

Superconformal field theories

A-Maximization

classically the trace of the energy–momentum tensor for a scale-invariant theory vanishes.

trace anomaly. r:

$$T_{\mu}^{\mu} = \frac{1}{g^3} \tilde{\beta} (F_{\mu\nu}^b)^2 - a (\tilde{R}_{\mu\nu\rho\sigma})^2 + \dots ,$$

central charge a

$\tilde{\beta}$ is the numerator of the exact NSVZ β function

$R_{\mu\nu\rho\sigma}$ is the curvature tensor

Cardy conjectured that a satisfies:

$$a_{IR} < a_{UV}$$

A-Maximization

in SCFT a can be determined by 't Hooft anomalies of SC R -charge

$$a = \frac{3}{32} (3\text{Tr}R^3 - \text{Tr}R)$$

$T_{\mu\nu}$ and the R -current ($J_{R\mu}$) in the same supermultiplet

In superspace, super-energy–momentum tensor $T_{\alpha\dot{\alpha}}(x, \theta, \bar{\theta})$ contains:
 $J_{R\mu}$, $J_{\alpha\mu}$ in the θ and $\bar{\theta}$ components and $T_{\mu\nu}$ in the θ^2 component

superconformal R -charge:

$$R = R_0 + \sum_i c_i Q_i$$

A-Maximization

superconformal symmetry relates different triangle anomalies $\langle J_R J_R J_i \rangle$ related to $\langle T T J_i \rangle$ by

$$9 \operatorname{Tr} R^2 Q_i = \operatorname{Tr} Q_i$$

two-point function

$$\langle J_i(x) J_k(0) \rangle \propto \tau_{ik} \frac{1}{x^4}$$

Unitarity $\Rightarrow \tau_{ik}$ to have positive definite eigenvalues

Superconformal symmetry \Rightarrow

$$\operatorname{Tr} R Q_i Q_k = -\frac{\tau_{ik}}{3}$$

$\Rightarrow \operatorname{Tr} R Q_i Q_k$ is negative definite

A-Maximization

Intriligator and Wecht: correct choice of the R -charge

$$R = R_0 + \sum_i c_i Q_i$$

maximizes a -charge:

$$\begin{aligned} \frac{\partial a}{\partial c_i} &= \frac{3}{32} (9 \operatorname{Tr} R^2 Q_i - \operatorname{Tr} Q_I) = 0 \\ \frac{\partial^2 a}{\partial c_i \partial c_k} &= \frac{27}{16} \operatorname{Tr} R Q_i Q_k < 0 \end{aligned}$$

The simplest chiral SCFT

	$SU(N)$	$SU(F)$	$SU(F + N - 4)$	$U(1)_R$
Q	\square	\square	$\mathbf{1}$	$R(Q)$
\bar{Q}	$\bar{\square}$	$\mathbf{1}$	\square	$R(\bar{Q})$
A	$\begin{array}{ c } \hline \square \\ \hline \end{array}$	$\mathbf{1}$	$\mathbf{1}$	$R(A)$

$F = 0$, breaks SUSY

$F = 1, 2$, runaway vacua

$F = 3$, quantum deformed moduli space

$F = 4$, s-confining

$F = 5$, splits into an IR free and IR fixed point sectors

$F > 5$?

Moduli Space

parameterized by mesons $M = Q\bar{Q}$, $H = \bar{Q}A\bar{Q}$ and baryons:

$$\begin{array}{cc} N \text{ even} & N \text{ odd} \\ \bar{Q}^N & \bar{Q}^N \\ A^{N/2} & QA^{N-1/2} \\ Q^2 A^{(N-2)/2} & Q^3 A^{(N-3)/2} \\ \vdots & \vdots \\ Q^k A^{(N-k)/2} & Q^k A^{(N-k)/2} \end{array}$$

where $k \leq \min(N, F)$

R-charge

anomaly cancellation for large $F, N \Rightarrow$

$$R(A) = \frac{F}{N} (2 - R(Q) - (\frac{N}{F} + 1) R(\bar{Q}))$$

In general

$$\begin{aligned} R(Q) &= 2 - \frac{6}{N} + b(N - F) + c \\ R(\bar{Q}) &= \frac{6}{N} + bF - c \frac{F}{F+N-4} \\ R(A) &= -\frac{12}{N} - 2bF \end{aligned}$$

$$\begin{aligned} a &= 3(N^2 - 1 + FN(R(Q) - 1))^3 + N(F + N - 4)(R(\bar{Q}) - 1)^3 \\ &+ N(N - 1)/2(R(A) - 1)^3 - (N^2 - 1) - NF(R(Q) - 1) \\ &- N(F + N - 4)(R(\bar{Q}) - 1) - N(N - 1)\frac{1}{2}(R(A) - 1) \end{aligned}$$

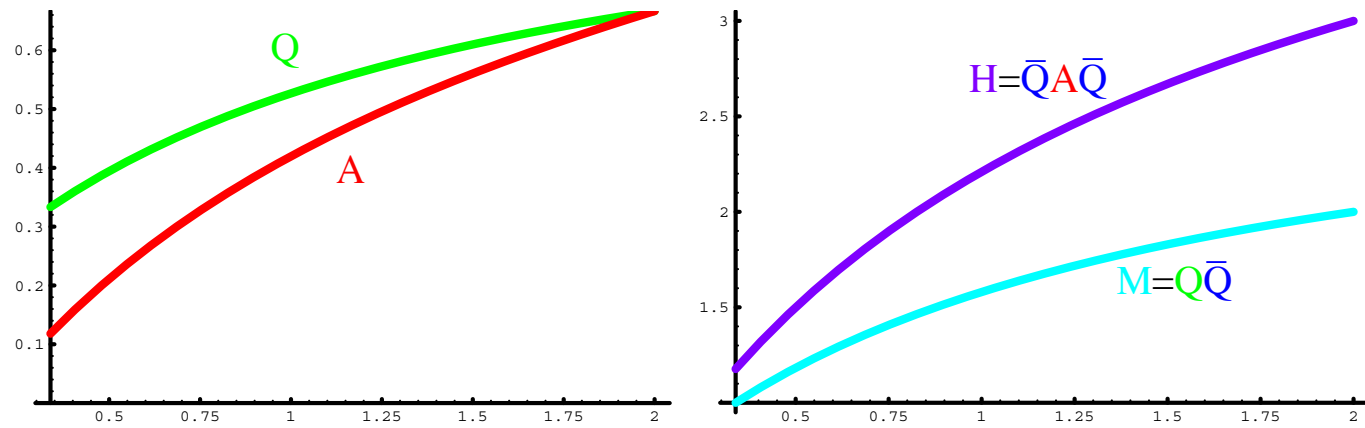
R-charge

a-maximization gives for F, N :

$$R(Q) = R(\bar{Q}) = -\frac{12 - 9\left(\frac{N}{F}\right)^2 + \sqrt{\left(\frac{N}{F}\right)^2\left(-4 + \frac{N}{F}\left(73\frac{N}{F} - 4\right)\right)}}{3\left(-4 + \left(\frac{N}{F} - 4\right)\frac{N}{F}\right)}$$

even though theory is chiral

R-charge



- (a) The R -charges of the fundamental fields, with $R(Q) = R(\bar{Q})$
(b) The corresponding dimensions of the meson operators.

Two Free Mesons

reduce F from the Banks–Zaks fixed point at $F \sim 2N$
meson $M = Q\bar{Q}$ goes free at

$$F = F_1 = \frac{9N}{4(4+\sqrt{7})} \approx 0.3386 N$$

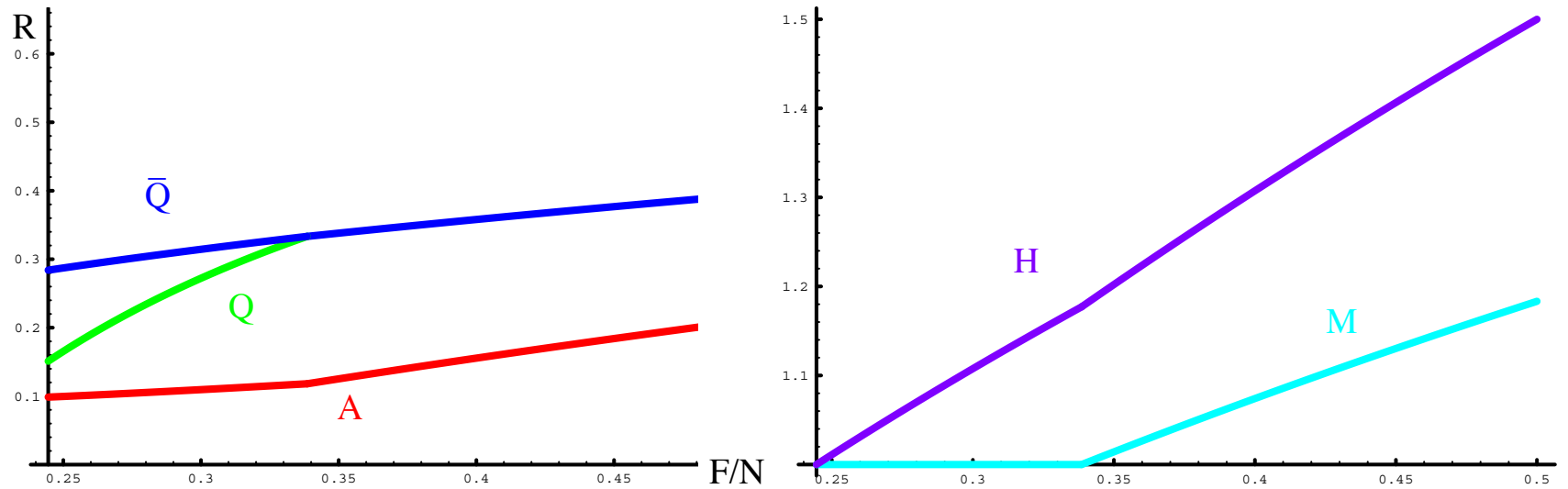
meson $H = \bar{Q}A\bar{Q}$ is still interacting

Kutasov: assume only one accidental $U(1)$ for the free meson M then

$$\begin{aligned} a_{\text{int}} &= a - a(R(M)) \\ &= a - \frac{3}{32}F(F + N - 4)(3(R(Q) + R(\bar{Q}) - 1))^3 \\ &\quad - (R(Q) + R(\bar{Q}) - 1) \end{aligned}$$

meson $H = \bar{Q}A\bar{Q}$ goes free at $F = F_2 \approx 0.2445N$

Two Free Mesons



- (a) The R -charges of the fundamental fields
- (b) The corresponding dimensions of the meson operators

Dual Description

for N odd and $F \geq 5$:

	$SU(F - 3)$	$Sp(2F - 8)$	$SU(F)$	$SU(N + F - 4)$
\tilde{y}	$\overline{\square}$	\square	1	1
\bar{p}	$\overline{\square}$	1	1	1
q	\square	1	$\overline{\square}$	1
a	\square \square	1	1	1
l	1	\square	1	$\overline{\square}$
B_1	1	1	\square	1
M	1	1	\square	\square
H	1	1	1	\square \square

with a superpotential

$$W = c_1 Mql\tilde{y} + c_2 Hll + B_1 q\bar{p} + a\tilde{y}\tilde{y}$$

$M = Q\bar{Q}$ and $H = \bar{Q}A\bar{Q}$ are mapped to elementary fields

Dual Description

even N and $F \geq 5$:

	$SU(F - 3)$	$Sp(2F - 8)$	$SU(F)$	$SU(N + F - 4)$	$SU(2)$
\tilde{y}	$\overline{\square}$	\square	1	1	1
\bar{p}	$\overline{\square}$	1	1	1	\square
q	\square	1	$\overline{\square}$	1	1
a	$\begin{array}{c} \square \\ \square \end{array}$	1	1	1	1
l	1	\square	1	$\overline{\square}$	1
S	1	1	\square	1	\square
B_0	1	1	1	1	1
M	1	1	\square	\square	1
H	1	1	1	$\begin{array}{c} \square \\ \square \end{array}$	1

with a superpotential

$$W = c_1 Mql\tilde{y} + c_2 Hll + Sq\bar{p} + a\tilde{y}\tilde{y} + B_0 a\bar{p}^2$$

Dual Description

M goes free at $F = F_1$, $c_1 \rightarrow 0$ chiral operator $ql\tilde{y}$ has dimension 2

H goes free at $F = F_2$, $c_2 \rightarrow 0$ chiral operator ll has dimension 2 corresponding to R -charge $4/3$

l has R -charge $2/3$ and dimension 1? is l free?

self-consistent if l is a gauge-invariant operator

recall SUSY QCD:

$$W = c M \phi \bar{\phi}$$

when M goes free, the coupling $c \rightarrow 0$ the chiral operator $\phi\bar{\phi}$ has dimension 2

a -maximization $\Rightarrow \phi, \bar{\phi}$ are free if we assume accidental axial symmetry for dual quarks

accidental axial symmetry only if dual gauge group is IR-free

dual β function \Rightarrow dual loses asymptotic freedom for $F < 3N/2$

dual quarks are free

Dual β function

with c_1 and c_2 set to zero

$$W = B_1 q \bar{p} + a \tilde{y} \tilde{y}$$

$Sp(2F - 8)$ has \tilde{y} and l with gauge interactions

$$\beta(g) = -\frac{g^3}{16\pi^2} \left[3(2F - 6) - (F - 3)(1 - \gamma_{\tilde{y}|g=0}) - (N + F - 4) \right] + \mathcal{O}(g^5),$$

nonperturbative $SU(F-3)$ and superpotential corrections through anomalous dimension $\gamma_{\tilde{y}}$

$Sp(2F - 8)$ is IR free if

$$N - 4F + 11 - (F - 3)\gamma_{\tilde{y}|g=0} > 0$$

Dual β function

superpotential has dimension 3 \Rightarrow

$$\gamma_a + 2\gamma_{\tilde{y}} = 0 \quad (*)$$

$a^{(F-3)/2}$ is a gauge-invariant operator \Rightarrow

$$\frac{F-3}{2} + \frac{F-3}{4}\gamma_a \geq 1 \quad (**)$$

Combining eqns (*) and (**) we have that for large F , $\gamma_{\tilde{y}} \leq 1$
large F and large N limit with $F < N/5$:

$$\beta \propto N - 4F + 11 - (F - 3)\gamma_{\tilde{y}} > 0 ,$$

$Sp(2F - 8)$ is IR free

assuming l is free for $F < F_2$ we can check that $Sp(2F - 8)$ becomes IR free at $F = F_2$

$F < F_2$: Mixed Phase

theory splits into two sectors in the IR, free magnetic sector:

	$Sp(2F - 8)$	$SU(F)$	$SU(N + F - 4)$
l	\square	$\mathbf{1}$	$\overline{\square}$
M	$\mathbf{1}$	\square	\square
H	$\mathbf{1}$	$\mathbf{1}$	$\begin{matrix} \square \\ \square \end{matrix}$

interacting superconformal sector: is

	$SU(F - 3)$	$Sp(2F - 8)$	$SU(F)$
\tilde{y}	$\overline{\square}$	\square	$\mathbf{1}$
\bar{p}	$\overline{\square}$	$\mathbf{1}$	$\mathbf{1}$
q	\square	$\mathbf{1}$	$\overline{\square}$
a	$\begin{matrix} \square \\ \square \end{matrix}$	$\mathbf{1}$	$\mathbf{1}$
B_1	$\mathbf{1}$	$\mathbf{1}$	\square

$$W = B_1 q \bar{p} + a \tilde{y} \tilde{y}$$

$\mathcal{N} = 1$: Open Questions

- how **nonperturbative effects** make $\gamma_Q \neq \gamma_{\bar{Q}}$ only for $F < F_1$
- mixed-phase first conjectured in **theories with an adjoint**, still not proven
- **SO with spinors**

$$\mathcal{N} = 2$$

$SU(N)$ with $\mathcal{N} = 2$ SUSY and F hypermultiplets in \square

$$\beta(g) = -\frac{g^3}{16\pi^2} \frac{(3N - N(1 - 2\beta(g)/g) - F)}{1 - Ng^2/8\pi^2}$$

adjoint in supermultiplet with gluon and gluino

$$\Rightarrow Z = 1/g^2 \Rightarrow \gamma(g) = 2\beta(g)/g$$

$\gamma_Q = 0$, non-renormalization of the superpotential \Rightarrow non-renormalization of the Kähler function, both related to a prepotential

solving for $\beta(g)$

$$\beta(g) = -\frac{g^3}{16\pi^2} (2N - F)$$

exact at one-loop

$\mathcal{N} = 2$ SCFT

$$\beta(g) = -\frac{g^3}{16\pi^2} (2N - F)$$

vanishes for $F = 2N$

β vanishes independent of $g \Rightarrow$ line of fixed points

Seiberg–Witten analysis $\Rightarrow a_D^i$ have no logarithmic corrections

classical relations between a^i and a_D^j are exact

theory with $F = 2N$ hypermultiplets is **nonperturbatively conformal**

Argyres–Douglas fixed points

massless electrically and magnetically charged particles at the same point in the moduli space

electric charge: $g_{\text{IR}} \rightarrow 0$, magnetic charge: $g_{\text{IR}} \rightarrow \infty$

IR fixed point? Argyres and Douglas: yes!

$\mathcal{N} = 2$ $SU(2)$ with one flavor

adjust mass and VEV so monopole and dyon points coincide

for $m = 3\Lambda_1/4$ and $u = 3\Lambda_1^2/4$

$$y^2 = \left(x - \frac{\Lambda_1^2}{4}\right)^3$$

all three roots coincide

Seiberg–Witten analysis shows that a_D has no logarithmic corrections,
theory is conformal

Argyres–Douglas fixed points

charges in $U(1)$ theories with IR fixed points do not produce long-range fields
using

$$d \geq \frac{1}{2}[C_2(\mathbf{r}) + C_2(V) - C_2(\mathbf{r}')] .$$

$F_{\mu\nu}$, is in a $(1, 0) + (0, 1)$ of $SO(4)$, has a scaling dimension $d \geq 2$
at an interacting IR fixed point generically $d > 2$
conformal symmetry and dimensional analysis \Rightarrow fields fall off as $1/x^d$

Other SCFTs

have several different interactions and are superconformal
lines (or manifolds) of fixed points
if there are n interactions and only p independent β functions
then $n - p$ dimensional manifold of fixed points

moving in manifold \leftrightarrow changing coupling of an exactly marginal operator
operator in \mathcal{L} has scaling dimension 4, independent of couplings

can also happen in $\mathcal{N} = 1$ theories

$\mathcal{N} = 4$ SUSY gauge theory

$\mathcal{N} = 1$ SUSY gauge theory with three chiral supermultiplets in the adjoint with a particular superpotential

$\equiv \mathcal{N} = 2$ SUSY gauge theory with an adjoint hypermultiplet

In general $\mathcal{N} = 4$ theories have a global $SU(4)_R \times U(1)_R$ R -symmetry restricted to vector supermultiplet does not transform under the $U(1)_R$

λ , and the three adjoint fermions, ψ , transform as a **4** of the $SU(4)_R$ real adjoint scalars ϕ transform as a **6** of $SU(4)_R$

in terms of $\mathcal{N} = 1$ fields, the $SU(4)_R$ symmetry is not manifest only $SU(3) \times U(1)$ subgroup is apparent

for canonically normalized $\mathcal{N} = 1$ superfields the superpotential is

$$W_{\mathcal{N}=4} = -i\sqrt{2}Y \text{Tr} \Phi_1 [\Phi_2, \Phi_3] = \frac{Y}{3\sqrt{2}} \epsilon_{ijk} f^{abc} \Phi_i^c \Phi_j^a \Phi_k^b$$

where $a, \dots, e = 1, \dots, N^2 - 1$ are the adjoint gauge indices

$i, \dots, m = 1, 2, 3$ are $SU(3)$ flavor indices, and $\Phi_i = T^a \Phi_i^a$

for $\mathcal{N} = 4$ SUSY, $Y = g$

$\mathcal{N} = 4$ SUSY gauge theory

Lagrangian is given by

$$\begin{aligned} \mathcal{L}_{\mathcal{N}=4} = & -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} - i\bar{\lambda}^a \sigma^\mu D_\mu \lambda^a - i\bar{\psi}_i^a \sigma^\mu D_\mu \psi_i^a + D^\mu \phi_i^{\dagger a} D_\mu \phi_i^a \\ & - \sqrt{2} g f^{abc} (\phi_i^{\dagger c} \lambda^a \psi_i^b - \bar{\psi}_i^c \bar{\lambda}^a \phi_i^b) - \frac{Y}{\sqrt{2}} \epsilon_{ijk} f^{abc} (\phi_i^c \psi_j^a \psi_k^b + \bar{\psi}_i^c \bar{\psi}_j^a \phi_k^{\dagger b}) \\ & + \frac{g^2}{2} (f^{abc} \phi_i^b \phi_i^{\dagger c}) (f^{ade} \phi_j^d \phi_j^{\dagger e}) - \frac{Y^2}{2} \epsilon_{ijk} \epsilon_{ilm} (f^{abc} \phi_j^b \phi_k^c) (f^{ade} \phi_l^{\dagger d} \phi_m^{\dagger e}) \end{aligned}$$

$SU(N)$ gauge theory with $\mathcal{N} = 2$ SUSY and A adjoint hypermultiplets has

$$\beta(g) = -\frac{g^3}{16\pi^2} (2 - 2A)N$$

$\Rightarrow \mathcal{N} = 4$ gauge theory has $\beta = 0 \leftrightarrow$ SCFT

Quivers and Mooses

Theories with gauge groups connected by bifundamentals called “quiver” theories and “mooses” matter content can be represented by a quiver/moose diagram in certain cases a quiver/moose theory can be considered as a latticization (a.k.a “deconstruction”) along a discretized extra dimension

$\mathcal{N} = 4$ and orbifolds

mod-out by a discrete subgroup Γ of the gauge and global symmetries

→ “daughter” theory

→ quiver/moose

large N limit of an $SU(N)$ dominated by planar diagrams

if Γ embedded in gauge group using regular representation N times then the planar diagrams of daughter \propto planar diagrams of full theory (up to a rescaling of the gauge coupling)

large N limit, daughters of the $\mathcal{N} = 4$ gauge theory are conformal

orbifolding in different ways break different amounts of SUSY:

$$\begin{aligned} SU(4)_R \supset \Gamma, \quad SU(3) \not\supset \Gamma &\Rightarrow \mathcal{N} = 0 \\ SU(3) \supset \Gamma, \quad SU(2) \not\supset \Gamma &\Rightarrow \mathcal{N} = 1 \\ SU(2) \supset \Gamma, &\Rightarrow \mathcal{N} = 2 . \end{aligned}$$

unbroken SUSY \leftrightarrow size of the R -symmetry subgroup invariant under Γ

$\mathcal{N} = 4$ and orbifolds

simplest case: permutation group $\Gamma = Z_k$ embedded in the gauge group
regular representation of Z_k :

$$\gamma^{\mathbf{a}} = \text{diag}(\omega^0, \omega^a, \omega^{2a}, \dots, \omega^{(k-1)a})$$

where $\omega = e^{2\pi i/k}$ and $a = 0, 1, \dots, k-1$, embed Z_k in $SU(kN)$ by
defining

$$\gamma_{\mathbf{N}}^{\mathbf{a}} = \text{diag}(\mathbf{1}_{\mathbf{N}}, \mathbf{1}_{\mathbf{N}}\omega^a, \mathbf{1}_{\mathbf{N}}\omega^{2a}, \dots, \mathbf{1}_{\mathbf{N}}\omega^{(k-1)a})$$

so adjoint transforms as

$$\mathbf{Ad} \rightarrow \gamma_{\mathbf{N}}^{\mathbf{a}} \mathbf{Ad} (\gamma_{\mathbf{N}}^{\mathbf{a}})^{\dagger}$$

$\mathcal{N} = 4$ and orbifolds

parts of the $kN \times kN$ matrix of gauge fields left invariant are

$$A_{inv} = \text{diag}(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \dots, \mathbf{A}_k) ,$$

where \mathbf{A}_i is adjoint under the i th $SU(N)$ subgroup of $SU(kN)$
orbifolded gauge group is $\prod_{i=1}^k SU(N)_i$

Orbifold example

example the Z_6 orbifold where the embedding of Z_6 in the global $SU(4)$ R -symmetry is such that the four fermion fields transform as:

$$(\psi_1, \psi_2, \psi_3, \psi_4) \rightarrow (\omega^a \psi_1, \omega^{-2a} \psi_2, \omega^{3a} \psi_3, \omega^{4a} \psi_4)$$

under a global transformation
adjoint fermion ψ_3 that transforms as

$$\psi_3 \rightarrow \omega^{3a} \gamma_{\mathbf{N}}^{\mathbf{a}} \psi_3 (\gamma_{\mathbf{N}}^{\mathbf{a}})^{\dagger}$$

Orbifold example

invariant pieces of ψ_3 :

$$\begin{array}{cccccc} 0 & 0 & 0 & \psi_{14} & 0 & 0 \\ 0 & 0 & 0 & 0 & \psi_{25} & 0 \\ 0 & 0 & 0 & 0 & 0 & \psi_{36} \\ \psi_{41} & 0 & 0 & 0 & 0 & 0 \\ 0 & \psi_{52} & 0 & 0 & 0 & 0 \\ 0 & 0 & \psi_{63} & 0 & 0 & 0 \end{array}$$

bifundamentals transforming as $(\square, \bar{\square})$ under $SU(N)_i \times SU(N)_j$
similar analysis for the remaining fermion and scalar fields

Orbifolds and the Hierarchy Problem

proposed that orbifold theories solve the hierarchy problem if physics was conformal above 1 TeV

exactly conformal theory has no quadratic divergences

consider the effective theory below some scale μ
calculate the one-loop β functions, set the $\beta = 0$
in daughter theories where matter fields are distinct bifundamentals,
fixed points for Y , and λ_i , approach $\mathcal{N} = 4$ SUSY:
 $Y = \lambda_i = g$ as $N \rightarrow \infty$

Orbifolds and the Hierarchy Problem

at fixed point the one-loop scalar mass is given by

$$m_\phi^2 = \left[N c_i \lambda_i + 3 \frac{N^2 - 1}{N} g^2 - 8 N Y^2 \right] \frac{\mu^2}{16\pi^2}$$

large N limit $\sum_i c_i = 5$: no quadratic divergence
leading order in $1/N$:

$$m_\phi^2 = \frac{3g^2}{N} \frac{\mu^2}{16\pi^2}$$

to get $m_\phi = 1$ TeV, with $\mu = M_{\text{Pl}}$ we need $N = 10^{28}$

if scalar mass term is relevant operator in low-energy effective theory
below SUSY breaking scale, a large mass is generated