Introduction to Supersymmetry

Unreasonable effectiveness of the SM

$$\mathcal{L}_{\text{Yukawa}} = -\frac{y_t}{\sqrt{2}} H^0 \overline{t_L} t_R + h.c.$$

$$H^0 = \langle H^0 \rangle + h^0 = v + h^0$$

$$m_t = \frac{y_t\, v}{\sqrt{2}}$$

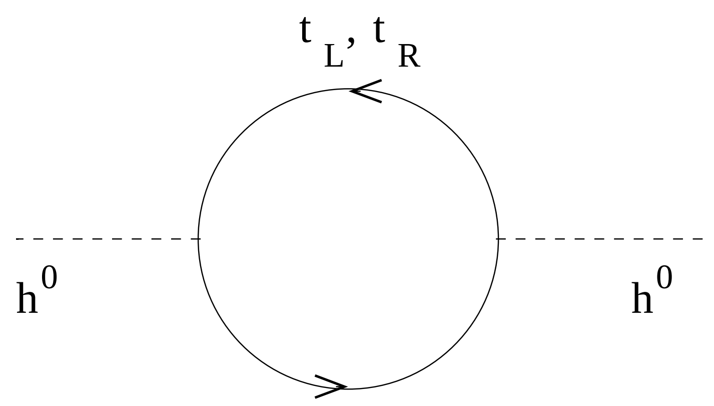


Figure 1: The top loop contribution to the Higgs mass term.

$$
\begin{aligned}
-i\delta m_h^2|_{\rm top} &= (-1)N_c \int \tfrac{d^4k}{(2\pi)^4}\,{\rm Tr}\left[\tfrac{-iy_t}{\sqrt{2}}\tfrac{i}{\not{k}-m_t}\left(\tfrac{-iy_t^*}{\sqrt{2}}\right)\tfrac{i}{\not{k}-m_t}\right] \\
&= -2N_c|y_t|^2 \int \tfrac{d^4k}{(2\pi)^4}\tfrac{k^2+m_t^2}{(k^2-m_t^2)^2}
\end{aligned}
$$

$$
k_0 \to ik_4,\; k^2 \to -k_E^2
$$

$$
-i\delta m_h^2|_{\rm top} = \tfrac{iN_c|y_t|^2}{8\pi^2}\int_0^{\Lambda^2} dk_E^2\,\tfrac{k_E^2(k_E^2-m_t^2)}{(k_E^2+m_t^2)^2}
$$

$$
x = k_E^2 + m_t^2
$$

$$
\begin{aligned}
\delta m_h^2|_{\rm top} &= -\tfrac{N_c|y_t|^2}{8\pi^2}\int_{m_t^2}^{\Lambda^2} dx\left(1-\tfrac{3m_t^2}{x}+\tfrac{2m_t^4}{x^2}\right) \\
&= -\tfrac{N_c|y_t|^2}{8\pi^2}\left[\Lambda^2 - 3\,m_t^2\,\ln\left(\tfrac{\Lambda^2+m_t^2}{m_t^2}\right)+\ldots\right]
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{L}_{\text{scalar}} &= -\frac{\lambda}{2}(h^0)^2(|\phi_L|^2 + |\phi_R|^2) - h^0(\mu_L|\phi_L|^2 + \mu_R|\phi_R|^2) \\
&\quad -m_L^2|\phi_L|^2 - m_R^2|\phi_R|^2
\end{aligned}
$$

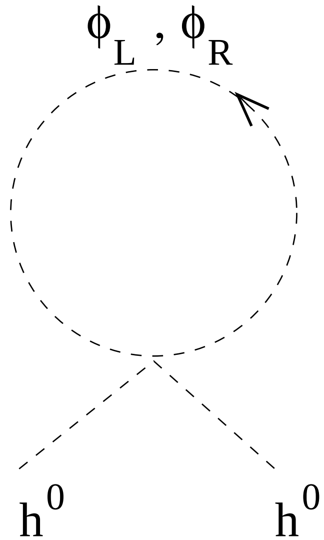


Figure 2: Scalar boson contribution to the Higgs mass term via the quartic coupling.

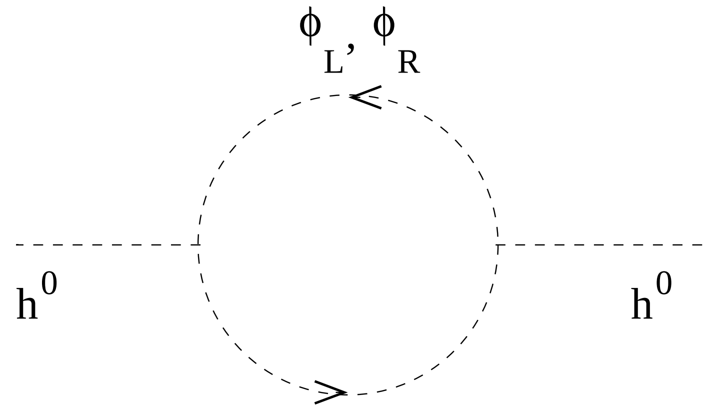


Figure 3: Scalar boson contribution to the Higgs mass term via the trilinear coupling.

$$-i\delta m_h^2|_2 = -i\lambda N \int \frac{d^4k}{(2\pi)^4} \left[\frac{i}{k^2-m_L^2} + \frac{i}{k^2-m_R^2}\right]$$

$$\delta m_h^2|_2 = \frac{\lambda N}{16\pi^2}\left[2\Lambda^2 - m_L^2 \ln\left(\frac{\Lambda^2+m_L^2}{m_L^2}\right) - m_R^2 \ln\left(\frac{\Lambda^2+m_R^2}{m_R^2}\right) + \ldots\right].$$

$$-i\delta m_h^2|_3 = N \int \frac{d^4k}{(2\pi)^4} \left[\left(-i\mu_L \frac{i}{k^2-m_L^2}\right)^2 + \left(-i\mu_R \frac{i}{k^2-m_R^2}\right)^2\right]$$

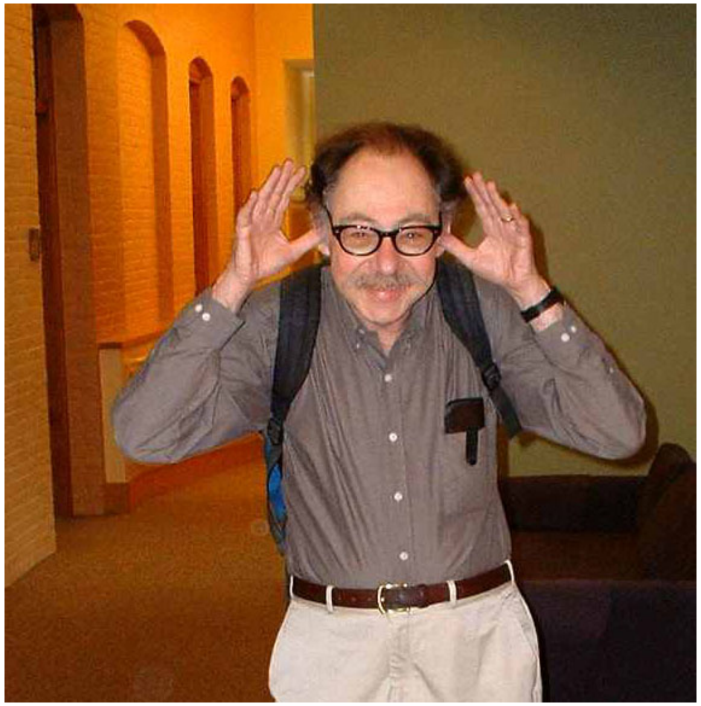
$$\delta m_h^2|_3 = -\frac{N}{16\pi^2}\left[\mu_L^2 \ln\left(\frac{\Lambda^2+m_L^2}{m_L^2}\right) + \mu_R^2 \ln\left(\frac{\Lambda^2+m_R^2}{m_R^2}\right) + \ldots\right].$$

If $N = N_c$ and $\lambda = |y_t|^2$ then Λ^2 cancels

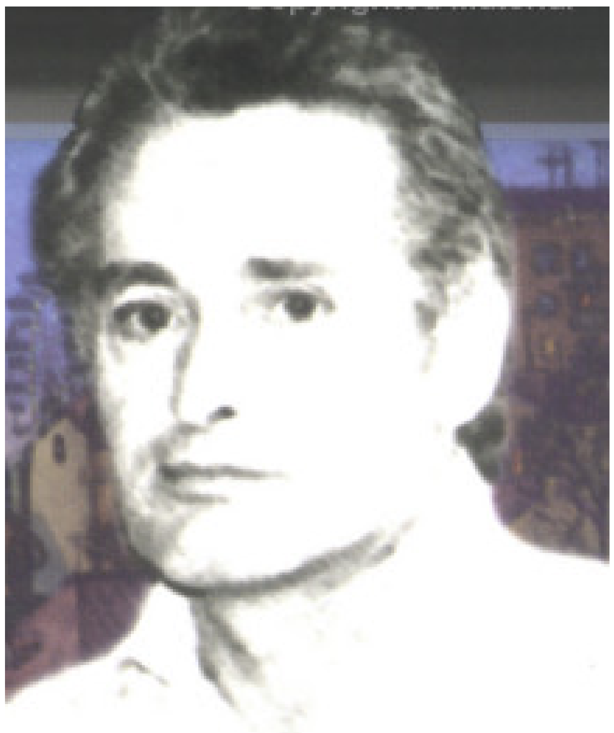
If $m_t = m_L = m_R$ and $\mu_L^2 = \mu_R^2 = 2\lambda m_t^2 \log\Lambda$ are canceled as well

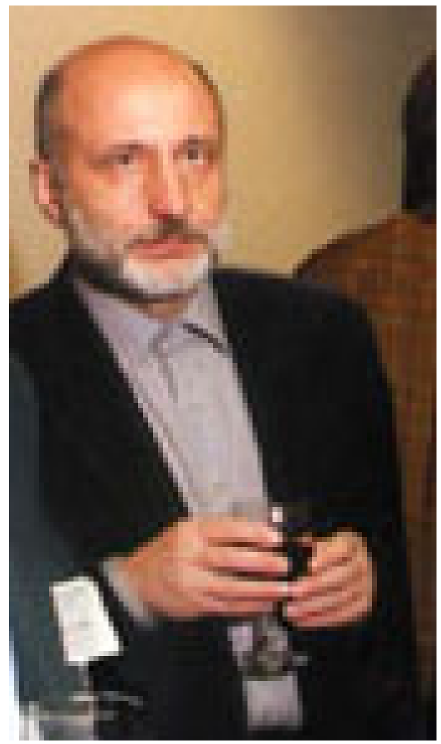
SUSY will guarantee these relations

Coleman-Mandula



Golfand-Lichtman





Haag-Lopuszanski-Sohnius

SUSY algebra

$$\{Q_\alpha, Q^\dagger_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu,$$

$$\sigma^\mu_{\alpha\dot\alpha} \quad = \quad (1, \sigma^i) \;\; \overline{\sigma}^{\mu\dot\alpha\alpha} = (1, -\sigma^i)$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$[P_\mu, Q_\alpha] = [P_\mu, Q^\dagger_{\dot\alpha}] = 0$$

$$[Q_\alpha, R] = Q_\alpha \quad [Q^\dagger_{\dot\alpha}, R] = -Q^\dagger_{\dot\alpha}$$

$$H = P^0 = \tfrac{1}{4}(Q_1 Q_1^\dagger + Q_1^\dagger Q_1 + Q_2 Q_2^\dagger + Q_2^\dagger Q_2)$$

$$\begin{aligned} (-1)^{\mathbf{F}} \left|\text{boson}\right\rangle &= +1 \left|\text{boson}\right\rangle \\ (-1)^{\mathbf{F}} \left|\text{fermion}\right\rangle &= -1 \left|\text{fermion}\right\rangle \end{aligned}$$

$$\{(-1)^{\mathbf{F}}, Q_\alpha\} = 0$$

$$\textstyle\sum_i |i\rangle\langle i| = 1$$

so

$$\begin{aligned} \textstyle\sum_i \langle i|(-1)^{\mathbf{F}} P^0 |i\rangle &= \tfrac{1}{4}\left(\textstyle\sum_i \langle i|(-1)^{\mathbf{F}} Q Q^\dagger |i\rangle + \textstyle\sum_i \langle i|(-1)^{\mathbf{F}} Q^\dagger Q|i\rangle\right) \\ &= \tfrac{1}{4}\left(\textstyle\sum_i \langle i|(-1)^{\mathbf{F}} Q Q^\dagger |i\rangle + \textstyle\sum_{ij} \langle i|(-1)^{\mathbf{F}} Q^\dagger |j\rangle\langle j|Q|i\rangle\right) \\ &= \tfrac{1}{4}\left(\textstyle\sum_i \langle i|(-1)^{\mathbf{F}} Q Q^\dagger |i\rangle + \textstyle\sum_{ij} \langle j|Q|i\rangle\langle i|(-1)^{\mathbf{F}} Q^\dagger |j\rangle\right) \\ &= \tfrac{1}{4}\left(\textstyle\sum_i \langle i|(-1)^{\mathbf{F}} Q Q^\dagger |i\rangle + \textstyle\sum_{j} \langle j|Q(-1)^{\mathbf{F}} Q^\dagger |j\rangle\right) \\ &= \tfrac{1}{4}\left(\textstyle\sum_i \langle i|(-1)^{\mathbf{F}} Q Q^\dagger |i\rangle - \textstyle\sum_{j} \langle j|(-1)^{\mathbf{F}} Q Q^\dagger |j\rangle\right) \\ &= 0. \end{aligned}$$

SUSY:

$$Q_\alpha|0\rangle = 0$$

implies that the vacuum energy vanishes

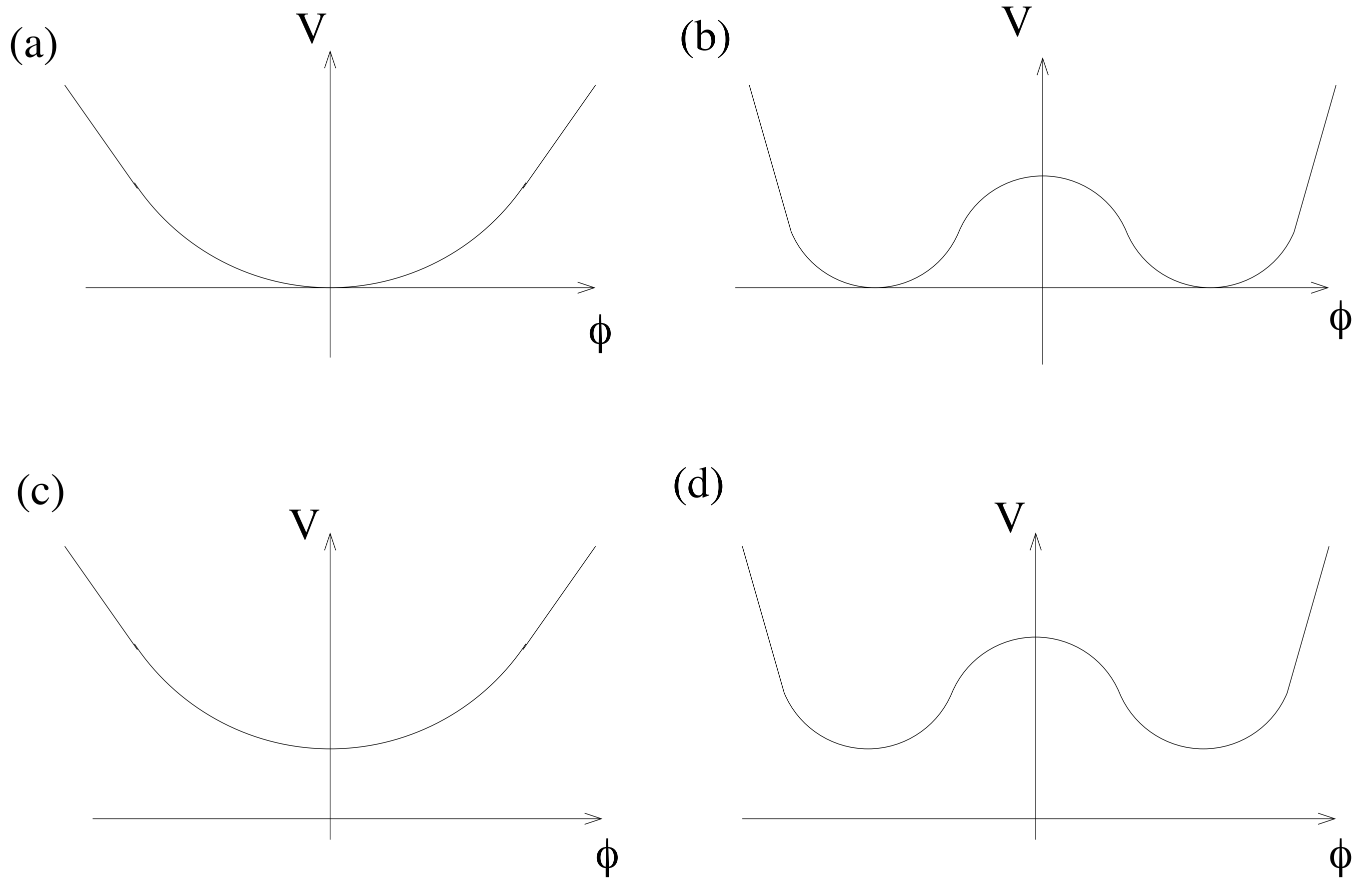
$$\langle 0|H|0\rangle = 0$$

SUSY breaking:

$$Q_\alpha|0\rangle \neq 0$$

and the vacuum energy is positive

$$\langle 0|H|0\rangle \neq 0$$



(a)
V
ϕ
(b)
V
ϕ
(c)
V
ϕ
(d)
V
ϕ

SUSY representations

massive particle rest frame: $p_\mu = (m, \vec{0})$.

$$
\begin{aligned}
\{Q_\alpha, Q^\dagger_{\dot\alpha}\} &= 2\, m\, \delta_{\alpha\dot\alpha} \\
\{Q_\alpha, Q_\beta\} &= 0 \\
\{Q^\dagger_{\dot\alpha}, Q^\dagger_{\dot\beta}\} &= 0
\end{aligned}
$$

Clifford vacuum:

$$
|\Omega_s\rangle = Q_1 Q_2 |m, s', s'_3\rangle, \\
Q_1|\Omega_s\rangle = Q_2|\Omega_s\rangle = 0
$$

massive multiplet:

$$
|\Omega_s\rangle \\
Q_1^\dagger|\Omega_s\rangle, Q_2^\dagger|\Omega_s\rangle \\
Q_1^\dagger Q_2^\dagger|\Omega_s\rangle
$$

massive "chiral" multiplet:

state	s_3
$\vert\Omega_0\rangle$	0
$Q_1^\dagger\vert\Omega_0\rangle, Q_2^\dagger\vert\Omega_0\rangle$	$\pm\frac{1}{2}$
$Q_1^\dagger Q_2^\dagger\vert\Omega_0\rangle$	0

massive vector multiplet:

state	s_3
$\vert\Omega_{\frac{1}{2}}\rangle$	$\pm\frac{1}{2}$
$Q_1^\dagger\vert\Omega_{\frac{1}{2}}\rangle, Q_2^\dagger\vert\Omega_{\frac{1}{2}}\rangle$	$0, 1, 0, -1$
$Q_1^\dagger Q_2^\dagger\vert\Omega_{\frac{1}{2}}\rangle$	$\pm\frac{1}{2}$

Massless particles

frame: $p_\mu = (E, 0, 0, -E)$

$$\begin{aligned} \{Q_1, Q_1^\dagger\} &= 4E \\ \{Q_2, Q_2^\dagger\} &= 0 \\ \{Q_\alpha, Q_\beta\} &= 0 \\ \{Q_{\dot\alpha}^\dagger, Q_{\dot\beta}^\dagger\} &= 0 \end{aligned}$$

Clifford vacuum:

$$|\Omega_\lambda\rangle = Q_1|E, \lambda'\rangle,$$
$$Q_1|\Omega_\lambda\rangle = 0$$

$$\langle\Omega_\lambda|Q_2Q_2^\dagger|\Omega_\lambda\rangle + \langle\Omega_\lambda|Q_2^\dagger Q_2|\Omega_\lambda\rangle = 0$$

$$\langle\Omega_\lambda|Q_2Q_2^\dagger|\Omega_\lambda\rangle = 0$$

massless supermultiplet

state	helicity
$\|\Omega_\lambda\rangle$	λ
$Q_1^\dagger\|\Omega_\lambda\rangle$	$\lambda + \frac{1}{2}$

CPT invariance requires:

state	helicity
$\|\Omega_{-\lambda-\frac{1}{2}}\rangle$	$-\lambda - \frac{1}{2}$
$Q_1^\dagger\|\Omega_{-\lambda-\frac{1}{2}}\rangle$	$-\lambda$

massless chiral multiplet

state	helicity
$\vert\Omega_0\rangle$	0
$Q_1^\dagger\vert\Omega_0\rangle$	$\frac{1}{2}$

include CPT conjugate states:

state	helicity
$\vert\Omega_{-\frac{1}{2}}\rangle$	$-\frac{1}{2}$
$Q_1^\dagger\vert\Omega_{-\frac{1}{2}}\rangle$	0

massless vector multiplet

state	helicity
$\vert\Omega_{\frac{1}{2}}\rangle$	$\frac{1}{2}$
$Q_1^\dagger\vert\Omega_{\frac{1}{2}}\rangle$	1

and its CPT conjugate:

state	helicity
$\vert\Omega_{-1}\rangle$	-1
$Q_1^\dagger\vert\Omega_{-1}\rangle$	$-\frac{1}{2}$

Superpartners

fermion	$\leftrightarrow$	sfermion
quark	$\leftrightarrow$	squark
gauge boson	$\leftrightarrow$	gaugino
gluon	$\leftrightarrow$	gluino

Extended SUSY

$$\begin{aligned} \{Q^a_\alpha, Q^\dagger_{\dot\alpha b}\} &= 2\sigma^\mu_{\alpha\dot\alpha} P_\mu \delta^a_b \\ \{Q^a_\alpha, Q^b_\beta\} &= 0 \\ \{Q^\dagger_{\dot\alpha a}, Q^\dagger_{\dot\beta b}\} &= 0 \end{aligned}$$

where

$$a, b = 1, \ldots, \mathcal{N}$$

$$U(\mathcal{N})_R \text{ R-symmetry}$$

massless multiplets: $p_\mu = (E, 0, 0, -E)$

$$\begin{aligned} \{Q^a_1, Q^\dagger_{1b}\} &= 4E\delta^a_b, \\ \{Q^a_2, Q^\dagger_{2b}\} &= 0. \end{aligned}$$

general massless multiplet

state	helicity	degeneracy
$\lvert\Omega_\lambda\rangle$	λ	1
$Q^\dagger_{1a}\lvert\Omega_\lambda\rangle$	$\lambda+\frac{1}{2}$	$\mathcal{N}$
$Q^\dagger_{1a}Q^\dagger_{1b}\lvert\Omega_\lambda\rangle$	$\lambda+1$	$\mathcal{N}(\mathcal{N}-1)/2$
$\vdots$	$\vdots$	$\vdots$
$Q^\dagger_{11}Q^\dagger_{12}\ldots Q^\dagger_{1\mathcal{N}}\lvert\Omega_\lambda\rangle$	$\lambda+\mathcal{N}/2$	1

$\mathcal{N} = 2$ massless vector multiplet

state	helicity	degeneracy
$\vert\Omega_{-1}\rangle$	-1	1
$Q^\dagger\vert\Omega_{-1}\rangle$	$-\frac{1}{2}$	2
$Q^\dagger Q^\dagger\vert\Omega_{-1}\rangle$	0	1

with the addition of the CPT conjugate:

state	helicity	degeneracy
$\vert\Omega_0\rangle$	0	1
$Q^\dagger\vert\Omega_0\rangle$	$\frac{1}{2}$	2
$Q^\dagger Q^\dagger\vert\Omega_0\rangle$	1	1

built from one $\mathcal{N} = 1$ vector multiplet and one $\mathcal{N} = 1$ chiral multiplet.

$\mathcal{N} = 2$ Hypermultiplet

state	helicity	degeneracy	
$\lvert\Omega_{-\frac{1}{2}}\rangle$	$-\frac{1}{2}$	1	χ_α
$Q^\dagger\lvert\Omega_{-\frac{1}{2}}\rangle$	0	2	ϕ
$Q^\dagger Q^\dagger\lvert\Omega_{-\frac{1}{2}}\rangle$	$\frac{1}{2}$	1	$\psi^{\dagger\dot{\alpha}}$

gauge-invariant mass term: $\psi^\alpha \chi_\alpha$

$\mathcal{N} = 2$ is vector-like

$\mathcal{N} = 3$ massless supermultiplet

state	helicity	degeneracy
$\vert\Omega_{-1}\rangle$	-1	1
$Q^\dagger\vert\Omega_{-1}\rangle$	$-\frac{1}{2}$	3
$Q^\dagger Q^\dagger\vert\Omega_{-1}\rangle$	0	3
$Q^\dagger Q^\dagger Q^\dagger\vert\Omega_{-1}\rangle$	$\frac{1}{2}$	1

plus CPT conjugate

state	helicity	degeneracy
$\vert\Omega_{-\frac{1}{2}}\rangle$	$-\frac{1}{2}$	1
$Q^\dagger\vert\Omega_{-\frac{1}{2}}\rangle$	0	3
$Q^\dagger Q^\dagger\vert\Omega_{-\frac{1}{2}}\rangle$	$\frac{1}{2}$	3
$Q^\dagger Q^\dagger Q^\dagger\vert\Omega_{-\frac{1}{2}}\rangle$	1	1

$\mathcal{N} = 3$ is vector-like

$\mathcal{N} = 4$ massless vector supermultiplet

state	helicity	$\mathbf{R}$
$\|\Omega_{-1}\rangle$	-1	$\mathbf{1}$
$Q^\dagger\|\Omega_{-1}\rangle$	$-\frac{1}{2}$	$\mathbf{4}$
$Q^\dagger Q^\dagger\|\Omega_{-1}\rangle$	0	$\mathbf{6}$
$Q^\dagger Q^\dagger Q^\dagger\|\Omega_{-1}\rangle$	$\frac{1}{2}$	$\overline{\mathbf{4}}$
$Q^\dagger Q^\dagger Q^\dagger Q^\dagger\|\Omega_{-1}\rangle$	1	$\mathbf{1}$

vector-like theory

Massive Supermultiplets

$$\{Q^a_\alpha, Q^\dagger_{\dot\alpha b}\} \quad = \quad 2\,m\,\delta_{\alpha\dot\alpha}\delta^a_b$$

state	spin
$\vert\Omega_s\rangle$	s
$Q^\dagger_{\dot\alpha a}\vert\Omega_s\rangle$	$s+\frac{1}{2}$
$Q^\dagger_{\dot\alpha a}Q^\dagger_{\dot\beta b}\vert\Omega_s\rangle$	$s+1$
$\vdots$	
$Q^\dagger_{11}Q^\dagger_{21}Q^\dagger_{12}Q^\dagger_{22}\ldots Q^\dagger_{1\mathcal{N}}Q^\dagger_{2\mathcal{N}}\vert\Omega_\lambda\rangle$	s

$\mathcal{N} = 2$ massive supermultiplet

state	$(d_R, 2j+1)$
$\vert\Omega_0\rangle$	$(1,1)$
$Q^\dagger\vert\Omega_0\rangle$	$(2,2)$
$Q^\dagger Q^\dagger\vert\Omega_0\rangle$	$(3,1)+(1,3)$
$Q^\dagger Q^\dagger Q^\dagger\vert\Omega_0\rangle$	$(2,2)$
$Q^\dagger Q^\dagger Q^\dagger Q^\dagger\vert\Omega_0\rangle$	$(1,1)$

16 states: five of spin 0, four of spin $\frac{1}{2}$, and one of spin 1.

$\mathcal{N} = 4$ massive supermultiplet

state	$(\mathbf{R}, 2j+1)$
$\lvert\Omega_0\rangle$	$(\mathbf{1}, 1)$
$Q^\dagger\lvert\Omega_0\rangle$	$(\mathbf{4}, 2)$
$Q^\dagger Q^\dagger\lvert\Omega_0\rangle$	$(\mathbf{10}, 1) + (\mathbf{6}, 3)$
$Q^\dagger Q^\dagger Q^\dagger\lvert\Omega_0\rangle$	$(\overline{\mathbf{20}}, 2) + (\overline{\mathbf{4}}, 4)$
$Q^\dagger Q^\dagger Q^\dagger Q^\dagger\lvert\Omega_0\rangle$	$(\mathbf{20'}, 1) + (\mathbf{15}, 3) + (\mathbf{1}, 5)$
$Q^\dagger Q^\dagger Q^\dagger Q^\dagger Q^\dagger\lvert\Omega_0\rangle$	$(\mathbf{20}, 2) + (\mathbf{4}, 4)$
$Q^\dagger Q^\dagger Q^\dagger Q^\dagger Q^\dagger Q^\dagger\lvert\Omega_0\rangle$	$(\overline{\mathbf{10}}, 1) + (\mathbf{6}, 3)$
$Q^\dagger Q^\dagger Q^\dagger Q^\dagger Q^\dagger Q^\dagger Q^\dagger\lvert\Omega_0\rangle$	$(\overline{\mathbf{4}}, 2)$
$Q^\dagger Q^\dagger Q^\dagger Q^\dagger Q^\dagger Q^\dagger Q^\dagger Q^\dagger\lvert\Omega_0\rangle$	$(\mathbf{1}, 1)$

which contains 256 states, including eight spin $\frac{3}{2}$ states and one spin 2 state

Central Charges

$$\begin{aligned}\{Q^a_\alpha, Q^\dagger_{\dot\alpha b}\} &= 2\sigma^\mu_{\alpha\dot\alpha} P_\mu \delta^a_b \\ \{Q^a_\alpha, Q^b_\beta\} &= 2\sqrt{2}\epsilon_{\alpha\beta} Z^{ab} \\ \{Q^\dagger_{\dot\alpha a}, Q^\dagger_{\dot\beta b}\} &= 2\sqrt{2}\epsilon_{\dot\alpha\dot\beta} Z^*_{ab}\end{aligned}$$

where

$$\epsilon = i\sigma^2$$

for $\mathcal{N} = 2$

$$\begin{aligned}\{Q^a_\alpha, Q^\dagger_{\dot\alpha b}\} &= 2\sigma^\mu_{\alpha\dot\alpha} P_\mu \delta^a_b \\ \{Q^a_\alpha, Q^b_\beta\} &= 2\sqrt{2}\epsilon_{\alpha\beta}\epsilon^{ab} Z \\ \{Q^\dagger_{\dot\alpha a}, Q^\dagger_{\dot\beta b}\} &= 2\sqrt{2}\epsilon_{\dot\alpha\dot\beta}\epsilon_{ab} Z\end{aligned}$$

Defining

$$\begin{aligned} A_\alpha &= \tfrac{1}{2}\left[Q^1_\alpha + \epsilon_{\alpha\beta}\left(Q^2_\beta\right)^\dagger\right] \\ B_\alpha &= \tfrac{1}{2}\left[Q^1_\alpha - \epsilon_{\alpha\beta}\left(Q^2_\beta\right)^\dagger\right] \end{aligned}$$

reduces the algebra to

$$\begin{aligned} \{A_\alpha, A^\dagger_\beta\} &= \delta_{\alpha\beta}(M + \sqrt{2}Z) \\ \{B_\alpha, B^\dagger_\beta\} &= \delta_{\alpha\beta}(M - \sqrt{2}Z) \end{aligned}$$

$$\langle M, Z|B_\alpha B^\dagger_\alpha|M, Z\rangle + \langle M, Z|B^\dagger_\alpha B_\alpha|M, Z\rangle = (M - \sqrt{2}Z) \ ,$$

$$M \geq \sqrt{2}Z$$

for $M = \sqrt{2}Z$ (short multiplets): B_α produces states of zero norm

$M > \sqrt{2}Z$ (long multiplets)

short (BPS) multiplet:

state	$2j+1$
$\vert\Omega_0\rangle$	1
$A^\dagger\vert\Omega_0\rangle$	2
$(A^\dagger)^2\vert\Omega_0\rangle$	1

state	$2j+1$
$\vert\Omega_{\frac{1}{2}}\rangle$	2
$A^\dagger\vert\Omega_{\frac{1}{2}}\rangle$	$1+3$
$(A^\dagger)^2\vert\Omega_{\frac{1}{2}}\rangle$	2

short multiplet has 8 states as opposed to 32 states for the corresponding long multiplet

BPS state:

$$M = \sqrt{2}Z$$

is exact