Seiberg duality for SUSY QCD

Phases of gauge theories

$$\begin{array}{ll} \text{Coulomb :} & V(R) \sim \frac{1}{R} \\ \text{Free electric :} & V(R) \sim \frac{1}{R\ln(R\Lambda)} \\ \text{Free magnetic :} & V(R) \sim \frac{\ln(R\Lambda)}{R} \\ \text{Higgs :} & V(R) \sim \text{constant} \\ \text{Confining :} & V(R) \sim \sigma R\ . \end{array}$$

$$\text{electric–magnetic duality:} \quad \begin{array}{rcl} \text{electron} & \leftrightarrow & \text{monopole} \\ \text{free electric} & \leftrightarrow & \text{free magnetic} \\ \text{Coulomb phase} & \leftrightarrow & \text{Coulomb phase} \end{array}$$

Mandelstam and 't Hooft conjectured duality: Higgs $\leftrightarrow$ confining
dual confinement: Meissner effect arising from a monopole condensate

analogous examples occur in SUSY gauge theories

The moduli space for $F \geq N$

	$SU(N)$	$SU(F)$	$SU(F)$	$U(1)$	$U(1)_R$
Φ, Q	$\Box$	$\Box$	$\mathbf{1}$	1	$\frac{F-N}{F}$
$\overline{\Phi}$, $\overline{Q}$	$\overline{\Box}$	$\mathbf{1}$	$\overline{\Box}$	-1	$\frac{F-N}{F}$

$\langle\Phi\rangle$ and $\langle\overline{\Phi}\rangle$ in the form

$$\langle\Phi\rangle = \begin{pmatrix} v_1 & & & 0 & \dots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & v_N & 0 & \dots & 0 \end{pmatrix}, \quad \langle\overline{\Phi}\rangle = \begin{pmatrix} \overline{v}_1 & & \\ & \ddots & \\ & & \overline{v}_N \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & 0 \end{pmatrix}$$

vacua are physically distinct, different VEVs correspond to different masses for the gauge bosons

Classical moduli space for $F \geq N$

VEV for a single flavor: $SU(N) \to SU(N-1)$
generic point in the moduli space: $SU(N)$ completely broken
$2NF - (N^2 - 1)$ massless chiral supermultiplets

gauge-invariant description "mesons," "baryons" and superpartners:

$$\begin{aligned} M_i^j &= \overline{\Phi}^{jn} \Phi_{ni} \\ B_{i_1,\ldots,i_N} &= \Phi_{n_1 i_1} \ldots \Phi_{n_N i_N} \epsilon^{n_1,\ldots,n_N} \\ \overline{B}^{i_1,\ldots,i_N} &= \overline{\Phi}^{n_1 i_1} \ldots \overline{\Phi}^{n_N i_N} \epsilon_{n_1,\ldots,n_N} \end{aligned}$$

constraints relate M and B, since the M has F^2 components, B and $\overline{B}$ each have $\begin{pmatrix} F \\ N \end{pmatrix}$ components, and all three constructed out of the same $2NF$ underlying squark fields
classically

$$B_{i_1,\ldots,i_N} \overline{B}^{j_1,\ldots,j_N} = M^{j_1}_{[i_1} \ldots M^{j_N}_{i_N]}$$

where [ ] denotes antisymmetrization

Classical moduli space for $F \geq N$

up to flavor transformations:

$$\begin{aligned}
\langle M \rangle &= \begin{pmatrix} v_1\bar{v}_1 & & & & & \\ & \ddots & & & & \\ & & v_N\bar{v}_N & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} \\
\langle B_{1,\ldots,N} \rangle &= v_1 \ldots v_N \\
\langle \overline{B}^{1,\ldots,N} \rangle &= \bar{v}_1 \ldots \bar{v}_N
\end{aligned}$$

all other components set to zero
rank $M \leq N$, if less than N, then B or $\overline{B}$ (or both) vanish
if the rank of M is k, then $SU(N)$ is broken to $SU(N-k)$
with $F-k$ massless flavors

Quantum moduli space for $F \geq N$

from ADS superpotential

$$M_i^j = (m^{-1})_i^j \left(\det m \Lambda^{3N-F}\right)^{1/N}$$

Givir large masses, m_H, to flavors N through F
matching gauge coupling gives

$$\Lambda^{3N-F} \det m_H = \Lambda_{N,N-1}^{2N+1}$$

low-energy effective theory has $N-1$ flavors and an ADS superpotential.
give small masses, m_L, to the light flavors:

$$\begin{aligned} M_i^j &= (m_L^{-1})_i^j \left(\det m_L \Lambda_{N,N-1}^{2N+1}\right)^{1/N} \\ &= (m_L^{-1})_i^j \left(\det m_L \det m_H \Lambda^{3N-F}\right)^{1/N} \end{aligned}$$

masses are holomorphic parameters of the theory, this relationship can only break down at isolated singular points

Quantum moduli space for $F \geq N$

$$M_i^j = (m^{-1})_i^j \left(\det m \Lambda^{3N-F}\right)^{1/N}$$

For $F \geq N$ we can take $m_j^i \to 0$ with components of M finite or zero
vacuum degeneracy is not lifted and there is a quantum moduli space
classical constraints between M, B, and $\overline{B}$ may be modified

parameterize the quantum moduli space by M, B, and $\overline{B}$
VEVs $\gg \Lambda$ perturbative regime
M, B, and $\overline{B} \to 0$ strong coupling
naively expect a singularity from gluons becoming massless

IR fixed points

$F \geq 3N$ lose asymptotic freedom: weakly coupled low-energy effective theory

For F just below $3N$ we have an IR fixed point (Banks-Zaks)

exact NSVZ β function:

$$\beta(g) \quad = \quad -\frac{g^3}{16\pi^2}\frac{(3N-F(1-\gamma))}{1-Ng^2/8\pi^2}$$

where γ is the anomalous dimension of the quark mass term

$$\gamma \quad = \quad -\frac{g^2}{8\pi^2}\frac{N^2-1}{N} + \mathcal{O}(g^4)$$

$$16\pi^2\beta(g) \quad = \quad -g^3\,(3N-F) - \frac{g^5}{8\pi^2}\left(3N^2 - 2FN + \frac{F}{N}\right) + \mathcal{O}(g^7)$$

IR fixed points

Large N with $F = 3N - \epsilon N$

$$16\pi^2\beta(g) \quad = \quad -g^3\epsilon N - \frac{g^5}{8\pi^2}\left(3(N^2-1) + \mathcal{O}(\epsilon)\right) + \mathcal{O}(g^7)$$

approximate solution of $\beta = 0$ where there first two terms cancel at

$$g_*^2 = \frac{8\pi^2}{3}\frac{N}{N^2-1}\,\epsilon$$

$\mathcal{O}(g^7)$ terms higher order in ϵ

without masses, gauge theory is scale-invariant for $g = g_*$

scale-invariant theory of fields with spin ≤ 1 is conformally invariant

SUSY algebra $\rightarrow$ superconformal algebra

particular R-charge enters the superconformal algebra, denote by $R_{\rm sc}$ dimensions of scalar component of gauge-invariant chiral and antichiral superfields:

$$\begin{aligned} d &= \tfrac{3}{2}R_{\rm sc}, \quad \text{for chiral superfields} \\ d &= -\tfrac{3}{2}R_{\rm sc}, \quad \text{for antichiral superfields} \end{aligned}$$

Chiral Ring

charge of a product of fields is the sum of the individual charges:

$$R_{\rm sc}[\mathcal{O}_1\mathcal{O}_2] = R_{\rm sc}[\mathcal{O}_1] + R_{\rm sc}[\mathcal{O}_2]$$

so for chiral superfields dimensions simply add:

$$D[\mathcal{O}_1\mathcal{O}_2] = D[\mathcal{O}_1] + D[\mathcal{O}_2]$$

More formally we can say that the chiral operators form a chiral ring.

ring: set of elements on which addition and multiplication are defined, with a zero and an a minus sign

in general, the dimension of a product of fields is affected by renormalizations that are independent of the renormalizations of the individual fields

Fixed Point Dimensions

R-symmetry of a SUSY gauge theory seems ambiguous since we can always form linear combinations with other $U(1)$'s

for the fixed point of SUSY QCD, $R_{\rm sc}$ is unique since we must have

$$R_{\rm sc}[Q] = R_{\rm sc}[\overline{Q}]$$

denote the anomalous dimension at the fixed point by γ_* then

$$D[M] = D[\Phi\overline{\Phi}] = 2 + \gamma_* \quad = \quad \tfrac{3}{2} 2 \tfrac{(F-N)}{F} = 3 - \tfrac{3N}{F}$$

and the anomalous dimension of the mass operator at the fixed point is

$$\gamma_* = 1 - \tfrac{3N}{F}$$

check that the exact β function vanishes:

$$\beta \propto 3N - F(1 - \gamma_*) = 0$$

Fixed Point Dimensions

For a scalar field in a conformal theory we also have

$$D(\phi) \geq 1 \ ,$$

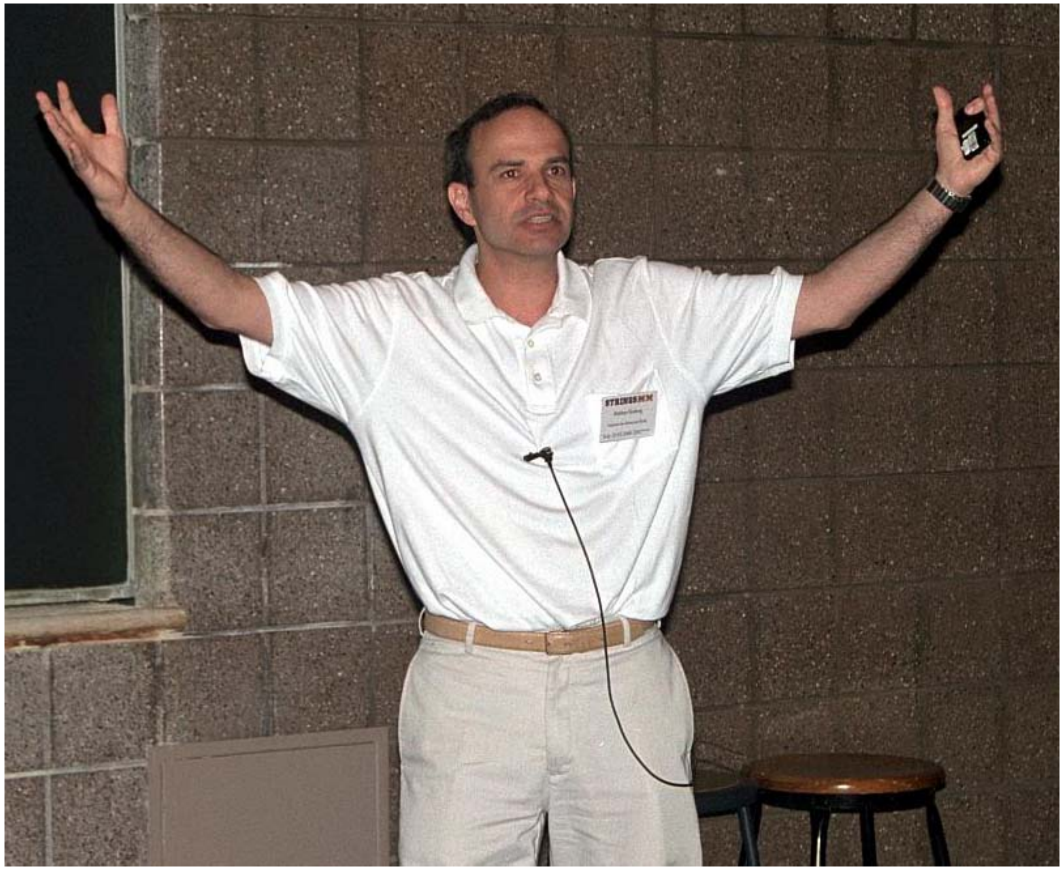
with equality for a free field Requiring $D[M] \geq 1 \Rightarrow$

$$F \geq \frac{3}{2}N$$

IR fixed point (non-Abelian Coulomb phase) is an interacting conformal theory for $\frac{3}{2}N < F < 3N$

no particle interpretation, but anomalous dimensions are physical quantities

Seiberg



Duality

conformal theory global symmetries unbroken
't Hooft anomaly matching should apply to low-energy degrees of freedom
anomalies of the M, B, and $\overline{B}$ do not match to quarks and gaugino

Seiberg found a nontrivial solution to the anomaly matching using a "dual" $SU(F-N)$ gauge theory with a "dual" gaugino, "dual" quarks and a gauge singlet "dual mesino":

	$SU(F-N)$	$SU(F)$	$SU(F)$	$U(1)$	$U(1)_R$
q	$\square$	$\overline{\square}$	$\mathbf{1}$	$\frac{N}{F-N}$	$\frac{N}{F}$
$\overline{q}$	$\overline{\square}$	$\mathbf{1}$	$\square$	$-\frac{N}{F-N}$	$\frac{N}{F}$
mesino	$\mathbf{1}$	$\square$	$\overline{\square}$	0	$2\,\frac{F-N}{F}$

Anomaly Matching

global symmetry	anomaly = dual anomaly
$SU(F)^3$	$-(F-N)+F=N$
$U(1)SU(F)^2$	$\frac{N}{F-N}(F-N)\frac{1}{2}=\frac{N}{2}$
$U(1)_R SU(F)^2$	$\frac{N-F}{F}(F-N)\frac{1}{2}+\frac{F-2N}{F}F\frac{1}{2}=-\frac{N^2}{2F}$
$U(1)^3$	$0=0$
$U(1)$	$0=0$
$U(1)U(1)_R^2$	$0=0$
$U(1)_R$	$\left(\frac{N-F}{F}\right)2(F-N)F+\left(\frac{F-2N}{F}\right)F^2+(F-N)^2-1$
	$=-N^2-1$
$U(1)_R^3$	$\left(\frac{N-F}{F}\right)^3 2(F-N)F+\left(\frac{F-2N}{F}\right)^3 F^2+(F-N)^2-1$
	$=-\frac{2N^4}{F^2}+N^2-1$
$U(1)^2U(1)_R$	$\left(\frac{N}{F-N}\right)^2\frac{N-F}{F}2F(F-N)=-2N^2$

Dual Superpotential

$$W = \lambda \widetilde{M}_i^j \phi_j \overline{\phi}^i$$

where ϕ represents the "dual" squark and $\widetilde{M}$ is the dual meson

ensures that the two theories have the same number of degrees of freedom, $\widetilde{M}$ eqm removes the color singlet $\phi\overline{\phi}$ degrees of freedom

dual baryon operators:

$$\begin{aligned} b^{i_1,\ldots,i_{F-N}} &= \phi^{n_1 i_1} \ldots \phi^{n_{F-N} i_{F-N}} \epsilon_{n_1,\ldots,n_{F-N}} \\ \overline{b}_{i_1,\ldots,i_{F-N}} &= \overline{\phi}_{n_1 i_1} \ldots \overline{\phi}_{n_{F-N} i_{F-N}} \epsilon^{n_1,\ldots,n_{F-N}} \end{aligned}$$

moduli spaces have a simple mapping

$$\begin{aligned} &M \leftrightarrow \widetilde{M} \\ &B_{i_1,\ldots,i_N} \leftrightarrow \epsilon_{i_1,\ldots,i_N,j_1,\ldots j_{F-N}} b^{j_1,\ldots,j_{F-N}} \\ &\overline{B}^{i_1,\ldots,i_N} \leftrightarrow \epsilon^{i_1,\ldots,i_N,j_1,\ldots j_{F-N}} \overline{b}_{j_1,\ldots,j_{F-N}} \end{aligned}$$

Dual β function

$$\beta(\widetilde{g}) \propto -\widetilde{g}^3(3\widetilde{N} - F) = -\widetilde{g}^3(2F - 3N)$$

dual theory loses asymptotic freedom when $F \leq 3N/2$
the dual theory leaves the conformal regime to become IR free at exactly the point where the meson of the original theory becomes a free field

strong coupling $\leftrightarrow$ weak coupling

Dual Banks–Zaks

$$F = 3\widetilde{N} - \epsilon\widetilde{N} = \frac{3}{2}\left(1 + \frac{\epsilon}{6}\right) N$$

perturbative fixed point at

$$\begin{aligned} \widetilde{g}_*^2 &= \frac{8\pi^2}{3}\frac{\widetilde{N}}{\widetilde{N}^2-1}\left(1+\frac{F}{\widetilde{N}}\right)\epsilon \\ \lambda_*^2 &= \frac{16\pi^2}{3\widetilde{N}}\epsilon \end{aligned}$$

where $D(\widetilde{M}\overline{\phi}\phi) = 3$ (marginal) since W has R-charge 2

If $\lambda = 0$, then $\widetilde{M}$ is free with dimension 1

If $\widetilde{g}$ near pure Banks-Zaks and $\lambda \approx 0$ then we can calculate the dimension of $\phi\overline{\phi}$ from the $R_{\rm sc}$ charge for $F > 3N/2$:

$$D(\phi\overline{\phi}) = \frac{3(F-\widetilde{N})}{F} = \frac{3N}{F} < 2 \ .$$

$\widetilde{M}\overline{\phi}\phi$ is a relevant operator, $\lambda = 0$ unstable fixed point, flows toward λ_*

Duality

SUSY QCD has an interacting IR fixed point for $3N/2 < F < 3N$ dual description has an interacting fixed point in the same region

theory weakly coupled near $F = 3N$ goes to stronger coupling as $F \downarrow$
dual weakly coupled near $F = 3N/2$ goes to stronger coupling as $F \uparrow$
For $F \leq 3N/2$ asymptotic freedom is lost in the dual:

$$\begin{aligned} \widetilde{g}_*^2 &= 0 \\ \lambda_*^2 &= 0 \end{aligned}$$

$\widetilde{M}$ has no interactions, dimension 1, accidental $U(1)$ symmetry in the IR

in this range IR is a theory of free massless composite gauge bosons, quarks, mesons, and superpartners

to go below $F = N + 2$ requires new considerations since there is no dual gauge group $SU(F - N)$

Integrating out a flavor

give a mass to one flavor

$$W_{\text{mass}} = m\overline{\Phi}^F \Phi_F$$

In dual theory

$$W_d = \lambda \widetilde{M}_i^j \overline{\phi}^i \phi_j + m\widetilde{M}_F^F$$

common to write

$$\lambda \widetilde{M} = \frac{M}{\mu}$$

trade the coupling λ for a scale μ and use the same symbol, M, for fields in the two different theories

$$W_d = \frac{1}{\mu} M_i^j \overline{\phi}^i \phi_j + m M_F^F$$

Integrating out a flavor

The equation of motion for M_F^F is:

$$\frac{\partial W_d}{\partial M_F^F} = \frac{1}{\mu}\overline{\phi}^F \phi_F + m = 0$$

dual squarks have VEVs:

$$\overline{\phi}^F \phi_F = -\mu m$$

along such a D-flat direction we have a theory with one less color, one less flavor, and some singlets

Integrating out a flavor

	$SU(F-N-1)$	$SU(F-1)$	$SU(F-1)$
q'	$\square$	$\overline{\square}$	$\mathbf{1}$
$\overline{q}'$	$\overline{\square}$	$\mathbf{1}$	$\square$
M'	$\mathbf{1}$	$\square$	$\overline{\square}$
q''	$\mathbf{1}$	$\overline{\square}$	$\mathbf{1}$
$\overline{q}''$	$\mathbf{1}$	$\mathbf{1}$	$\square$
S	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
M_j^F	$\mathbf{1}$	$\square$	$\mathbf{1}$
M_F^j	$\mathbf{1}$	$\mathbf{1}$	$\overline{\square}$
M_F^F	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$

$$W_{\rm eff} = \frac{1}{\mu}\left(\langle\overline{\phi}^F\rangle M_F^j \phi_j'' + \langle\phi_F\rangle M_i^F \overline{\phi}''^i + M_F^F S\right) + \frac{1}{\mu} M' \overline{\phi}' \phi'$$

integrate out M_F^j, ϕ_j'', M_i^F, $\overline{\phi}''^i$, M_F^F, and S since, leaves just the dual of $SU(N)$ with $F-1$ flavors which has a superpotential

$$W = \frac{1}{\mu} M' \overline{\phi}' \phi'$$

Consistency Checks

- global anomalies of the quarks and gauginos match those of the dual quarks, dual gauginos, and "mesons."
- Integrating out a flavor gives $SU(N)$ with $F-1$ flavors, with dual $SU(F-N-1)$ and $F-1$ flavors. Starting with the dual of the original theory, the mapping of the mass term is a linear term for the "meson" which forces the dual squarks to have a VEV and Higgses the theory down to $SU(F-N-1)$ with $F-1$ flavors.
- The moduli spaces have the same dimensions and the gauge invariant operators match.

Classically, the final consistency check is not satisfied

Consistency Checks

moduli space of complex dimension

$$2FN - (N^2 - 1)$$

$2FN$ chiral superfields and $N^2 - 1$ complex D-term constraints

dual has F^2 chiral superfields (M) and the equations of motion set the dual squarks to zero when M has rank F

duality: weak $\leftrightarrow$ strong also classical $\leftrightarrow$ quantum

original theory: $\text{rank}(M) \leq N$ classically

dual theory: $F_{eff} = F - \text{rank}(M)$ light dual quarks

If $\text{rank}(M) > N$ then $F_{eff} < \widetilde{N} = F - N$, $\Rightarrow$ ADS superpotential $\Rightarrow$ no vacuum with $\text{rank}(M) > N$

in dual, $\text{rank}(M) \leq N$ is enforced by nonperturbative quantum effects

Consistency Checks

rank constraint ⇒ number of complex degrees of freedom in M to $F^2 - \widetilde{N}^2$ since rank N $F \times F$ matrix can be written with an $(F-N) \times (F-N)$ block set to zero.

when M has N large eigenvalues, $F_{eff} = \widetilde{N}$ light dual quarks $2F_{eff}\widetilde{N} - (\widetilde{N}^2 - 1) = \widetilde{N}^2 + 1$ complex degrees of freedom
M eqm removes $\widetilde{N}^2$ color singlet degrees of freedom
dual quark equations of motion enforce that an $\widetilde{N} \times \widetilde{N}$ corner of M is set to zero

two moduli spaces match:

$$2FN - (N^2 - 1) = F^2 - \widetilde{N}^2 + \widetilde{N}^2 + 1 - \widetilde{N}^2 = F^2 - \widetilde{N}^2 + 1$$

once nonperturbative effects are taken into account

$F = N$: confinement with χSB

For $F = N$ 't Hooft anomaly matching works with just M, B, and $\overline{B}$ confining: all massless degrees of freedom are color singlet particles

For $F = N$ flavors the baryons are flavor singlets:

$$
\begin{aligned}
B &= \epsilon^{i_1,\ldots,i_F} B_{i_1,\ldots,i_F} \\
\overline{B} &= \epsilon_{i_1,\ldots,i_F} \overline{B}^{i_1,\ldots,i_F}
\end{aligned}
$$

classical constraint:

$$\det M = B\overline{B}$$

With quark masses:

$$\langle M^j_i \rangle = (m^{-1})^j_i \left(\det m\Lambda^{3N-F}\right)^{1/N}$$

Confinement with χSB

Taking a determinant of this equation (using $F = N$)

$$\det\langle M\rangle = \det\left(m^{-1}\right)\det m\Lambda^{2N} = \Lambda^{2N}$$

independent of the masses

$\det m \neq 0$ sets $\langle B\rangle = \langle\overline{B}\rangle = 0$, can integrate out all the fields that have baryon number

classical constraint is violated!

Holomorphy and the Symmetries

flavor invariants are:

	$U(1)_A$	$U(1)$	$U(1)_R$
$\det M$	$2N$	0	0
B	N	N	0
$\overline{B}$	N	$-N$	0
Λ^{2N}	$2N$	0	0

R-charge of the squarks, $(F-N)/F$, vanishes since $F=N$ generalized form of the constraint with correct $\Lambda \to 0$ and B,$\overline{B} \to 0$ limits is

$$\det M - \overline{B}B = \Lambda^{2N}\left(1+\sum_{pq} C_{pq}\frac{\left(\Lambda^{2N}\right)^p(\overline{B}B)^q}{(\det M)^{p+q}}\right)$$

with $p, q > 0$. For $\langle \overline{B}B \rangle \gg \Lambda^{2N}$ the theory is perturbative, but with $C_{pq} \neq 0$ we find solutions of the form

$$\det M \approx \left(\overline{B}B\right)^{(q-1)/(p+q)}$$

which do not reproduce the weak coupling $\Lambda \to 0$ limit

Quantum Constraint

$$\det M - \overline{B}B = \Lambda^{2N}$$

correct form to be an instanton effect

$$e^{-S_{\rm inst}} \propto \Lambda^b = \Lambda^{2N}$$

Quantum Constraint

cannot take $M = B = \overline{B} = 0$

cannot go to the origin of moduli space (“deformed” moduli space)
global symmetries are at least partially broken everywhere

Enhanced Symmetry Points

$M_i^j = \Lambda^2 \delta_i^j$, $B = \overline{B} = 0$

$SU(F) \times SU(F) \times U(1) \times U(1)_R \rightarrow SU(F)_d \times U(1) \times U(1)_R$

chiral symmetry breaking, as in non-supersymmetric QCD

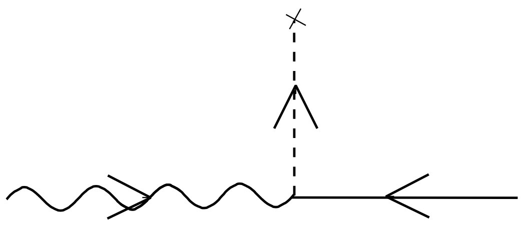
$M = 0$, $B\overline{B} = -\Lambda^{2N}$

$SU(F) \times SU(F) \times U(1) \times U(1)_R \rightarrow SU(F) \times SU(F) \times U(1)_R$

baryon number spontaneously broken

Smooth Moduli Space

For large VEVs : perturbative Higgs phase, squark VEVs give masses to quarks and gauginos



no point in the moduli space where gluons become light
$\Rightarrow$ no singular points

theory exhibits "complementarity": can go smoothly from a Higgs phase (large VEVs) to a confining phase (VEVs of $\mathcal{O}(\Lambda)$) without going through a phase transition

$F = N$: Consistency Checks

with F flavors and rank$(M) = N$, dual has confinement with χSB

$$\det(\phi\overline{\phi}) - \overline{b}b = \widetilde{\Lambda}_{eff}^{2\widetilde{N}}$$

M eqm sets $\phi\overline{\phi} = 0$
matching dual gauge coupling:

$$\widetilde{\Lambda}_{eff}^{2\widetilde{N}} = \widetilde{\Lambda}^{3\widetilde{N}-F}\det{}' M$$

where $\det' M$ is the product of the N nonzero eigenvalues of M
combining gives

$$\overline{B}B \propto \det{}' M$$

classical constraint of the original theory is reproduced in the dual by a nonperturbative effect

$F = N$: consistency checks

$$\det M - \overline{B}B = \Lambda^{2N}$$

is eqm of

$$W_{\text{constraint}} = X\left(\det M - \overline{B}B - \Lambda^{2N}\right)$$

with Lagrange multiplier field X

add mass for the Nth flavor

$$M = \begin{pmatrix} \widetilde{M}_i^j & N^j \\ P_i & Y \end{pmatrix}$$

where $\widetilde{M}$ is an $(N-1) \times (N-1)$ matrix

$F = N$: consistency checks

$$W = X\left(\det M - \overline{B}B - \Lambda^{2N}\right) + mY$$

$$\begin{aligned} \frac{\partial W}{\partial B} &= -X\overline{B} = 0 \quad \frac{\partial W}{\partial N^j} = X\,\mathrm{cof}(N^j) = 0 \\ \frac{\partial W}{\partial \overline{B}} &= -XB = 0 \quad \frac{\partial W}{\partial P_i} = X\,\mathrm{cof}(P_i) = 0 \\ \frac{\partial W}{\partial Y} &= X\det\widetilde{M} + m = 0 \end{aligned}$$

where $\mathrm{cof}(M^i_j)$ is the cofactor of the matrix element M^i_j
solution:

$$X = -m\left(\det\widetilde{M}\right)^{-1}$$

$$B = \overline{B} = N^j = P_i = 0$$

plugging solution into X eqm gives

$$\frac{\partial W}{\partial X} = Y\det\widetilde{M} - \Lambda^{2N} = 0$$

Effective Superpotential: $F \to N-1$

$$W_{\text{eff}} = \frac{m\,\Lambda^{2N}}{\det \widetilde{M}}$$

matching relation for the holomorphic gauge coupling:

$$m\Lambda^{2N} = \Lambda_{N,N-1}^{2N+1}$$

so

$$W_{\text{eff}} = \frac{\Lambda_{N,N-1}^{2N+1}}{\det \widetilde{M}}$$

ADS superpotential for $SU(N)$ with $N-1$ flavors

Enhanced Symmetry Point

$M_i^j = \Lambda^2 \delta_i^j$, $B = \overline{B} = 0$
Φ and $\overline{\Phi}$ VEVs break $SU(N) \times SU(F) \times SU(F) \to SU(F)_d$
quarks transform as $\Box \times \overline{\Box} = \mathbf{1} + \mathbf{Ad}$ under $SU(F)_d$
gluino transforms as $\mathbf{Ad}$ under $SU(F)_d$

	$SU(F)_d$	$U(1)$	$U(1)_R$
$M - \mathrm{Tr}M$	$\mathbf{Ad}$	0	0
$\mathrm{Tr}M$	$\mathbf{1}$	0	0
B	$\mathbf{1}$	N	0
$\overline{B}$	$\mathbf{1}$	$-N$	0

$\mathrm{Tr}M$ gets a mass with the Lagrange multiplier field X

Enhanced Symmetry Points: Anomalies

global symmetry	elem. anomaly	=	comp. anomaly
$U(1)^2 U(1)_R$	$-2FN$	=	$-2N^2$
$U(1)_R$	$-2FN + N^2 - 1$	=	$-(F^2 - 1) - 1 - 1$
$U(1)_R^3$	$-2FN + N^2 - 1$	=	$-(F^2 - 1) - 1 - 1$
$U(1)_R SU(F)_d^2$	$-2N + N$	=	$-N$

agree because $F = N$

Enhanced Symmetry Points

At $M = 0$, $B\overline{B} = -\Lambda^{2N}$ only the $U(1)$ symmetry is broken

	$SU(F)$	$SU(F)$	$U(1)_R$
M	$\Box$	$\overline{\Box}$	0
B	$\mathbf{1}$	$\mathbf{1}$	0
$\overline{B}$	$\mathbf{1}$	$\mathbf{1}$	0

linear combination $B + \overline{B}$ gets mass with Lagrange multiplier field X

global symmetry	elem. anomaly	$=$	comp. anomaly
$SU(F)^3$	N	$=$	F
$U(1)_R SU(F)^2$	$-N\frac{1}{2}$	$=$	$-F\frac{1}{2}$
$U(1)_R$	$-2FN + N^2 - 1$	$=$	$-F^2 - 1$
$U(1)_R^3$	$-2FN + N^2 - 1$	$=$	$-F^2 - 1$

agree because $F = N$

$F = N + 1$: s-confinement

For $F = N + 1$ 't Hooft anomaly matching works with M, B, and $\overline{B}$ confining

does not require χSB, can go to the origin of moduli space

theory develops a dynamical superpotential

	$SU(F)$	$SU(F)$	$U(1)$	$U(1)_R$
M	$\square$	$\overline{\square}$	0	$\frac{2}{F}$
B	$\overline{\square}$	$\mathbf{1}$	N	$\frac{N}{F}$
$\overline{B}$	$\mathbf{1}$	$\square$	$-N$	$\frac{N}{F}$

For $F = N + 1$ baryons are flavor antifundamentals since they are antisymmetrized in $N = F - 1$ colors:

$$B^i = \epsilon^{i_1,\ldots,i_N,i} B_{i_1,\ldots,i_N}$$
$$\overline{B}_i = \epsilon_{i_1,\ldots,i_N,i} \overline{B}^{i_1,\ldots,i_N}$$

$F = N + 1$: Classical Constraints

$$(M^{-1})^i_j \det M = B^i \overline{B}_j$$
$$M^j_i B^i = M^j_i \overline{B}_j = 0$$

with quark masses:

$$\begin{aligned} \langle M^j_i \rangle &= (m^{-1})^j_i \left(\det m \Lambda^{2N-1}\right)^{1/N} \\ \langle B^i \rangle &= \langle \overline{B}_j \rangle = 0 \end{aligned}$$

taking determinant gives

$$(M^{-1})^i_j \det M = m^i_j \Lambda^{2N-1} \ .$$

Thus, we see that the classical constraint is satisfied as $m^i_j \to 0$
taking limit in different ways covers the classical moduli space
classical and quantum moduli spaces are the same
chiral symmetry remains unbroken at $M = B = \overline{B} = 0$

Most General Superpotential

$$W = \frac{1}{\Lambda^{2N-1}} \left[\alpha B^i M_i^j \overline{B}_j + \beta \det M + \det M \, f\left(\frac{\det M}{B^i M_i^j \overline{B}_j} \right) \right]$$

where f is an as yet unknown function
only $f = 0$ reproduces the classical constraints:

$$\begin{aligned}
\frac{\partial W}{\partial M_i^j} &= \frac{1}{\Lambda^{2N-1}} \left[\alpha B^i \overline{B}_j + \beta (M^{-1})^i_j \det M \right] = 0 \\
\frac{\partial W}{\partial B^i} &= \frac{1}{\Lambda^{2N-1}} \alpha M_i^j \overline{B}_j = 0 \\
\frac{\partial W}{\partial \overline{B}_j} &= \frac{1}{\Lambda^{2N-1}} \alpha B^i M_i^j = 0
\end{aligned}$$

provided that $\beta = -\alpha$

$F = N + 1$ Superpotential

to determine α, add a mass for one flavor

$$W = \frac{\alpha}{\Lambda^{2N-1}} \left[B^i M_i^j \overline{B}_j - \det M \right] + mX$$

$$M = \begin{pmatrix} M'^i_j & Z^i \\ Y_j & X \end{pmatrix} \; , \; B = \left(U^i, B' \right) \; , \; \overline{B} = \begin{pmatrix} \overline{U}_j \\ \overline{B}' \end{pmatrix}$$

$$\begin{aligned}
\frac{\partial W}{\partial Y} &= \frac{\alpha}{\Lambda^{2N-1}} \left(B'\overline{U} - \mathrm{cof}(Y) \right) = 0 \\
\frac{\partial W}{\partial Z} &= \frac{\alpha}{\Lambda^{2N-1}} \left(U\overline{B}' - \mathrm{cof}(Z) \right) = 0 \\
\frac{\partial W}{\partial U} &= \frac{\alpha}{\Lambda^{2N-1}} Z\overline{B}' = 0 \\
\frac{\partial W}{\partial \overline{U}} &= \frac{\alpha}{\Lambda^{2N-1}} B'\overline{Y} = 0 \\
\frac{\partial W}{\partial X} &= \frac{\alpha}{\Lambda^{2N-1}} \left(B'\overline{B}' - \det M' \right) + m = 0
\end{aligned}$$

$F = N + 1$ Superpotential

solution of eqms:

$$\begin{aligned} Y &= Z = U = \overline{U} = 0 \\ \det M' - B'\overline{B}' &= \frac{m\Lambda^{2N-1}}{\alpha} = \frac{1}{\alpha}\Lambda_{N,N}^{2N} \end{aligned}$$

correct quantum constraint for $F = N$ flavors if and only if $\alpha = 1$

Plugging back in superpotential with $m\Lambda^{2N-1} = \Lambda_{N,N}^{2N}$:

$$W_{\text{eff}} = \frac{X}{\Lambda^{2N-1}}\left(B'\overline{B}' - \det M' + \Lambda_{N,N}^{2N}\right)$$

Holding $\Lambda_{N,N}$ fixed as $m \to \infty \Rightarrow \Lambda \to 0$
X becomes Lagrange multiplier
reproduce the superpotential for $F = N$

$F = N + 1$ Superpotential

superpotential for confined SUSY QCD with $F = N + 1$ flavors is:

$$W = \frac{1}{\Lambda^{2N-1}} \left[B^i M_i^j \overline{B}_j - \det M \right]$$

$M = B = \overline{B} = 0$ is on the quantum moduli space, possible singular behavior since naively gluons and gluinos should become massless

actually M, B, $\overline{B}$ become massless: confinement without χSB

$F = N + 1$ Anomalies

global symmetry	elem. anomaly	=	comp. anomaly
$SU(F)^3$	N	=	$F - 1$
$U(1)SU(F)^2$	$N\frac{1}{2}$	=	$N\frac{1}{2}$
$U(1)_R SU(F)^2$	$-\frac{N}{F}\frac{N}{2}$	=	$\frac{2-F}{F}\frac{F}{2} + \frac{N-F}{2F}$
$U(1)_R$	$-\frac{N}{F}2NF + N^2 - 1$	=	$\frac{2-F}{F}F^2 + 2(N - F)$
$U(1)_R^3$	$-\left(\frac{N}{F}\right)^3 2NF + N^2 - 1$	=	$\left(\frac{2-F}{F}\right)^3 F^2 + \left(\frac{N-F}{F}\right)^3 2F$,

agree because $F = N + 1$

Connection to $F > N + 1$

dual theory for $F = N + 2$:

	$SU(2)$	$SU(N+2)$	$SU(N+2)$	$U(1)$	$U(1)_R$
q	$\square$	$\overline{\square}$	$\mathbf{1}$	$\frac{N}{2}$	$\frac{N}{N+2}$
$\bar{q}$	$\overline{\square}$	$\mathbf{1}$	$\square$	$-\frac{N}{2}$	$\frac{N}{N+2}$
M	$\mathbf{1}$	$\square$	$\overline{\square}$	0	$\frac{4}{N+2}$

.

$$W = \frac{1}{\mu} M \overline{\phi} \phi$$

mass for one flavor produces adual squark VEV

$$\langle \overline{\phi}^F \phi_F \rangle = -\mu m$$

completely breaks the $SU(2)$

$F = N + 2 \to F = N + 1$

massless spectrum of the low-energy effective theory:

	$SU(N+1)$	$SU(N+1)$	$U(1)$	$U(1)_R$
q'	$\overline{\square}$	$\mathbf{1}$	N	$\frac{N}{N+1}$
$\overline{q'}$	$\mathbf{1}$	$\square$	$-N$	$\frac{N}{N+1}$
M'	$\square$	$\overline{\square}$	0	$\frac{2}{N+1}$

Comparing with the confined spectrum we identify

$$q'^i = cB^i \ , \ \ \overline{q'}_j = \overline{c}\overline{B}_j$$

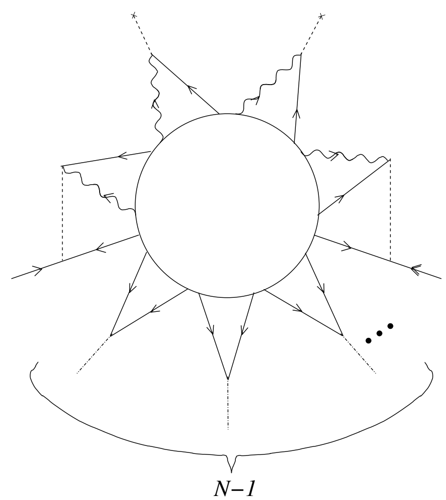
where c and $\overline{c}$ are rescalings

$$W_{\text{tree}} = \frac{c\overline{c}}{\mu} B^i M'^j_i \overline{B}_j$$

$F = N + 2 \rightarrow F = N + 1$

broken $SU(2) \Rightarrow$ instantons generate superpotential

$$W_{\rm inst.} \quad = \quad \frac{\widetilde{\widetilde{\Lambda}}^{b}_{N,N+2}}{\langle \overline{\phi}^{F} \phi_{F} \rangle} \det \left(\frac{M'}{\mu} \right) = - \frac{\widetilde{\Lambda}^{4-N}_{N,N+2}}{m} \frac{\det M'}{\mu^{N+2}}$$



two mesinos (external straight lines) and $N-1$ mesons (dash-dot lines). instanton has 4 gaugino legs (internal wavy lines) and $N+2$ quark and antiquark legs (internal straight lines)

$F = N + 2 \to F = N + 1$

effective superpotential agrees with the result for $F = N + 1$:

$$W_{\text{eff}} = \frac{1}{\Lambda^{2N-1}} \left[B^i M'^j_i \overline{B}_j - \det M' \right]$$

if and only if

$$c\bar{c} = \frac{\mu}{\Lambda^{2N-1}} \ , \quad \frac{\widetilde{\Lambda}^{4-N}_{N,N+2}}{\mu^{N+2} m} = \frac{1}{\Lambda^{2N-1}}$$

second relation follows from

$$\widetilde{\Lambda}^{3\widetilde{N}-F} \Lambda^{3N-F} = (-1)^{F-N} \mu^F$$

Intrinsic Scales

$$\widetilde{\Lambda}^{3\widetilde{N}-F}\Lambda^{3N-F} = (-1)^{F-N}\mu^F \ (*)$$

consider generic values of $\langle M \rangle$ in dual, dual quarks are massive pure $SU(\tilde{N} = F - N)$ gauge theory.

$$\widetilde{\Lambda}_L^{3\widetilde{N}} = \widetilde{\Lambda}^{3\widetilde{N}-F}\det\left(\frac{M}{\mu}\right)$$

gaugino condensation:

$$\begin{aligned} W_L &= \widetilde{N}\widetilde{\Lambda}_L^3 = (F-N)\left(\frac{\widetilde{\Lambda}^{3\widetilde{N}-F}\det M}{\mu^F}\right)^{1/(F-N)} \\ &= (N-F)\left(\frac{\Lambda^{3N-F}}{\det M}\right)^{1/(N-F)} \end{aligned}$$

where we have used eqn (*) Adding mass term $m_j^i M_i^j$ gives:

$$M_i^j = (m^{-1})_i^j \left(\det m\Lambda^{3N-F}\right)^{1/N}$$

which is the correct result

Dual of Dual

assume that $\widetilde{\widetilde{\Lambda}} = \Lambda$, (*) implies

$$\Lambda^{3N-F}\widetilde{\Lambda}^{3\widetilde{N}-F} = (-1)^{F-\widetilde{N}}\widetilde{\mu}^F$$

since $F - \widetilde{N} = N$, we must have for consistency

$$\widetilde{\mu} = -\mu$$

composite meson of the dual quarks:

$$N^i_j \equiv \overline{\phi}^i \phi_j$$

dual–dual squarks as d, dual–dual superpotential is

$$W_{dd} = \frac{N^j_i}{\widetilde{\mu}}\overline{d}^i d_j + \frac{M^i_j}{\mu} N^j_i$$

Dual of Dual

equations of motion give

$$\frac{\partial W}{\partial M^i_j} = \frac{1}{\mu} N^j_i = 0$$
$$\frac{\partial W}{\partial N^j_i} = \frac{1}{\widetilde{\mu}} \overline{d}^i d_j + \frac{1}{\mu} M^i_j = 0$$

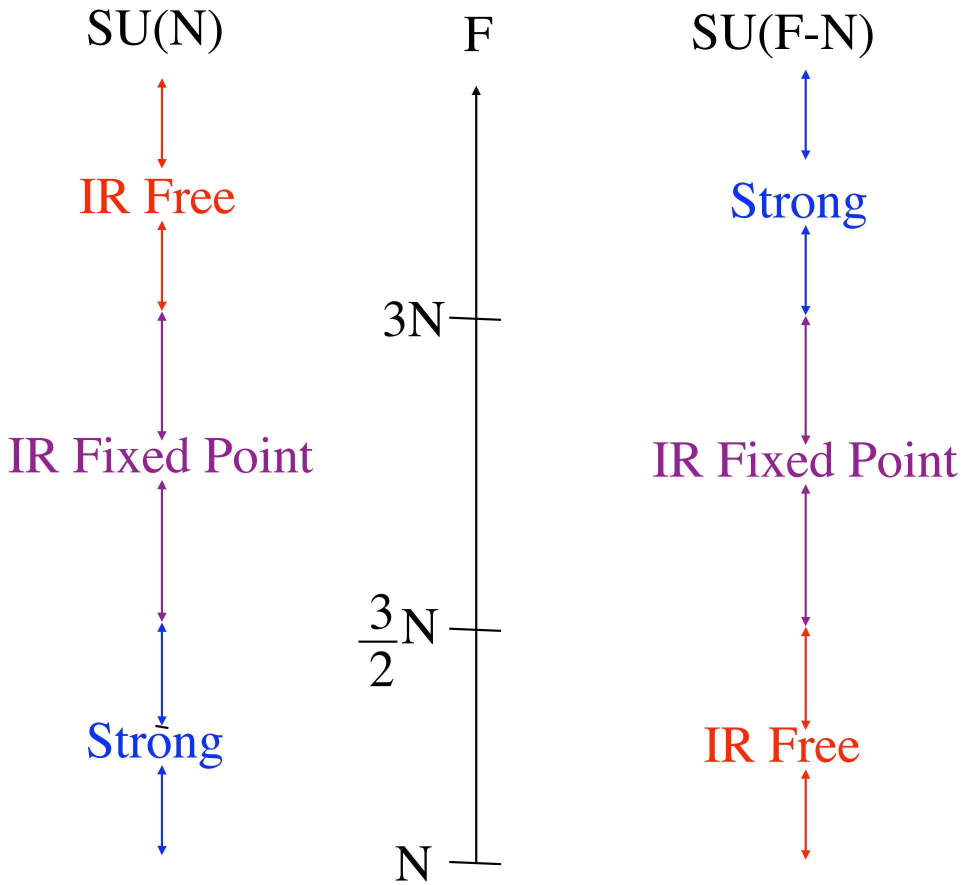
So, since $\widetilde{\mu} = -\mu$, we can identify the original squarks with the dual–dual squarks:

$$\Phi_j = d_j \ .$$

Plugging into the dual–dual superpotential (it vanishes

dual of the dual of SUSY QCD is just SUSY QCD

Duality for SUSY SU(N)

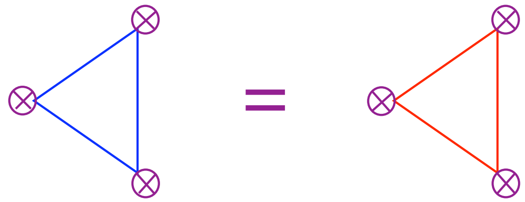


F=N+1 → confinement without χSB

F=N → confinement with χSB

Duality Consistency Checks

- **Anomaly Matching**



$Q,\bar{Q}$: SU(N) $\qquad$ $q,\bar{q}$, M: SU(F-N)

- **Identical Space of Vacua**

$Q\bar{Q} \longleftrightarrow M$

$Q^N, \bar{Q}^N \longleftrightarrow q^{F-N}, \bar{q}^{\,F-N}$

- **Deformations**

SU(N), F $\longleftrightarrow$ SU(F-N), F

$W = m\, Q_F \bar{Q}_F$ $\qquad$ $W = Mq\bar{q} + mM_{FF}$

$\downarrow$ $\qquad$ $\downarrow$ $\langle q \rangle \neq 0$, $\langle \bar{q} \rangle \neq 0$

SU(N), F-1 $\longleftrightarrow$ SU(F-1-N), F-1