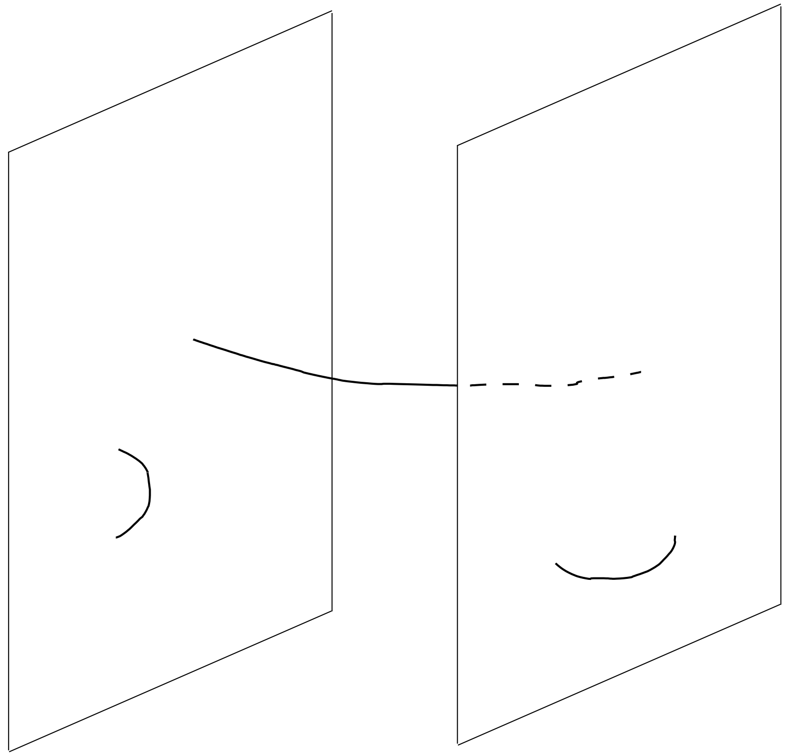
Introduction to AdS/CFT

D-branes

Type IIA string theory: Dp-branes p even (0,2,4,6,8)

Type IIB string theory: Dp-branes p odd (1,3,5,7,9)

10D Type IIB



two parallel D3-branes
low-energy effective description: Higgsed $\mathcal{N} = 4$ SUSY gauge theory

Two parallel D3-branes

lowest energy string stretched between D3-branes: $m \propto LT$
$L \longrightarrow 0$ massless particle $\subset$ 4D effective theory
Dirichlet BC's $\longrightarrow$ gauge boson and superpartners
D3-branes are BPS invariant under half of the SUSY charges
$\Rightarrow$ low-energy effective theory is $\mathcal{N} = 4$ SUSY gauge theory

six extra dimensions, move branes apart in six different ways
moduli space $\leftrightarrow$ $\langle \phi \rangle$ six scalars in the $\mathcal{N} = 4$ SUSY gauge multiplet

moduli space is encoded geometrically

N parallel D3-branes

low-energy effective theory is an $\mathcal{N} = 4$, $U(N)$ gauge theory
N^2 ways to connect oriented strings

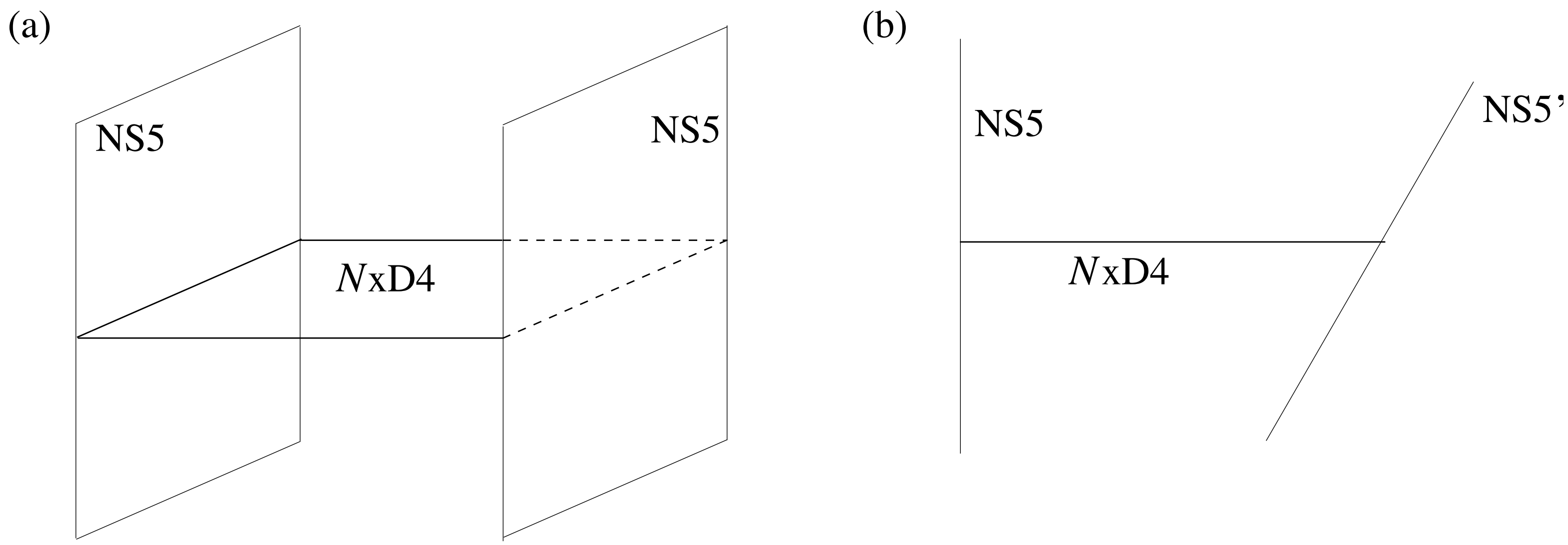
Moving one of the branes $\rightarrow$ mass for $2N - 1$ of the gauge bosons
$\leftrightarrow \langle\phi\rangle$ breaks $U(N) \rightarrow U(N-1)$

gauge coupling related to string coupling g_s

$$g^2 = 4\pi g_s$$

Type IIA D4-branes

5D gauge theory, compactify 1 dimension



D4-brane shares three spatial directions with the 5-brane

$$g_4^2 = \frac{g_5^2}{L}$$

Type IIA D4-branes

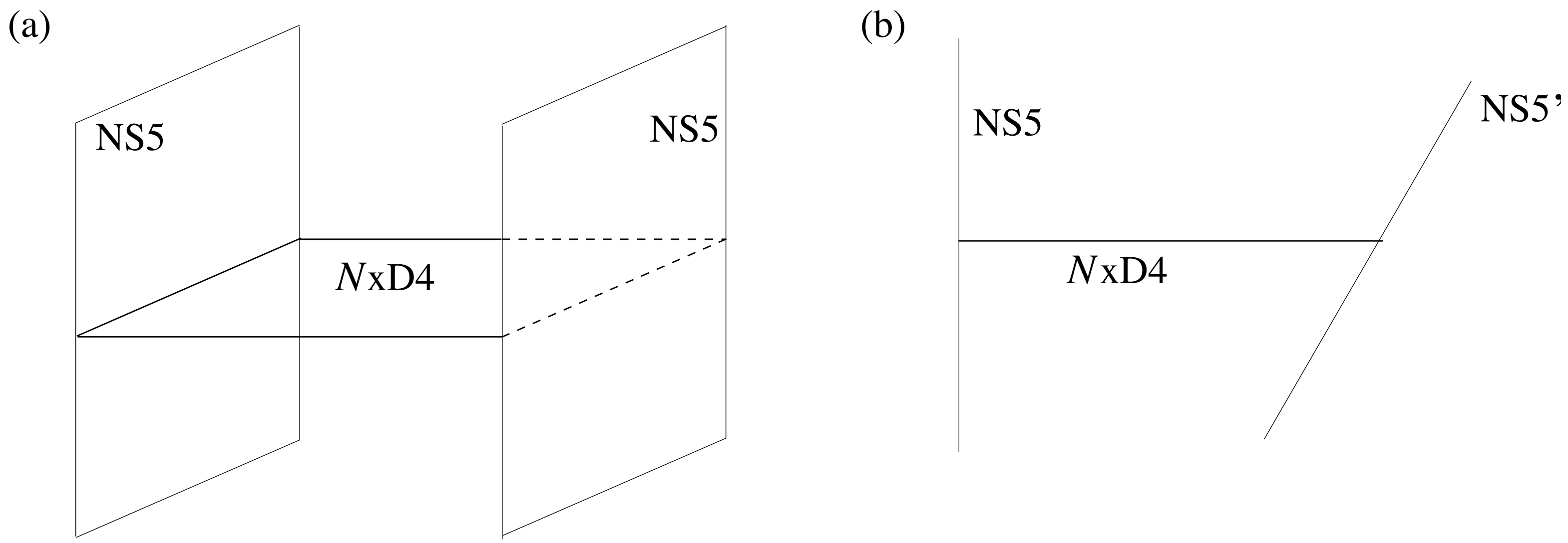
3D end of the D4-brane has two coordinates on the 5-brane
$\leftrightarrow$ two real scalars

two sets of parallel BPS states: D4-branes and 5-branes
each set invariant under one half of the SUSYs
low-energy effective theory has $\mathcal{N} = 2$ SUSY

two real scalars $\leftrightarrow$ scalar component of $\mathcal{N} = 2$ vector supermultiplet

moduli space is reproduced by the geometry

D-brane constructions



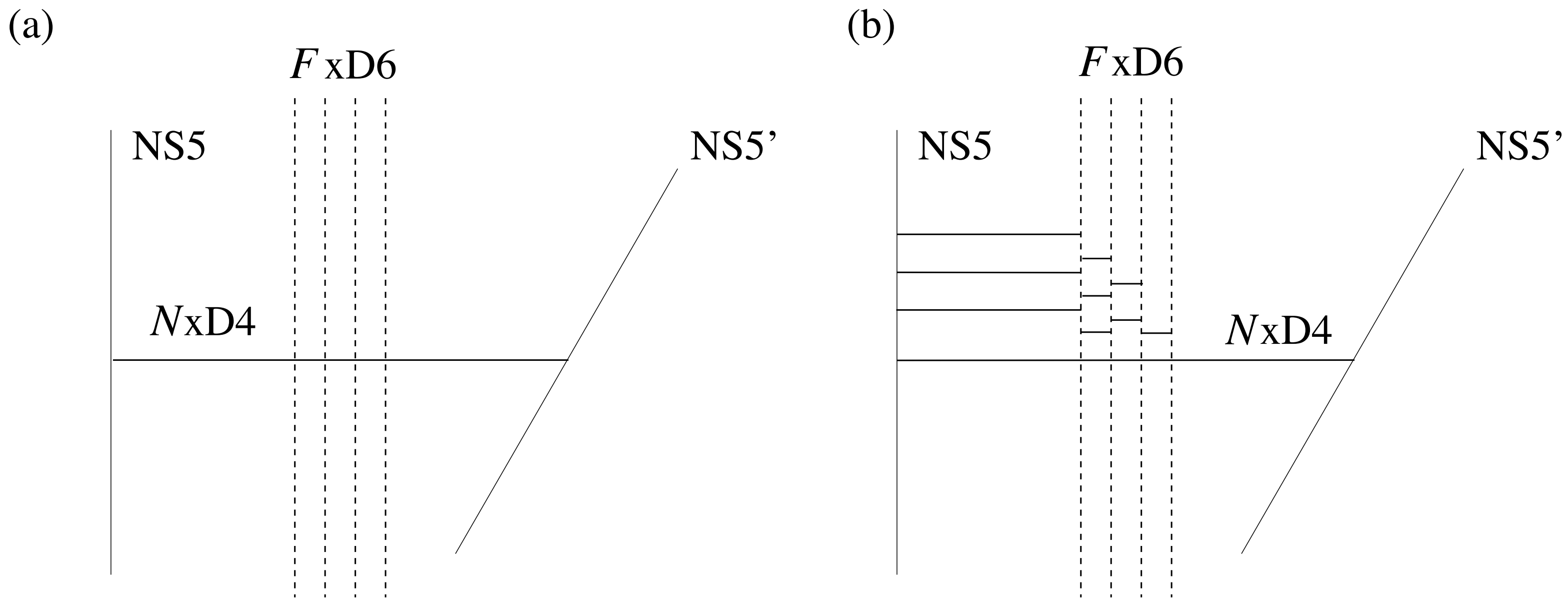
(a) $\mathcal{N} = 2$ SUSY (b) non-parallel NS5-branes $\leftrightarrow$ $\mathcal{N} = 1$ SUSY

rotate one of the NS5-branes $\rightarrow$ D4-branes can't move $\leftrightarrow$ massive scalar breaks $\mathcal{N} = 2 \rightarrow \mathcal{N} = 1$ SUSY

the non-parallel NS5-branes preserve different SUSYs

Adding Flavors

F D6-branes || one of NS5-branes along 2D of the NS5 $\perp$ D4-branes



(a) $SU(N)$ $\mathcal{N}=1$, F flavors. (b) Higgsing the gauge group

strings between D4 and D6 have $SU(N)$ color index and $SU(F)$ flavor index, two orientations $\rightarrow$ chiral supermultiplet and conjugate

Adding Flavors

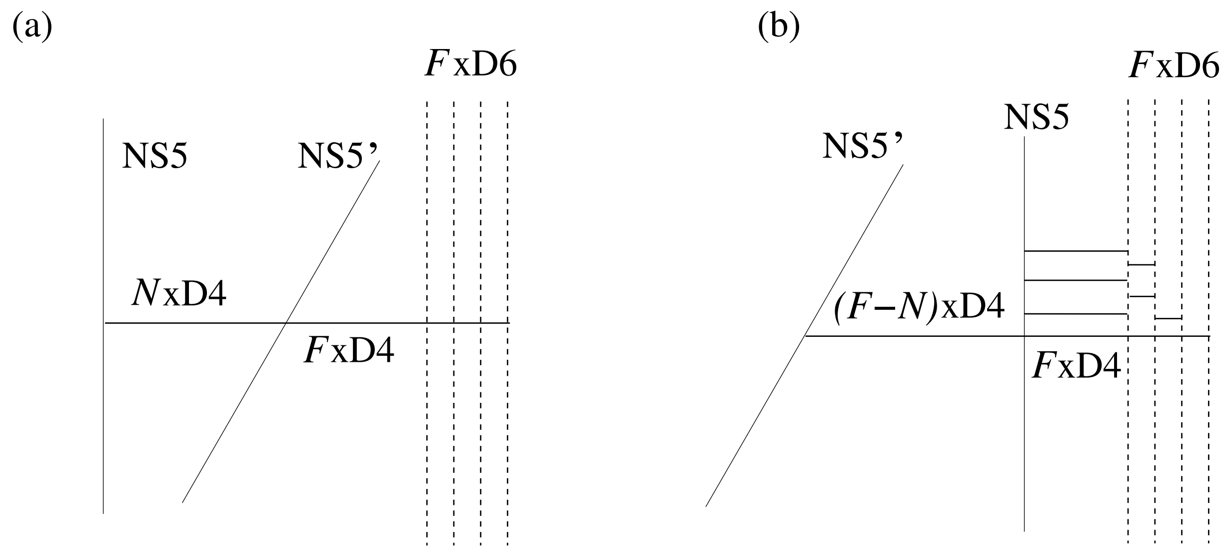
Moving D6 in $\perp$ direction, string between D6 and D4 has finite length $\leftrightarrow$ adding a mass term for flavor

break the D4-branes at D6-brane and move section of the D4 between $||$ NS5 and D6-brane $\leftrightarrow$ squark VEV $\langle \phi \rangle \neq 0$, $\langle \overline{\phi} \rangle \neq 0$ $\leftrightarrow$ Higgsing

counting # of ways of moving segments
$\rightarrow$ dimension of the the moduli space $= 2NF - N^2$
correct result for classical $U(N)$ gauge theory

Seiberg Duality

(a) move NS5' through the D6 (b) move NS5' around the NS5



N D4s between NS5s join up, leaving $(F-N)$ D4s, $\#R - \#L$ fixed

$\leftrightarrow SU(F-N)$ $\mathcal{N}=1$ SUSY gauge theory with F flavors

D4s between || NS5 and D6-branes move without Higgsing $SU(F-N)$

ways of moving $= F^2$ complex dof $\leftrightarrow$ meson in classical limit

dual quarks $\leftrightarrow$ strings from $(F-N)$ D4s to F D4s

stretched to finite length $\leftrightarrow$ meson VEV $\rightarrow$ dual quark mass

Lift to M-theory

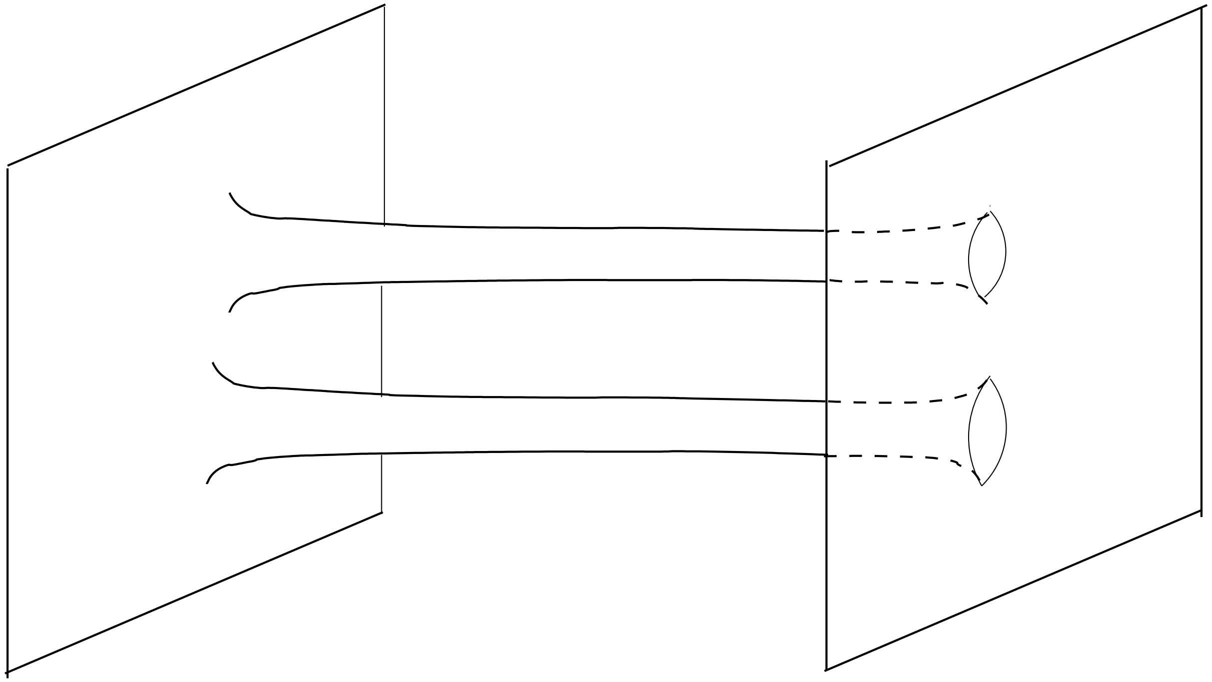
to get quantum corrections
Type IIA string theory $\leftrightarrow$ compactification of M-theory on a circle

$$g_s = (R_{10} M_{\rm Pl})^{3/2}$$

finite string coupling g_s $\leftrightarrow$ to a finite radius R_{10}
eg. $\mathcal{N} = 2$ $SU(2)$ gauge theory $\leftrightarrow$ two D4-branes between || NS5s
NS5 is low-energy description of M5-brane
D4 is low-energy description of M5-brane wrapped on circle

Lift to M-theory

D4s ending on NS5s $\rightarrow$ single M5



M-theory curve describes a 6D space, 4D spacetime
remaining 2D given by the elliptic curve of Seiberg-Witten
larger gauge groups, more D4-branes, surface has more handles

M-theory brane bending

M5s not $||$, bend toward or away from each other depending on the # branes "pulling" on either side
move one D4 $\leftrightarrow$ Higgsing by a $v = \langle \phi \rangle$
probe $g(v)$

$$g_4^2 = \frac{g_5^2}{L}$$

bending of M5-brane $\leftrightarrow$ to running coupling
at large v bending reproduces β

M-theory not completely developed

not understood:
get quantum moduli space for $\mathcal{N} = 1$ $SU(N)$ rather than $U(N)$
dimension of dual quantum moduli space reduced
from F^2 to $F^2 - ((F - N)^2 - 1)$

N D3 branes of Type IIB

$E \ll 1/\sqrt{\alpha'}$, effective theory:

$$S_{\rm eff} = S_{\rm brane} + S_{\rm bulk} + S_{\rm int}$$

$S_{\rm brane}$ = gauge theory
$S_{\rm bulk}$ = closed string loops = Type IIB sugra + higher dimension ops
10D graviton fluctuations h:

$$g_{MN} = \eta_{MN} + \kappa_{\rm IIB}\, h_{MN}$$

where $\kappa_{\rm IIB} \sim g_s \alpha'^2$, 10D Newton's constant, has mass dimension -4

$$S_{\rm bulk} = \frac{1}{2\kappa_{\rm IIB}^2} \int \sqrt{g} R \sim \int (\partial h)^2 + \kappa_{\rm IIB} (\partial h)^2 h + \ldots$$

$E \to 0 \equiv$ drop terms with positive powers of $\kappa_{\rm IIB}$, leaves kinetic term
all terms in $S_{\rm int}$ can be neglected $\to$ free graviton
Equivalently, hold E, g_s, N fixed take $\alpha' \to 0$ ($\kappa_{\rm IIB} \to 0$)
$\to$ free IIB sugra and 4D $SU(N)$, $\mathcal{N} = 4$ SUSY gauge theory

Supergravity Approximation

low-energy effective theory: Type IIB supergravity with N D3-branes, source for gravity, warps the 10D space
solution for the metric:

$$\begin{aligned} ds^2 &= f^{-1/2}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + f^{1/2}\left(dr^2 + r^2 d\Omega_5^2\right) \\ f &= 1 + \left(\frac{R}{r}\right)^4 \;,\quad R^4 = 4\pi g_s \alpha'^2 N \end{aligned}$$

where r is radial distance from branes, and R is curvature radius
observer at r measures red-shifted E_r, observer at $r = \infty$ measures

$$E = \sqrt{g_{tt}}\, E_r = f^{-1/4} E_r$$

$E \to 0 \leftrightarrow$ keep states with $r \to 0$ or bulk states with $\lambda \to \infty$
two sectors decouple since long wavelengths cannot probe short-distances
agreement with previous analysis
states with $r \to 0 \leftrightarrow$ gauge theory, bulk states $\leftrightarrow$ free Type IIB sugra

Near-Horizon Limit

study the states near D-branes, $r \to 0$, by change of coordinate

$$u = \frac{r}{\alpha'}$$

hold finite as $\alpha' \to 0$
low-energy (near-horizon) limit:

$$\frac{ds^2}{\alpha'} = \frac{u^2}{\sqrt{4\pi g_s N}} \left(dt^2 + dx_i^2\right) + \sqrt{4\pi g_s N} \left(\frac{du^2}{u^2} + d\Omega_5^2\right)$$

metric of $\mathrm{AdS}_5 \times S^5$
identify the gauge theory with supergravity near horizon limit
Maldacena's conjecture: Type IIB string theory on $\mathrm{AdS}_5 \times S^5 \equiv$ 4D $SU(N)$ gauge theory with $\mathcal{N} = 4$ SUSY, a CFT

so much circumstantial evidence, called AdS/CFT correspondence

Supergravity Approximation

Sugra on $\mathrm{AdS}_5 \times S^5$ is good approximation string theory when g_s is weak and $R/\alpha'^{1/2}$ is large:

$$g_s \ll 1 \;, \;\; g_s N \gg 1$$

Perturbation theory is a good description of a gauge theory when

$$g^2 \ll 1 \;, \;\; g^2 N \ll 1$$

AdS/CFT correspondence:
weakly coupled gravity $\leftrightarrow$ large N, strongly coupled gauge theory

hard to prove but also potentially quite useful

$\mathrm{AdS}_5 \times S^5$

S^5 can be embedded in a flat 6D space with constraint:

$$R^2 = \sum_{i=1}^{6} Y_i^2 \ ,$$

S^5 space with constant positive curvature,
$SO(6)$ isometry $\leftrightarrow$ $SU(4)_R$ symmetry of $\mathcal{N} = 4$ gauge theory

AdS_5 can be embedded in 6D:

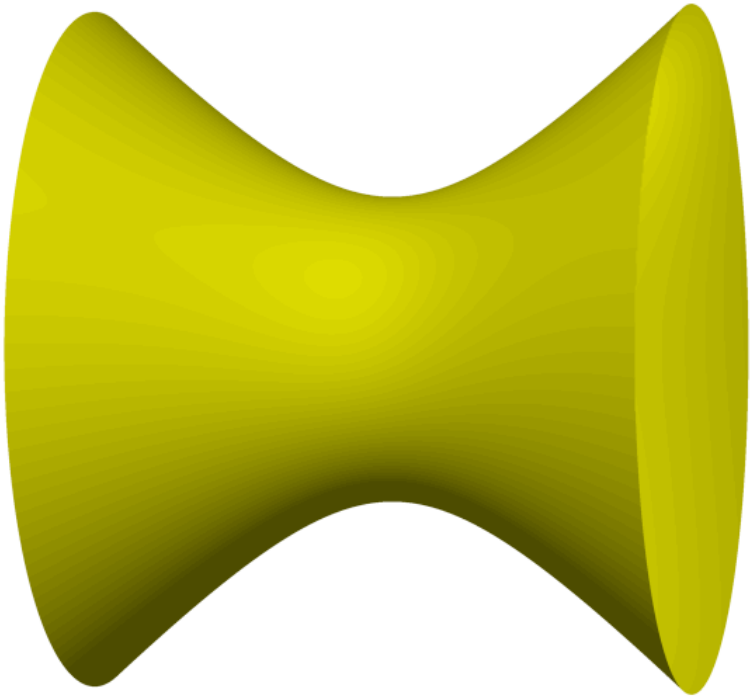
$$ds^2 = -dX_0^2 - dX_5^2 + \sum_{i=1}^{4} dX_i^2$$

with the constraint:

$$R^2 = X_0^2 + X_5^2 - \left(\sum_{i=1}^{4} X_i^2\right)$$

AdS_5 space with a constant negative curvature and $\Lambda < 0$
isometry is $SO(4,2)$ $\leftrightarrow$ conformal symmetry in 3+1 D

AdS Space



hyperboloid embedded in a higher dimensional space

AdS_5

change to "global" coordinates:

$$\begin{aligned} X_0 &= R\cosh\rho\,\cos\tau \quad X_5 = R\cosh\rho\,\sin\tau \\ X_i &= R\sinh\rho\,\Omega_i,\;\; i=1,\ldots,4\;,\;\; \sum_i \Omega_i^2 = 1 \end{aligned}$$

$$ds^2 = R^2(-\cosh^2\rho\,d\tau^2 + d\rho^2 + \sinh^2\rho\,d\Omega^2)$$

periodic coordinate τ going around the "waist" at $\rho = 0$
while $\rho \geq 0$ is the $\perp$ coordinate in the horizontal direction

to get causal (rather than periodic) structure
cut hyperboloid at $\tau = 0$, paste together an infinite number of copies so that τ runs from $-\infty$ to $+\infty$
causal universal covering spacetime

AdS$_5$: "Poincaré coordinates"

$$\begin{aligned} X_0 &= \tfrac{1}{2u}\left(1 + u^2(R^2 + \vec{x}^2 - t^2)\right), \quad X_5 = R\,u\,t \\ X_i &= R\,u\,x_i, \; i = 1, \ldots, 3 \; ; \quad X_4 = \tfrac{1}{2u}\left(1 - u^2(R^2 - \vec{x}^2 + t^2)\right) \end{aligned}$$

$$ds^2 = R^2\left(\frac{du^2}{u^2} + u^2(-dt^2 + d\vec{x}^2)\right)$$

cover half of the space covered by the global coordinates
Wick rotate to Euclidean

$$\tau \to \tau_E = -i\tau \; , \quad \text{or} \;\; t \to t_E = -it$$

$$\begin{aligned} ds_E^2 &= R^2\left(\cosh^2\rho d\tau_E^2 + d\rho^2 + \sinh^2\rho d\Omega^2\right) \\ &= R^2\left(\frac{du^2}{u^2} + u^2(dt_E^2 + d\vec{x}^2)\right) \end{aligned}$$

AdS$_5$: “Poincaré coordinates”

another coordinate choice (also referred to as Poincaré coordinates)

$$u = \frac{1}{z}\ , \ \ x_4 = t_E$$

metric is conformally flat:

$$ds_E^2 = \frac{R^2}{z^2}\left(dz^2 + \sum_{i=1}^4 dx_i^2\right)$$

boundary of this space is R^4 at $z = 0$, Wick rotation of 4D Minkowski, and a point $z = \infty$

AdS/CFT correspondence

partition functions of CFT and the string theory are related

$$\langle \exp \int d^4x \phi_0(x) \mathcal{O}(x) \rangle_{\mathrm{CFT}} = Z_{\mathrm{string}} \left[\phi(x,z)|_{z=0} = \phi_0(x) \right]$$

$\mathcal{O} \subset$ CFT $\leftrightarrow \phi$ AdS$_5$ field, $\phi_0(x)$ is boundary value

For large N and $g^2 N$, use the supergravity approximation

$$Z_{\mathrm{string}} \approx e^{-S_{\mathrm{sugra}}[\phi(x,z)|_{z=0}=\phi_0(x)]}$$

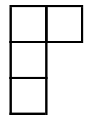
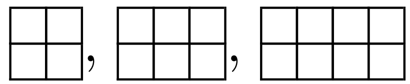
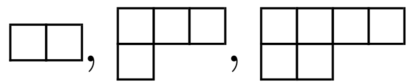
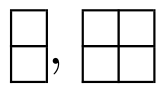
CFT Operators

$\mathcal{O} \subset$ CFT $\leftrightarrow \phi$ AdS$_5$ field
scaling dimensions of chiral operators can be calculated from R-charge

primary operators annihilated by lowering operators S_α and K_μ
descendant operators obtained by raising operators Q_α and P_μ
interested in the mapping of chiral primary operators

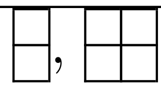
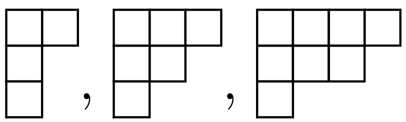
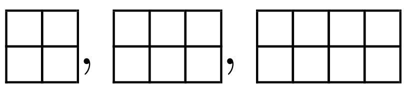
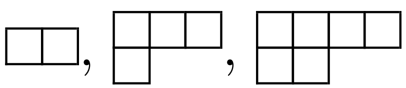
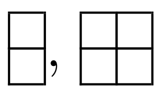
$\mathcal{N} = 4$ multiplet $SU(4)_R$ representations:
$(A_\mu, \mathbf{1})$, $(\lambda_\alpha, \square)$, $(\phi, \begin{array}{|c|}\hline \\ \hline \\ \hline\end{array})$

Chiral Primary Operators

Operator	$SU(4)_R$	Dimension
$T^{\mu\nu}$	$\mathbf{1}$	4
J_R^{μ}		3
$\mathrm{Tr}(\Phi^{I_1}...\Phi^{I_k}),\ k \geq 2$	$(0,k,0)$  , $\ldots$	k
$\mathrm{Tr}(W^{\alpha}W_{\alpha}\Phi^{I_1}...\Phi^{I_k})$	$(2,k,0)$  , $\ldots$	$k+3$
$\mathrm{Tr}\,\phi^k F^{\mu\nu}F_{\mu\nu} + ...$	$(0,k,0)$ $\mathbf{1}$,  , $\ldots$	$k+4$

Corresponding Type IIB KK modes

harmonics on S^5, masses determined by $SU(4)_R$ irrep

Spin	$SU(4)_R \sim SO(6)$	m^2R^2	Operator
2	$\mathbf{1}$, , ...	$k(k+4)\ ,\ \ k \geq 0$	k=0, $T^{\mu\nu}$
1	, ...	$(k-1)(k+1)\ ,\ \ k \geq 1$	$k=1,\ J^{\mu}_{R}$
0	, ...	$k(k-4)\ ,\ \ k \geq 2$	$\mathrm{Tr}(\Phi^{I_1}...\Phi^{I_k})$
0	, ...	$(k-1)(k+3)\ ,\ \ k \geq 0$	$\mathrm{Tr}(W^{\alpha}W_{\alpha}\Phi^{I_1}...\Phi^{I_k})$
0	$\mathbf{1}$, , ...	$k(k+4)\ ,\ \ k \geq 0$	$\mathrm{Tr}\,\phi^k F^{\mu\nu}F_{\mu\nu} + ...$

lowest states form graviton supermultiplet of $D=5$, gauged sugra

Waves on AdS$_5$

massive scalar field in AdS$_5$:

$$S = \frac{1}{2} \int d^4x \, dz \sqrt{g} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2)$$

Using the conformally flat Euclidean metric

$$ds_E^2 = \frac{R^2}{z^2} \left(dz^2 + \sum_{i=1}^{4} dx_i^2 \right)$$

and assuming a factorized solution:

$$\phi(x, z) = e^{ip.x} f(p\, z)$$

eqm reduces to

$$z^5 \partial_z \left(\frac{1}{z^3} \partial_z f \right) - z^2 p^2 f - m^2 R^2 f = 0$$

Waves on AdS_5

Writing $y = pz$ the solutions are modified Bessel functions:

$$f(y) = \begin{cases} y^2 I_{\Delta-2}(y) & \sim \quad y^{\Delta}, \text{ as } y \to 0 \\ y^2 K_{\Delta-2}(y) & \sim \quad y^{4-\Delta}, \text{ as } y \to 0 \end{cases} ,$$

Δ is determined by the mass

$$\Delta = 2 + \sqrt{4 + m^2 R^2}$$

$y^2 I_{\Delta-2}(y)$ blows up as $y \to \infty$: not normalizable

$$x \to \frac{x}{\rho} \ , \ \ p \to \rho p$$

then the scalar field transforms as

$$\phi(x, z) \to \rho^{4-\Delta} e^{ip.x} f(pz)$$

conformal weight $4 - \Delta$, $\leftrightarrow$ CFT $\mathcal{O}$ must have dimension Δ

bulk mass, $m \leftrightarrow$ scaling dimension, Δ

Propagators on AdS_5

propagate boundary ϕ_0 into the interior:

$$\phi(x,z) = c\int d^4x' \frac{z^\Delta}{(z^2+|x-x'|^2)^\Delta}\phi_0(x')$$

for small z the bulk field scales as $z^{4-\Delta}\phi_0(x)$

$$\partial_z\phi(x,z) = c\Delta\int d^4x' \frac{z^{\Delta-1}}{|x-x'|^{2\Delta}}\phi_0(x') + \mathcal{O}(z^{\Delta+1}) \qquad (*)$$

integrating action by parts + eqm yields:

$$S \;=\; \frac{1}{2}\int d^4x dz\, \partial_5\left(\frac{R^3}{z^3}\phi\partial_5\phi\right) = \frac{1}{2}\int d^4x \left(\frac{R^3}{z^3}\phi\partial_5\phi\right)|_{z=0}$$

Using the boundary condition $\phi(x,0) = \phi_0(x)$ and (*)

$$S = \frac{cR^3\Delta}{2}\int d^4x d^4x' \frac{\phi_0(x)\phi_0(x')}{|x-x'|^{2\Delta}}$$

Two-Point Function of CFT

for corresponding operator $\mathcal{O}$ derived from

$$\langle \exp \int d^4x \phi_0(x) \mathcal{O}(x) \rangle_{\rm CFT} \approx e^{-S_{\rm sugra}[\phi(x,z)|_{z=0}=\phi_0(x)]}$$

$$\langle \mathcal{O}(x)\mathcal{O}(x') \rangle \quad = \quad \frac{\delta^2 S}{\delta\phi_0(x)\,\delta\phi_0(x')} = \frac{cR^3\Delta}{|x-x'|^{2\Delta}}$$

correct scaling for dimension Δ in 4D CFT

Dimension $\leftrightarrow$ Mass

In AdS_{d+1}:

$$
\begin{aligned}
\text{scalars}: &\quad \Delta_\pm = \tfrac{1}{2}\left(d \pm \sqrt{d^2 + 4m^2R^2}\right) \\
\text{spinors}: &\quad \Delta = \tfrac{1}{2}\left(d + 2|m|R\right) \\
\text{vectors}: &\quad \Delta_\pm = \tfrac{1}{2}\left(d \pm \sqrt{(d-2)^2 + 4m^2R^2}\right) \qquad . \\
p\text{-forms}: &\quad \Delta_\pm = \tfrac{1}{2}\left(d \pm \sqrt{(d-2p)^2 + 4m^2R^2}\right) \\
\text{massless spin } 2: &\quad \Delta = d
\end{aligned}
$$

for scalar requiring $\Delta_\pm$ is real $\Rightarrow$ Breitenlohner–Freedman bound

$$-\tfrac{d^2}{4} < m^2R^2$$

Dimension $\leftrightarrow$ Mass

relation is expected to hold for stringy states:

$$m \sim \frac{1}{l_s} \leftrightarrow \Delta \sim (g^2 N)^{1/4}$$
$$m \sim \frac{1}{l_{\rm Pl}} \leftrightarrow \Delta \sim N^{1/4}$$

large N and large $g^2 N$ $\leftrightarrow$ very large dimension $\mathcal{M}$
neglected in the supergravity approximation

$(N+1)$ D3-branes

$SU(N+1)$, $\mathcal{N}=4$ SUSY gauge theory
pull one of the branes distance u away $SU(N+1) \to SU(N)$
stretched string states $\leftrightarrow$ massive gauge bosons

$$m_W = \frac{u}{\alpha'}$$

$\Box + \overline{\Box}$ of $SU(N)$

$u \to \infty \leftrightarrow$ static quark

consider static quark–antiquark pair at distance r on $\partial \mathrm{AdS}_5$
minimum action: string stretching from the quark to the antiquark

Wilson Loops

in AdS_5

$$\langle W(C)\rangle = e^{-\alpha(D)}$$

where D is surface of minimal area $\partial D = C$, surface $D \leftrightarrow$ to the world-

sheet of the string

$\alpha(D)$ is a regularized area
subtract a term $\propto$ the circumference of $C \leftrightarrow$ action of the widely separated static quarks

If C is a square in Euclidean, width r and height T (along the Euclidean time direction)

$$\langle W(C)\rangle = e^{-TV(r)}$$

Nonperturbative Coulomb potential

Using the conformally flat Euclidean metric

$$ds_E^2 = \frac{R^2}{z^2}\left(dz^2 + \sum_{i=1}^4 dx_i^2\right)$$

scale size of C by

$$x_i \longrightarrow \rho\, x_i$$

keep $\alpha(D)$ fixed by scaling D:

$$x_i \longrightarrow \quad \rho\, x_i \quad z \longrightarrow \quad \rho\, z$$

$\alpha(D)$ is independent of ρ, $\alpha(D) \not\propto C \sim \rho^2$

$$V(r) \sim -\frac{\sqrt{g^2 N}}{r}$$

$1/r$ behavior required by conformal symmetry
$\sqrt{g^2 N}$ behavior is different from perturbative result

Breaking SUSY: finite temperature

take Euclidean time ($t_E = -it$) to be periodic:

$$t_E \sim t_E + \beta \quad e^{itE} \to e^{-\beta E}$$

$\leftrightarrow$ finite temperature 4D gauge theory
periodic boundary conditions for bosons
antiperiodic boundary conditions for fermions
zero-energy boson modes, no zero-energy fermion modes
$\to$ SUSY is broken
Scalars will get masses from loop effects
gluons are protected by gauge symmetry
low-energy effective theory is pure non-SUSY Yang-Mills

high-temperature limit lose one dimension
$\to$ zero-temperature, non-SUSY, 3D Yang-Mills

AdS Finite Temperature

in AdS there is a at high T partition function dominated by a black hole metric with a horizon size $b = \pi T$

$$\frac{ds^2}{R^2} = \left(u^2 - \frac{b^4}{u^2}\right)^{-1} du^2 + \left(u^2 - \frac{b^4}{u^2}\right) d\tau^2 + u^2 dx^i dx^i$$

blackhole horizon $\leftrightarrow$ confinement in gauge theory

Finite Temperature and Confinement

$$\langle W(C) \rangle = e^{-\alpha(D)}$$

in black hole metric bounded by the horizon, $u = b$
minimal area of D is area at the horizon

$$\alpha(D) = R^2 b^2 \, \text{area}(C)$$

$\leftrightarrow$ area law confinement

$$V(r) = R^2 b^2 r$$

string tension is very large

$$\sigma \sim R^2 b^2 \sim \sqrt{g^2 N} \, \alpha' b^2$$

Glueballs

massless scalar field Φ in AdS_5, dilaton which couples to Tr F^2
Tr F^2 has nonzero overlap with gluon states

$\Phi \leftrightarrow 0^{++}$ glueball

with AdS black hole metric:

$$\partial_\mu \left[\sqrt{g} g^{\mu\nu} \partial_\nu \Phi\right] = 0 \ , \qquad \Phi = f(u) e^{ik.x}$$

$$u^{-1} \frac{d}{du} \left(\left(u^4 - b^4\right) u \frac{df}{du} \right) - k^2 f = 0$$

for large u, $f(u) \sim u^\lambda$ where $m^2 = 0 = \lambda(\lambda + 4)$ so as $u \to \infty$ either $f(u) \sim$ constant or $\sim u^{-4}$.

second solution is normalizable solution

need f to be regular at $u = b \Rightarrow df/du$ is finite

wave guide problem, bc in the direction $\perp$ to k

Glueball Mass Gap

no normalizable solutions for $k^2 \geq 0$
discrete eigenvalues solutions for $k^2 < 0$
3D glueball masses

$$M_i^2 = -k_i^2 > 0$$

mass gap as expected for confining gauge theory

4D Glueball Masses

M-theory 5-brane wrapped on two circles
one circle is small $\rightarrow$ Type IIA D4-branes on a circle
problem is that the supergravity limit $g \rightarrow 0$, $g^2 N \rightarrow \infty \not\leftrightarrow$ gauge theories we usually think about.

Strong coupling problem

QCD_3 intrinsic scale:

$$g_3^2 N = g^2 N T$$

hold fixed as $T \to \infty$ need $g^2 N \to 0$

QCD_4 intrinsic scale:

$$\Lambda_{\mathrm{QCD}} = \exp\left(\frac{-24\pi^2}{11\, g^2 N}\right) T$$

hold fixed as $T \to \infty$ need $g^2 N \to 0$

supergravity calculation works when extra SUSY states have masses $\sim$ glueballs

4D Glueball Masses

consider M5-branes wrapped on two circles where the M5-branes have some angular momentum a

$$\begin{aligned} ds^2_{\rm IIA} \quad &= \frac{2\pi\lambda A}{3u_0} u^3 \Delta^{1/2} \Big[\quad 4\big(-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2\big) + \frac{4A^2}{9u_0^2} \left(1 - \frac{u_0^6}{u^6\Delta}\right) d\theta_2^2 \\ &+ \frac{4\ du^2}{u^4(1 - \frac{a^4}{u^4} - \frac{u_0^6}{u^6})} d\theta^2 + \frac{\tilde{\Delta}}{u^2\Delta} \sin^2\theta d\varphi^2 \\ &+ \frac{1}{u^2\Delta} \cos^2\theta d\Omega_2^2 - \frac{4a^2 A u_0^2}{3u^6\Delta} \sin^2\theta d\theta_2 d\varphi \Big] \end{aligned}$$

$$\Delta \equiv 1 - \frac{a^4 \cos^2\theta}{u^4} \ , \qquad \tilde{\Delta} \equiv 1 - \frac{a^4}{u^4} \ ,$$

$$A \equiv \frac{u_0^4}{u_H^4 - \frac{1}{3}a^4} \ , \qquad u_H^6 - a^4 u_H^2 - u_0^6 = 0$$

horizon u_H, dilaton background $e^{2\Phi}$, temperature T_H

$$e^{2\Phi} = \frac{8\pi}{27} \frac{A^3\lambda^3 u^3 \Delta^{1/2}}{u_0^3} \frac{1}{N^2} \ , \qquad R = (2\pi T_H)^{-1} = \frac{A}{3u_0}$$

when $a/u_0 \gg 1$ $R \to 0$ shrinks to zero

4D Glueball Masses

0^{++} glueballs $\leftrightarrow$ $\mathrm{Tr}FF$, solve

$$\partial_\mu \left[\sqrt{g} e^{-2\Phi} g^{\mu\nu} \partial_\nu \Phi\right] = 0$$

0^{-+} glueballs $\leftrightarrow$ $\mathrm{Tr}F\tilde{F}$, solve

$$\partial_\nu \left[\sqrt{g} g^{\mu\rho} g^{\nu\sigma} (\partial_\rho A_\sigma - \partial_\sigma A_\rho)\right] = 0$$

discrete sets of eigenvalues, functions of a

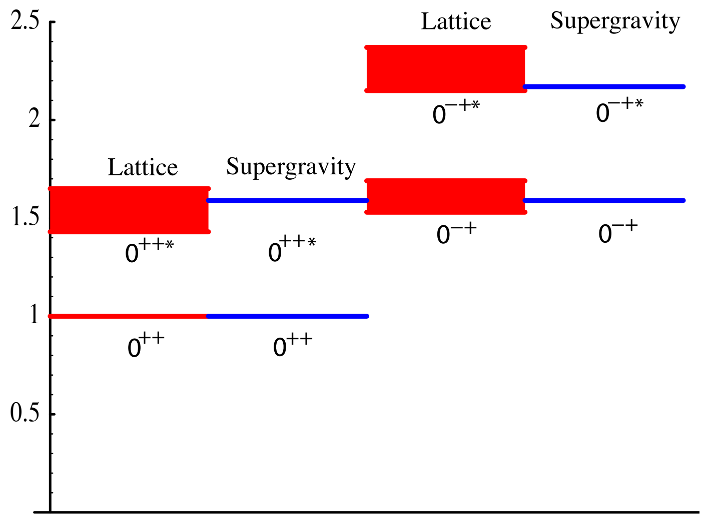
4D Glueball Masses: $a \to \infty$

state	lattice $N = 3$	SUGRA $a = 0$	SUGRA $a \to \infty$
0^{++}	1.61 ± 0.15	1.61 (input)	1.61 (input)
0^{++*}	2.48 ± 0.23	2.55	2.56
0^{-+}	2.59 $\pm$0.13	2.00	2.56
0^{-+*}	3.64 $\pm$0.18	2.98	3.49

circle KK modes decouple $\Rightarrow$ real 4D gauge theory
0^{++} glueball mass ratios change only slightly
S^4 KK modes do not decouple

$a/u_0 \gg 1$, approaches a SUSY limit

4D Glueball Mass



masses are within 4% of the lattice results

strong-coupling expansion off by between 7% and 28%

SUGRA results are much better than we have any reason to expect

Breaking SUSY: Orbifolds

Type IIB on $\mathrm{AdS}_5 \times S^5$ KK mode	$\leftrightarrow$	$\mathcal{N} = 4\,\mathrm{CFT}$ operator
$\downarrow$	orbifolding S^5	$\downarrow$
$\mathrm{AdS}_5 \times S^5/\Gamma$ invariant KK mode	$\leftrightarrow$	$\mathcal{N} < 4\,\mathrm{CFT}$ invariant operator

construct $\mathcal{N} = 1$ SUSY CFTs by orbifolding $\mathcal{N} = 4$with discrete group Γ embedded in $SU(N)$ using an N-fold copy of the regular representation

$\leftrightarrow$ Type IIB string theory on orbifold $\mathrm{AdS}_5 \times S^5/\Gamma$

For $\mathcal{N} = 1$, the $SO(6) \simeq SU(4)_R$ isometry of S^5 is broken to $U(1)_R \times \Gamma$

Z_3 Orbifold

$$X^{1,2,3} \to e^{2\pi i/3} X^{1,2,3} \ ,$$

X^i parameterize the $R^6 \perp$ to the D3-branes

	$SU(N)$	$SU(N)$	$SU(N)$	$U(1)_R$
U^i	$\square$	$\overline{\square}$	$\mathbf{1}$	$\frac{2}{3}$
V^i	$\mathbf{1}$	$\square$	$\overline{\square}$	$\frac{2}{3}$
W^i	$\overline{\square}$	$\mathbf{1}$	$\square$	$\frac{2}{3}$

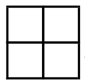
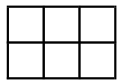
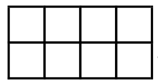
,

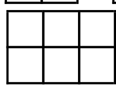
where $i = 1, 2, 3$, $SU(3)$ global symmetry is broken by the superpotential
orbifold fixed point $X^i = 0$
volume of S^5 is nonzero, manifold is non-singular
supergravity description still applicable

Z_3 Orbifold

KK modes of supergravity on $\text{AdS}_5 \times S^5/Z_3$ are Z_3 invariant

for example, the KK mode

Spin	$SU(4)_R \sim SO(6)$	m^2R^2	Operator
0	, , , $\ldots$	$k(k-4)\ ,\ \ k \geq 2$	$\mathrm{Tr}(\Phi^{I_1} \ldots \Phi^{I_k})$

with $k = 3$,  $= \mathbf{50}$ of $SU(4)_R$ couples to a dim 3 chiral primary op

$SU(4)_R \to SU(3) \times U(1)_R$ gives:

$$\mathbf{50} \to \mathbf{10}_2 + \overline{\mathbf{10}}_{-2} + \mathbf{15}_{2/3} + \overline{\mathbf{15}}_{-2/3}$$

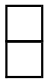
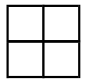
Z_3 on $\mathbf{3}$ of $SU(3)$: $(x^1, x^2, x^3) \to (e^{2\pi i/3}x^1,\ e^{2\pi i/3}x^2,\ e^{-4\pi i/3}x^3)$

$\mathbf{10}$ is contained in $\mathbf{3} \times \mathbf{3} \times \mathbf{3} \Rightarrow \mathbf{10}$ is invariant under the Z_3 projection,

$\mathbf{10}$ has correct R-charge

$\leftrightarrow$10 chiral primary operators $\mathrm{Tr}\, U^{i_1} V^{i_2} W^{i_3}$ symmetric in i_k

Z_3 Orbifold

Spin	$SU(4)_R \sim SO(6)$	m^2R^2	Operator
0	$\mathbf{1}$, , , $\ldots$	$k(k+4)\ ,\ \ k \geq 0$	$\mathrm{Tr}\, \phi^k F^{\mu\nu} F_{\mu\nu} + \ldots$

$k=0$, dilaton transforms as $\mathbf{1}$ invariant under the Z_3 projection couples to the marginal primary operator $\sum_{i=1}^{3} \mathrm{Tr}\, F_i^2$

result is independent of Γ
$\mathrm{Tr}\, F^2$ is marginal in any theory obtained by Γ projection on $\mathcal{N}=4$