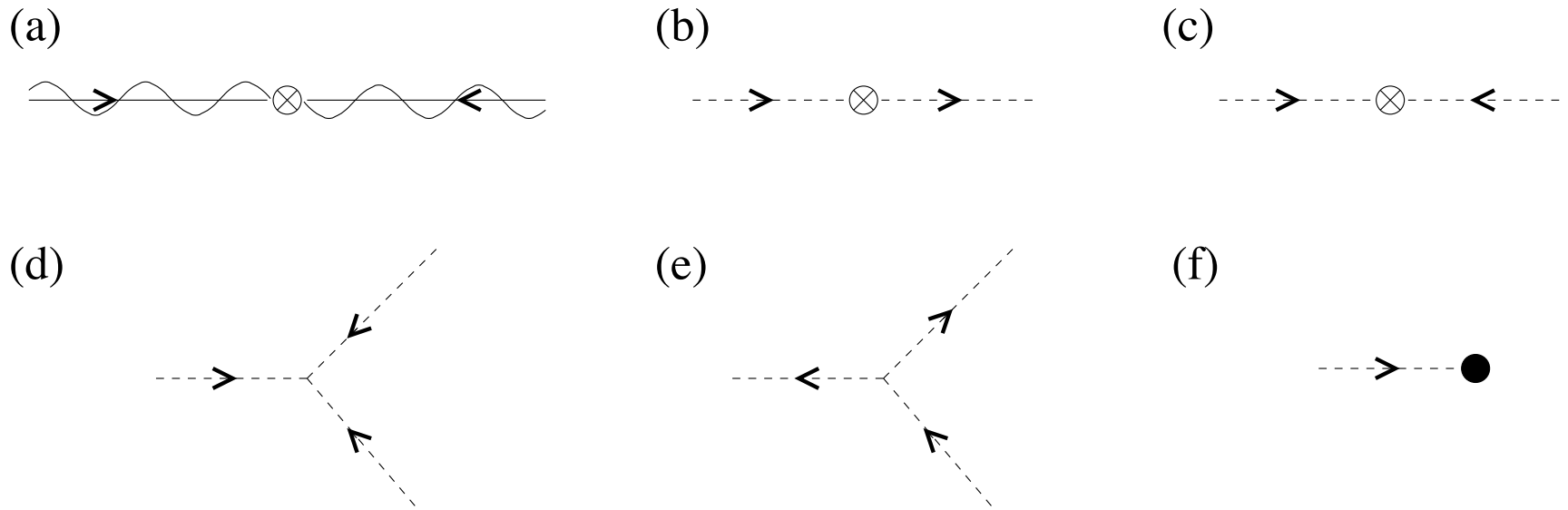


Figure 6:

Additional soft SUSY breaking interactions: (a) gaugino mass M_λ , (b) non-holomorphic mass m^2 , (c) holomorphic mass b^{ij} , (d) holomorphic trilinear coupling a^{ijk} , (e) non-holomorphic trilinear coupling c_i^{jk} , and (f) tadpole e^i .

Spurions

fictitious background fields that transform under a symmetry group

“spontaneously” breaks symmetry

chiral superfield Φ with a wavefunction renormalization factor Z

Take Z to be a real SUSY breaking spurion field

$$Z = 1 + b\theta^2 + b^*\bar{\theta}^2 + c\theta^2\bar{\theta}^2$$

$$\int d^4\theta Z\Phi^\dagger\Phi = \mathcal{L}_{\text{free}} + b\mathcal{F}^*\phi + b^*\phi^*\mathcal{F} + c\phi^*\phi$$

integrate out the auxiliary field \mathcal{F} :

$$\int d^4\theta Z\Phi^\dagger\Phi = \partial^\mu\phi^*\partial_\mu\phi + i\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi + (c - |b|^2)\phi^*\phi$$

soft SUSY breaking mass: $m^2 = |b|^2 - c$

Superpotential spurions

θ^2 component spurions in Yukawa couplings, masses, and the coefficient of $W_\alpha W^\alpha$ generate a , b , and M_λ soft SUSY breaking terms

$$\begin{aligned}\mathcal{L}_{\text{soft}} = & -\frac{1}{2}(M_\lambda \lambda^a \lambda^a + h.c.) - (m^2)_j^i \phi^{*j} \phi_i \\ & -\left(\frac{1}{2} b^{ij} \phi_i \phi_j + \frac{1}{6} a^{ijk} \phi_i \phi_j \phi_k + h.c.\right) \\ & -\frac{1}{2} c_i^{jk} \phi^{i*} \phi_j \phi_k + e^i \phi_i + h.c.\end{aligned}$$

The c term requires a term like

$$\int d^4\theta C_i^{jk} \Phi^{i\dagger} \Phi_j \Phi_k + h.c.$$

where C_i^{jk} has a nonzero $\theta^2 \bar{\theta}^2$ component