

Seiberg duality for SUSY QCD

Phases of gauge theories

$$\begin{aligned} \text{Coulomb :} & \quad V(R) \sim \frac{1}{R} \\ \text{Free electric :} & \quad V(R) \sim \frac{1}{R \ln(R\Lambda)} \\ \text{Free magnetic :} & \quad V(R) \sim \frac{\ln(R\Lambda)}{R} \\ \text{Higgs :} & \quad V(R) \sim \text{constant} \\ \text{Confining :} & \quad V(R) \sim \sigma R . \end{aligned}$$

$$\begin{array}{l} \text{electric-magnetic duality:} \\ \text{Coulomb phase} \end{array} \quad \begin{array}{l} \text{electron} \leftrightarrow \text{monopole} \\ \text{free electric} \leftrightarrow \text{free magnetic} \\ \text{Coulomb phase} \leftrightarrow \text{Coulomb phase} \end{array}$$

Mandelstam and 't Hooft conjectured duality: Higgs \leftrightarrow confining
dual confinement: Meissner effect arising from a monopole condensate

analogous examples occur in SUSY gauge theories

The moduli space for $F \geq N$

	$SU(N)$	$SU(F)$	$SU(F)$	$U(1)$	$U(1)_R$
Φ, Q	\square	\square	$\mathbf{1}$	1	$\frac{F-N}{F}$
$\bar{\Phi}, \bar{Q}$	$\bar{\square}$	$\mathbf{1}$	$\bar{\square}$	-1	$\frac{F-N}{F}$

$\langle \Phi \rangle$ and $\langle \bar{\Phi} \rangle$ in the form

$$\langle \Phi \rangle = \begin{pmatrix} v_1 & & & 0 & \dots & 0 \\ & \ddots & & \vdots & & \vdots \\ & & v_N & 0 & \dots & 0 \end{pmatrix}, \quad \langle \bar{\Phi} \rangle = \begin{pmatrix} \bar{v}_1 & & & & & \\ & \ddots & & & & \\ & & & & & \bar{v}_N \\ 0 & \dots & 0 & & & \\ \vdots & & & & & \vdots \\ 0 & \dots & 0 & & & 0 \end{pmatrix}$$

vacua are physically distinct, different VEVs correspond to different masses for the gauge bosons

Classical moduli space for $F \geq N$

VEV for a single flavor: $SU(N) \rightarrow SU(N - 1)$

generic point in the moduli space: $SU(N)$ completely broken

$2NF - (N^2 - 1)$ massless chiral supermultiplets

gauge-invariant description “mesons,” “baryons” and superpartners:

$$\begin{aligned} M_i^j &= \bar{\Phi}^{jn} \Phi_{ni} \\ B_{i_1, \dots, i_N} &= \Phi_{n_1 i_1} \dots \Phi_{n_N i_N} \epsilon^{n_1, \dots, n_N} \\ \bar{B}^{i_1, \dots, i_N} &= \bar{\Phi}^{n_1 i_1} \dots \bar{\Phi}^{n_N i_N} \epsilon_{n_1, \dots, n_N} \end{aligned}$$

constraints relate M and B , since the M has F^2 components, B and \bar{B} each have $\binom{F}{N}$ components, and all three constructed out of the same $2NF$ underlying squark fields classically

$$B_{i_1, \dots, i_N} \bar{B}^{j_1, \dots, j_N} = M_{[i_1}^{j_1} \dots M_{i_N]}^{j_N}$$

where $[\]$ denotes antisymmetrization

Quantum moduli space for $F \geq N$

from ADS superpotential

$$M_i^j = (m^{-1})_i^j (\det m \Lambda^{3N-F})^{1/N}$$

Give large masses, m_H , to flavors N through F
matching gauge coupling gives

$$\Lambda^{3N-F} \det m_H = \Lambda_{N,N-1}^{2N+1}$$

low-energy effective theory has $N - 1$ flavors and an ADS superpotential.
give small masses, m_L , to the light flavors:

$$\begin{aligned} M_i^j &= (m_L^{-1})_i^j \left(\det m_L \Lambda_{N,N-1}^{2N+1} \right)^{1/N} \\ &= (m_L^{-1})_i^j (\det m_L \det m_H \Lambda^{3N-F})^{1/N} \end{aligned}$$

masses are holomorphic parameters of the theory, this relationship can only break down at isolated singular points

Quantum moduli space for $F \geq N$

$$M_i^j = (m^{-1})_i^j (\det m \Lambda^{3N-F})^{1/N}$$

For $F \geq N$ we can take $m_j^i \rightarrow 0$ with components of M finite or zero
vacuum degeneracy is not lifted and there is a quantum moduli space
classical constraints between M , B , and \bar{B} may be modified

parameterize the quantum moduli space by M , B , and \bar{B}
VEVs $\gg \Lambda$ perturbative regime
 M , B , and $\bar{B} \rightarrow 0$ strong coupling
naively expect a singularity from gluons becoming massless

IR fixed points

$F \geq 3N$ lose asymptotic freedom: weakly coupled low-energy effective theory

For F just below $3N$ we have an IR fixed point (Banks-Zaks)
exact NSVZ β function:

$$\beta(g) = -\frac{g^3}{16\pi^2} \frac{(3N - F(1 - \gamma))}{1 - Ng^2/8\pi^2}$$

where γ is the anomalous dimension of the quark mass term

$$\gamma = -\frac{g^2}{8\pi^2} \frac{N^2 - 1}{N} + \mathcal{O}(g^4)$$

$$16\pi^2\beta(g) = -g^3(3N - F) - \frac{g^5}{8\pi^2} \left(3N^2 - 2FN + \frac{F}{N}\right) + \mathcal{O}(g^7)$$

IR fixed points

Large N with $F = 3N - \epsilon N$

$$16\pi^2\beta(g) = -g^3\epsilon N - \frac{g^5}{8\pi^2} (3(N^2 - 1) + \mathcal{O}(\epsilon)) + \mathcal{O}(g^7)$$

approximate solution of $\beta = 0$ where the first two terms cancel at

$$g_*^2 = \frac{8\pi^2}{3} \frac{N}{N^2 - 1} \epsilon$$

$\mathcal{O}(g^7)$ terms higher order in ϵ

without masses, gauge theory is scale-invariant for $g = g_*$

scale-invariant theory of fields with spin ≤ 1 is conformally invariant

SUSY algebra \rightarrow superconformal algebra

particular R -charge enters the superconformal algebra, denote by R_{sc}
dimensions of scalar component of gauge-invariant chiral and antichiral
superfields:

$$\begin{aligned} d &= \frac{3}{2} R_{\text{sc}}, & \text{for chiral superfields} \\ d &= -\frac{3}{2} R_{\text{sc}}, & \text{for antichiral superfields} \end{aligned}$$

Chiral Ring

charge of a product of fields is the sum of the individual charges:

$$R_{\text{sc}}[\mathcal{O}_1 \mathcal{O}_2] = R_{\text{sc}}[\mathcal{O}_1] + R_{\text{sc}}[\mathcal{O}_2]$$

so for chiral superfields dimensions simply add:

$$D[\mathcal{O}_1 \mathcal{O}_2] = D[\mathcal{O}_1] + D[\mathcal{O}_2]$$

More formally we can say that the chiral operators form a chiral ring.

ring: set of elements on which addition and multiplication are defined, with a zero and an a minus sign

in general, the dimension of a product of fields is affected by renormalizations that are independent of the renormalizations of the individual fields

Fixed Point Dimensions

R -symmetry of a SUSY gauge theory seems ambiguous since we can always form linear combinations with other $U(1)$'s
for the fixed point of SUSY QCD, R_{sc} is unique since we must have

$$R_{\text{sc}}[Q] = R_{\text{sc}}[\bar{Q}]$$

denote the anomalous dimension at the fixed point by γ_* then

$$D[M] = D[\Phi\bar{\Phi}] = 2 + \gamma_* = \frac{3}{2}2\frac{(F-N)}{F} = 3 - \frac{3N}{F}$$

and the anomalous dimension of the mass operator at the fixed point is

$$\gamma_* = 1 - \frac{3N}{F}$$

check that the exact β function vanishes:

$$\beta \propto 3N - F(1 - \gamma_*) = 0$$

Fixed Point Dimensions

For a scalar field in a conformal theory we also have

$$D(\phi) \geq 1 ,$$

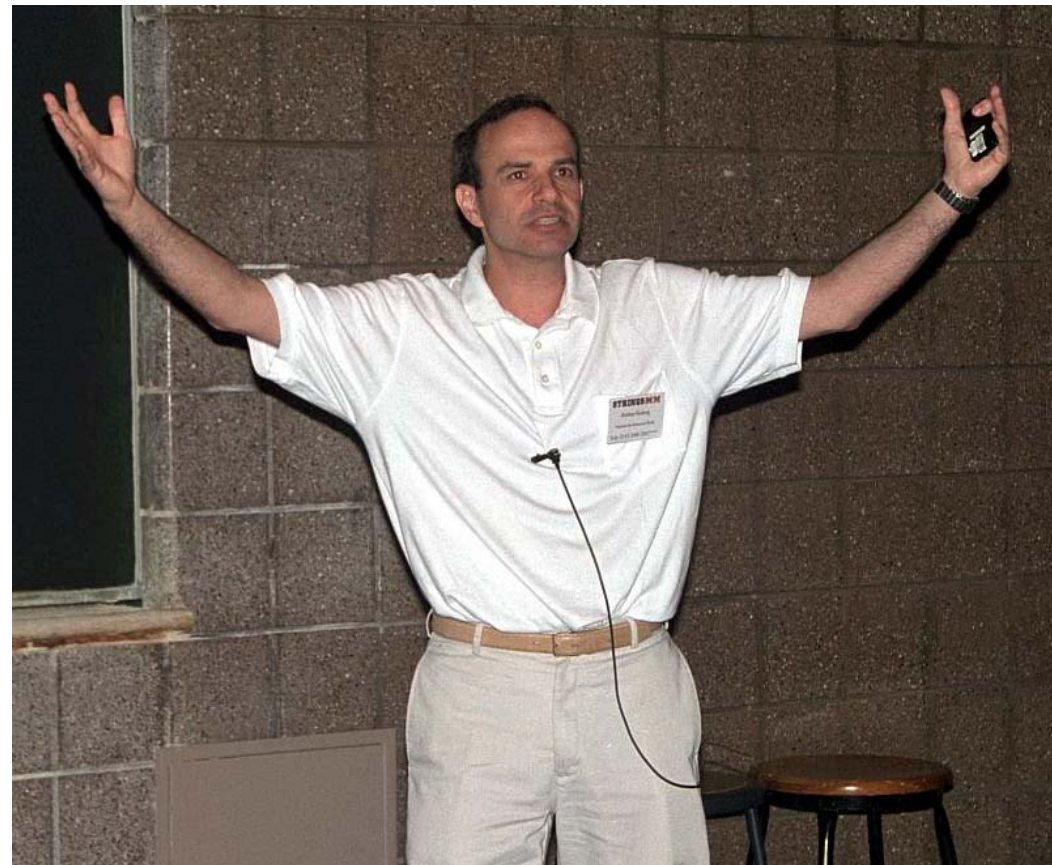
with equality for a free field Requiring $D[M] \geq 1 \Rightarrow$

$$F \geq \frac{3}{2}N$$

IR fixed point (non-Abelian Coulomb phase) is an interacting conformal theory for $\frac{3}{2}N < F < 3N$

no particle interpretation, but anomalous dimensions are physical quantities

Seiberg



Duality

conformal theory global symmetries unbroken

‘t Hooft anomaly matching should apply to low-energy degrees of freedom
 anomalies of the M , B , and \bar{B} do not match to quarks and gaugino

Seiberg found a nontrivial solution to the anomaly matching using a
 “dual” $SU(F - N)$ gauge theory with a “dual” gaugino, “dual” quarks
 and a gauge singlet “dual mesino”:

	$SU(F - N)$	$SU(F)$	$SU(F)$	$U(1)$	$U(1)_R$
q	\square	$\bar{\square}$	$\mathbf{1}$	$\frac{N}{F-N}$	$\frac{N}{F}$
\bar{q}	$\bar{\square}$	$\mathbf{1}$	\square	$-\frac{N}{F-N}$	$\frac{N}{F}$
mesino	$\mathbf{1}$	\square	$\bar{\square}$	0	$2 \frac{F-N}{F}$

Anomaly Matching

global symmetry	anomaly = dual anomaly
$SU(F)^3$	$-(F - N) + F = N$
$U(1)SU(F)^2$	$\frac{N}{F-N}(F - N)\frac{1}{2} = \frac{N}{2}$
$U(1)_R SU(F)^2$	$\frac{N-F}{F}(F - N)\frac{1}{2} + \frac{F-2N}{F}F\frac{1}{2} = -\frac{N^2}{2F}$
$U(1)^3$	$0 = 0$
$U(1)$	$0 = 0$
$U(1)U(1)_R^2$	$0 = 0$
$U(1)_R$	$\left(\frac{N-F}{F}\right) 2(F - N)F + \left(\frac{F-2N}{F}\right) F^2 + (F - N)^2 - 1$ $= -N^2 - 1$
$U(1)_R^3$	$\left(\frac{N-F}{F}\right)^3 2(F - N)F + \left(\frac{F-2N}{F}\right)^3 F^2 + (F - N)^2 - 1$ $= -\frac{2N^4}{F^2} + N^2 - 1$
$U(1)^2 U(1)_R$	$\left(\frac{N}{F-N}\right)^2 \frac{N-F}{F} 2F(F - N) = -2N^2$

Dual Superpotential

$$W = \lambda \widetilde{M}_i^j \phi_j \bar{\phi}^i$$

where ϕ represents the “dual” squark and \widetilde{M} is the dual meson
 ensures that the two theories have the same number of degrees of freedom, \widetilde{M} eqm removes the color singlet $\phi \bar{\phi}$ degrees of freedom
 dual baryon operators:

$$\begin{aligned} b^{i_1, \dots, i_{F-N}} &= \phi^{n_1 i_1} \dots \phi^{n_{F-N} i_{F-N}} \epsilon_{n_1, \dots, n_{F-N}} \\ \bar{b}_{i_1, \dots, i_{F-N}} &= \bar{\phi}_{n_1 i_1} \dots \bar{\phi}_{n_{F-N} i_{F-N}} \epsilon^{n_1, \dots, n_{F-N}} \end{aligned}$$

moduli spaces have a simple mapping

$$\begin{aligned} M &\leftrightarrow \widetilde{M} \\ B_{i_1, \dots, i_N} &\leftrightarrow \epsilon_{i_1, \dots, i_N, j_1, \dots, j_{F-N}} b^{j_1, \dots, j_{F-N}} \\ \bar{B}^{i_1, \dots, i_N} &\leftrightarrow \epsilon^{i_1, \dots, i_N, j_1, \dots, j_{F-N}} \bar{b}_{j_1, \dots, j_{F-N}} \end{aligned}$$

Dual β function

$$\beta(\tilde{g}) \propto -\tilde{g}^3(3\tilde{N} - F) = -\tilde{g}^3(2F - 3N)$$

dual theory loses asymptotic freedom when $F \leq 3N/2$

the dual theory leaves the conformal regime to become IR free at exactly the point where the meson of the original theory becomes a free field

strong coupling \leftrightarrow weak coupling

Dual Banks–Zaks

$$F = 3\tilde{N} - \epsilon\tilde{N} = \frac{3}{2} \left(1 + \frac{\epsilon}{6}\right) N$$

perturbative fixed point at

$$\begin{aligned}\tilde{g}_*^2 &= \frac{8\pi^2}{3} \frac{\tilde{N}}{\tilde{N}^2 - 1} \left(1 + \frac{F}{\tilde{N}}\right) \epsilon \\ \lambda_*^2 &= \frac{16\pi^2}{3\tilde{N}} \epsilon\end{aligned}$$

where $D(\tilde{M}\bar{\phi}\phi) = 3$ (marginal) since W has R -charge 2

If $\lambda = 0$, then \tilde{M} is free with dimension 1

If \tilde{g} near pure Banks-Zaks and $\lambda \approx 0$ then we can calculate the dimension of $\phi\bar{\phi}$ from the R_{sc} charge for $F > 3N/2$:

$$D(\phi\bar{\phi}) = \frac{3(F - \tilde{N})}{F} = \frac{3N}{F} < 2 .$$

$\tilde{M}\bar{\phi}\phi$ is a relevant operator, $\lambda = 0$ unstable fixed point, flows toward λ_*

Duality

SUSY QCD has an interacting IR fixed point for $3N/2 < F < 3N$
dual description has an interacting fixed point in the same region

theory weakly coupled near $F = 3N$ goes to stronger coupling as $F \downarrow$
dual weakly coupled near $F = 3N/2$ goes to stronger coupling as $F \uparrow$
For $F \leq 3N/2$ asymptotic freedom is lost in the dual:

$$\begin{aligned}\tilde{g}_*^2 &= 0 \\ \lambda_*^2 &= 0\end{aligned}$$

\widetilde{M} has no interactions, dimension 1, accidental $U(1)$ symmetry in the IR
in this range IR is a theory of free massless composite gauge bosons,
quarks, mesons, and superpartners

to go below $F = N + 2$ requires new considerations since there is no
dual gauge group $SU(F - N)$

Integrating out a flavor

give a mass to one flavor

$$W_{\text{mass}} = m \bar{\Phi}^F \Phi_F$$

In dual theory

$$W_d = \lambda \widetilde{M}_i^j \bar{\phi}^i \phi_j + m \widetilde{M}_F^F$$

common to write

$$\lambda \widetilde{M} = \frac{M}{\mu}$$

trade the coupling λ for a scale μ and use the same symbol, M , for fields in the two different theories

$$W_d = \frac{1}{\mu} M_i^j \bar{\phi}^i \phi_j + m M_F^F$$

Integrating out a flavor

The equation of motion for M_F^F is:

$$\frac{\partial W_d}{\partial M_F^F} = \frac{1}{\mu} \bar{\phi}^F \phi_F + m = 0$$

dual squarks have VEVs:

$$\bar{\phi}^F \phi_F = -\mu m$$

along such a D -flat direction we have a theory with one less color, one less flavor, and some singlets

Integrating out a flavor

	$SU(F - N - 1)$	$SU(F - 1)$	$SU(F - 1)$
q'	\square	$\overline{\square}$	$\mathbf{1}$
\overline{q}'	$\overline{\square}$	$\mathbf{1}$	\square
M'	$\mathbf{1}$	\square	$\overline{\square}$
q''	$\mathbf{1}$	$\overline{\square}$	$\mathbf{1}$
\overline{q}''	$\mathbf{1}$	$\mathbf{1}$	\square
S	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$
M_j^F	$\mathbf{1}$	\square	$\mathbf{1}$
M_F^j	$\mathbf{1}$	$\mathbf{1}$	$\overline{\square}$
M_F^F	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$

$$W_{\text{eff}} = \frac{1}{\mu} \left(\langle \overline{\phi}^F \rangle M_F^j \phi_j'' + \langle \phi_F \rangle M_i^F \overline{\phi}''^i + M_F^F S \right) + \frac{1}{\mu} M' \overline{\phi}' \phi'$$

integrate out M_F^j , ϕ_j'' , M_i^F , $\overline{\phi}''^i$, M_F^F , and S since, leaves just the dual of $SU(N)$ with $F - 1$ flavors which has a superpotential

$$W = \frac{1}{\mu} M' \overline{\phi}' \phi'$$

Consistency Checks

- global anomalies of the quarks and gauginos match those of the dual quarks, dual gauginos, and “mesons.”
- Integrating out a flavor gives $SU(N)$ with $F - 1$ flavors, with dual $SU(F - N - 1)$ and $F - 1$ flavors. Starting with the dual of the original theory, the mapping of the mass term is a linear term for the “meson” which forces the dual squarks to have a VEV and Higgses the theory down to $SU(F - N - 1)$ with $F - 1$ flavors.
- The moduli spaces have the same dimensions and the gauge invariant operators match.

Classically, the final consistency check is not satisfied

Consistency Checks

moduli space of complex dimension

$$2FN - (N^2 - 1)$$

$2FN$ chiral superfields and $N^2 - 1$ complex D -term constraints

dual has F^2 chiral superfields (M) and the equations of motion set the dual squarks to zero when M has rank F

duality: weak \leftrightarrow strong also classical \leftrightarrow quantum

original theory: $\text{rank}(M) \leq N$ classically

dual theory: $F_{eff} = F - \text{rank}(M)$ light dual quarks

If $\text{rank}(M) > N$ then $F_{eff} < \tilde{N} = F - N$, \Rightarrow ADS superpotential
 \Rightarrow no vacuum with $\text{rank}(M) > N$

in dual, $\text{rank}(M) \leq N$ is enforced by nonperturbative quantum effects

Consistency Checks

rank constraint \Rightarrow number of complex degrees of freedom in M to $F^2 - \tilde{N}^2$
since rank N $F \times F$ matrix can be written with an $(F - N) \times (F - N)$
block set to zero.

when M has N large eigenvalues, $F_{eff} = \tilde{N}$ light dual quarks
 $2F_{eff}\tilde{N} - (\tilde{N}^2 - 1) = \tilde{N}^2 + 1$ complex degrees of freedom
 M eqm removes \tilde{N}^2 color singlet degrees of freedom
dual quark equations of motion enforce that an $\tilde{N} \times \tilde{N}$ corner of M is
set to zero

two moduli spaces match:

$$2FN - (N^2 - 1) = F^2 - \tilde{N}^2 + \tilde{N}^2 + 1 - \tilde{N}^2 = F^2 - \tilde{N}^2 + 1$$

once nonperturbative effects are taken into account

$F = N$: confinement with χ SB

For $F = N$ 't Hooft anomaly matching works with just M , B , and \bar{B}
confining: all massless degrees of freedom are color singlet particles

For $F = N$ flavors the baryons are flavor singlets:

$$\begin{aligned} B &= \epsilon^{i_1, \dots, i_F} B_{i_1, \dots, i_F} \\ \bar{B} &= \epsilon_{i_1, \dots, i_F} \bar{B}^{i_1, \dots, i_F} \end{aligned}$$

classical constraint:

$$\det M = B \bar{B}$$

With quark masses:

$$\langle M_i^j \rangle = (m^{-1})_i^j (\det m \Lambda^{3N-F})^{1/N}$$

Confinement with χ SB

Taking a determinant of this equation (using $F = N$)

$$\det \langle M \rangle = \det (m^{-1}) \det m \Lambda^{2N} = \Lambda^{2N}$$

independent of the masses

$\det m \neq 0$ sets $\langle B \rangle = \langle \bar{B} \rangle = 0$, can integrate out all the fields that have baryon number

classical constraint is violated!

Holomorphy and the Symmetries

flavor invariants are:

	$U(1)_A$	$U(1)$	$U(1)_R$
$\det M$	$2N$	0	0
B	N	N	0
\bar{B}	N	$-N$	0
Λ^{2N}	$2N$	0	0

R -charge of the squarks, $(F - N)/F$, vanishes since $F = N$
 generalized form of the constraint with correct $\Lambda \rightarrow 0$ and $B, \bar{B} \rightarrow 0$
 limits is

$$\det M - \bar{B}B = \Lambda^{2N} \left(1 + \sum_{pq} C_{pq} \frac{(\Lambda^{2N})^p (\bar{B}B)^q}{(\det M)^{p+q}} \right)$$

with $p, q > 0$. For $\langle \bar{B}B \rangle \gg \Lambda^{2N}$ the theory is perturbative, but with $C_{pq} \neq 0$ we find solutions of the form

$$\det M \approx (\bar{B}B)^{(q-1)/(p+q)}$$

which do not reproduce the weak coupling $\Lambda \rightarrow 0$ limit

Quantum Constraint

$$\det M - \bar{B}B = \Lambda^{2N}$$

correct form to be an [instanton effect](#)

$$e^{-S_{\text{inst}}} \propto \Lambda^b = \Lambda^{2N}$$

Quantum Constraint

cannot take $M = B = \bar{B} = 0$



cannot go to the origin of moduli space (“deformed” moduli space)
global symmetries are at least partially broken everywhere

Enhanced Symmetry Points

$$M_i^j = \Lambda^2 \delta_i^j, B = \bar{B} = 0$$

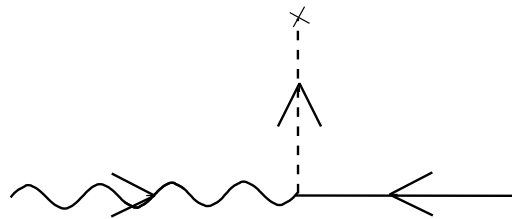
$SU(F) \times SU(F) \times U(1) \times U(1)_R \rightarrow SU(F)_d \times U(1) \times U(1)_R$
chiral symmetry breaking, as in non-supersymmetric QCD

$$M = 0, B\bar{B} = -\Lambda^{2N}$$

$SU(F) \times SU(F) \times U(1) \times U(1)_R \rightarrow SU(F) \times SU(F) \times U(1)_R$
baryon number spontaneously broken

Smooth Moduli Space

For large VEVs : perturbative Higgs phase, squark VEVs give masses to quarks and gauginos



no point in the moduli space where gluons become light
 \Rightarrow no singular points

theory exhibits “complementarity”: can go smoothly from a Higgs phase (large VEVs) to a confining phase (VEVs of $\mathcal{O}(\Lambda)$) without going through a phase transition

$F = N$: Consistency Checks

with F flavors and $\text{rank}(M) = N$, dual has confinement with χ SB

$$\det(\phi\bar{\phi}) - \bar{b}b = \tilde{\Lambda}_{eff}^{2\tilde{N}}$$

M eqm sets $\phi\bar{\phi} = 0$

matching dual gauge coupling:

$$\tilde{\Lambda}_{eff}^{2\tilde{N}} = \tilde{\Lambda}^{3\tilde{N}-F} \det' M$$

where $\det' M$ is the product of the N nonzero eigenvalues of M
combining gives

$$\bar{B}B \propto \det' M$$

classical constraint of the original theory is reproduced in the dual by a nonperturbative effect

$F = N$: consistency checks

$$\det M - \bar{B}B = \Lambda^{2N}$$

is eqm of

$$W_{\text{constraint}} = X (\det M - \bar{B}B - \Lambda^{2N})$$

with Lagrange multiplier field X
add mass for the N th flavor

$$M = \begin{pmatrix} \widetilde{M}_i^j & N^j \\ P_i & Y \end{pmatrix}$$

where \widetilde{M} is an $(N - 1) \times (N - 1)$ matrix

$F = N$: consistency checks

$$W = X (\det M - \bar{B}B - \Lambda^{2N}) + mY$$

$$\begin{aligned} \frac{\partial W}{\partial \bar{B}} &= -X\bar{B} = 0 & \frac{\partial W}{\partial N^j} &= X \operatorname{cof}(N^j) = 0 \\ \frac{\partial W}{\partial B} &= -XB = 0 & \frac{\partial W}{\partial P_i} &= X \operatorname{cof}(P_i) = 0 \\ \frac{\partial W}{\partial Y} &= X \det \widetilde{M} + m = 0 \end{aligned}$$

where $\operatorname{cof}(M_j^i)$ is the cofactor of the matrix element M_j^i
solution:

$$\begin{aligned} X &= -m \left(\det \widetilde{M} \right)^{-1} \\ B &= \bar{B} = N^j = P_i = 0 \end{aligned}$$

plugging solution into X eqm gives

$$\frac{\partial W}{\partial X} = Y \det \widetilde{M} - \Lambda^{2N} = 0$$

Effective Superpotential: $F \rightarrow N - 1$

$$W_{\text{eff}} = \frac{m \Lambda^{2N}}{\det \widetilde{M}}$$

matching relation for the holomorphic gauge coupling:

$$m \Lambda^{2N} = \Lambda_{N, N-1}^{2N+1}$$

so

$$W_{\text{eff}} = \frac{\Lambda_{N, N-1}^{2N+1}}{\det \widetilde{M}}$$

ADS superpotential for $SU(N)$ with $N - 1$ flavors

Enhanced Symmetry Point

$$M_i^j = \Lambda^2 \delta_i^j, B = \bar{B} = 0$$

Φ and $\bar{\Phi}$ VEVs break $SU(N) \times SU(F) \times SU(F) \rightarrow SU(F)_d$

quarks transform as $\square \times \bar{\square} = \mathbf{1} + \mathbf{Ad}$ under $SU(F)_d$

gluino transforms as \mathbf{Ad} under $SU(F)_d$

	$SU(F)_d$	$U(1)$	$U(1)_R$
$M - \text{Tr}M$	\mathbf{Ad}	0	0
$\text{Tr}M$	$\mathbf{1}$	0	0
B	$\mathbf{1}$	N	0
\bar{B}	$\mathbf{1}$	$-N$	0

$\text{Tr}M$ gets a mass with the Lagrange multiplier field X

Enhanced Symmetry Points: Anomalies

global symmetry	elem. anomaly	=	comp. anomaly
$U(1)^2 U(1)_R$	$-2FN$	=	$-2N^2$
$U(1)_R$	$-2FN + N^2 - 1$	=	$-(F^2 - 1) - 1 - 1$
$U(1)_R^3$	$-2FN + N^2 - 1$	=	$-(F^2 - 1) - 1 - 1$
$U(1)_R SU(F)_d^2$	$-2N + N$	=	$-N$

agree because $F = N$

Enhanced Symmetry Points

At $M = 0$, $B\bar{B} = -\Lambda^{2N}$ only the $U(1)$ symmetry is broken

	$SU(F)$	$SU(F)$	$U(1)_R$
M	\square	$\bar{\square}$	0
B	$\mathbf{1}$	$\mathbf{1}$	0
\bar{B}	$\mathbf{1}$	$\mathbf{1}$	0

linear combination $B + \bar{B}$ gets mass with Lagrange multiplier field X

global symmetry	elem. anomaly	=	comp. anomaly
$SU(F)^3$	N	=	F
$U(1)_R SU(F)^2$	$-N\frac{1}{2}$	=	$-F\frac{1}{2}$
$U(1)_R$	$-2FN + N^2 - 1$	=	$-F^2 - 1$
$U(1)_R^3$	$-2FN + N^2 - 1$	=	$-F^2 - 1$

agree because $F = N$

$F = N + 1$: s-confinement

For $F = N + 1$ 't Hooft anomaly matching works with M , B , and \bar{B} confining

does not require χ SB, can go to the origin of moduli space

theory develops a dynamical superpotential

	$SU(F)$	$SU(F)$	$U(1)$	$U(1)_R$
M	\square	$\bar{\square}$	0	$\frac{2}{F}$
B	$\bar{\square}$	$\mathbf{1}$	N	$\frac{N}{F}$
\bar{B}	$\mathbf{1}$	\square	$-N$	$\frac{N}{F}$

For $F = N + 1$ baryons are flavor antifundamentals since they are antisymmetrized in $N = F - 1$ colors:

$$\begin{aligned}
 B^i &= \epsilon^{i_1, \dots, i_N, i} B_{i_1, \dots, i_N} \\
 \bar{B}_i &= \epsilon_{i_1, \dots, i_N, i} \bar{B}^{i_1, \dots, i_N}
 \end{aligned}$$

$F = N + 1$: Classical Constraints

$$\begin{aligned}(M^{-1})_j^i \det M &= B^i \bar{B}_j \\ M_i^j B^i &= M_i^j \bar{B}_j = 0\end{aligned}$$

with quark masses:

$$\begin{aligned}\langle M_i^j \rangle &= (m^{-1})_i^j (\det m \Lambda^{2N-1})^{1/N} \\ \langle B^i \rangle &= \langle \bar{B}_j \rangle = 0\end{aligned}$$

taking determinant gives

$$(M^{-1})_j^i \det M = m_j^i \Lambda^{2N-1} .$$

Thus, we see that the classical constraint is satisfied as $m_j^i \rightarrow 0$
taking limit in different ways covers the classical moduli space
classical and quantum moduli spaces are the same
chiral symmetry remains unbroken at $M = B = \bar{B} = 0$

Most General Superpotential

$$W = \frac{1}{\Lambda^{2N-1}} \left[\alpha B^i M_i^j \bar{B}_j + \beta \det M + \det M f \left(\frac{\det M}{B^i M_i^j \bar{B}_j} \right) \right]$$

where f is an as yet unknown function

only $f = 0$ reproduces the classical constraints:

$$\begin{aligned} \frac{\partial W}{\partial M_i^j} &= \frac{1}{\Lambda^{2N-1}} \left[\alpha B^i \bar{B}_j + \beta (M^{-1})_j^i \det M \right] = 0 \\ \frac{\partial W}{\partial B^i} &= \frac{1}{\Lambda^{2N-1}} \alpha M_i^j \bar{B}_j = 0 \\ \frac{\partial W}{\partial \bar{B}_j} &= \frac{1}{\Lambda^{2N-1}} \alpha B^i M_i^j = 0 \end{aligned}$$

provided that $\beta = -\alpha$

$F = N + 1$ Superpotential

to determine α , add a mass for one flavor

$$W = \frac{\alpha}{\Lambda^{2N-1}} \left[B^i M_i^j \bar{B}_j - \det M \right] + mX$$

$$M = \begin{pmatrix} M_j^{i'} & Z^i \\ Y_j & X \end{pmatrix}, \quad B = (U^i, B'), \quad \bar{B} = \begin{pmatrix} \bar{U}_j \\ \bar{B}' \end{pmatrix}$$

$$\frac{\partial W}{\partial Y} = \frac{\alpha}{\Lambda^{2N-1}} (B' \bar{U} - \text{cof}(Y)) = 0$$

$$\frac{\partial W}{\partial Z} = \frac{\alpha}{\Lambda^{2N-1}} (U \bar{B}' - \text{cof}(Z)) = 0$$

$$\frac{\partial W}{\partial U} = \frac{\alpha}{\Lambda^{2N-1}} Z \bar{B}' = 0$$

$$\frac{\partial W}{\partial \bar{U}} = \frac{\alpha}{\Lambda^{2N-1}} B' \bar{Y} = 0$$

$$\frac{\partial W}{\partial X} = \frac{\alpha}{\Lambda^{2N-1}} (B' \bar{B}' - \det M') + m = 0$$

$F = N + 1$ Superpotential

solution of eqms:

$$\begin{aligned} Y &= Z = U = \bar{U} = 0 \\ \det M' - B' \bar{B}' &= \frac{m \Lambda^{2N-1}}{\alpha} = \frac{1}{\alpha} \Lambda_{N,N}^{2N} \end{aligned}$$

correct quantum constraint for $F = N$ flavors if and only if $\alpha = 1$

Plugging back in superpotential with $m \Lambda^{2N-1} = \Lambda_{N,N}^{2N}$:

$$W_{\text{eff}} = \frac{X}{\Lambda^{2N-1}} \left(B' \bar{B}' - \det M' + \Lambda_{N,N}^{2N} \right)$$

Holding $\Lambda_{N,N}$ fixed as $m \rightarrow \infty \Rightarrow \Lambda \rightarrow 0$
 X becomes Lagrange multiplier
reproduce the superpotential for $F = N$

$F = N + 1$ Superpotential

superpotential for confined SUSY QCD with $F = N + 1$ flavors is:

$$W = \frac{1}{\Lambda^{2N-1}} \left[B^i M_i^j \bar{B}_j - \det M \right]$$

$M = B = \bar{B} = 0$ is on the quantum moduli space, possible singular behavior since naively gluons and gluinos should become massless
actually M, B, \bar{B} become massless: confinement without χ SB

$F = N + 1$ Anomalies

global symmetry	elem. anomaly	=	comp. anomaly
$SU(F)^3$	N	=	$F - 1$
$U(1)SU(F)^2$	$N\frac{1}{2}$	=	$N\frac{1}{2}$
$U(1)_R SU(F)^2$	$-\frac{N}{F}\frac{N}{2}$	=	$\frac{2-F}{F}\frac{F}{2} + \frac{N-F}{2F}$
$U(1)_R$	$-\frac{N}{F}2NF + N^2 - 1$	=	$\frac{2-F}{F}F^2 + 2(N - F)$
$U(1)_R^3$	$-\left(\frac{N}{F}\right)^3 2NF + N^2 - 1$	=	$\left(\frac{2-F}{F}\right)^3 F^2 + \left(\frac{N-F}{F}\right)^3 2F$,

agree because $F = N + 1$

Connection to $F > N + 1$

dual theory for $F = N + 2$:

	$SU(2)$	$SU(N + 2)$	$SU(N + 2)$	$U(1)$	$U(1)_R$
q	\square	$\bar{\square}$	$\mathbf{1}$	$\frac{N}{2}$	$\frac{N}{N+2}$
\bar{q}	$\bar{\square}$	$\mathbf{1}$	\square	$-\frac{N}{2}$	$\frac{N}{N+2}$
M	$\mathbf{1}$	\square	$\bar{\square}$	0	$\frac{4}{N+2}$

$$W = \frac{1}{\mu} M \bar{\phi} \phi$$

mass for one flavor produces dual squark VEV

$$\langle \bar{\phi}^F \phi_F \rangle = -\mu m$$

completely breaks the $SU(2)$

$$F = N + 2 \longrightarrow F = N + 1$$

massless spectrum of the low-energy effective theory:

	$SU(N + 1)$	$SU(N + 1)$	$U(1)$	$U(1)_R$
q'	$\bar{\square}$	$\mathbf{1}$	N	$\frac{N}{N+1}$
\bar{q}'	$\mathbf{1}$	\square	$-N$	$\frac{N}{N+1}$
M'	\square	$\bar{\square}$	0	$\frac{2}{N+1}$

Comparing with the confined spectrum we identify

$$q'^i = cB^i, \quad \bar{q}'_j = \bar{c}\bar{B}_j$$

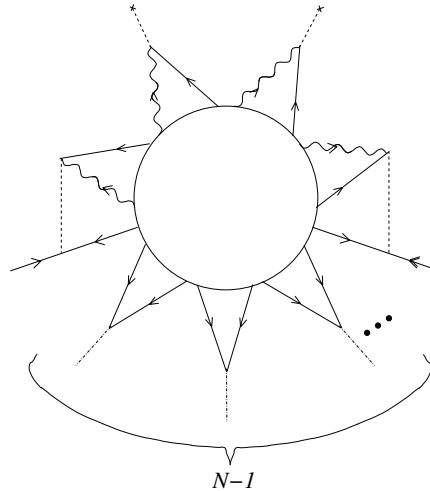
where c and \bar{c} are rescalings

$$W_{\text{tree}} = \frac{c\bar{c}}{\mu} B^i M_i'^j \bar{B}_j$$

$$F = N + 2 \rightarrow F = N + 1$$

broken $SU(2) \Rightarrow$ instantons generate superpotential

$$W_{\text{inst.}} = \frac{\tilde{\Lambda}_{N,N+2}^b}{\langle \bar{\phi}^F \phi_F \rangle} \det \left(\frac{M'}{\mu} \right) = - \frac{\tilde{\Lambda}_{N,N+2}^{4-N}}{m} \frac{\det M'}{\mu^{N+2}}$$



two mesinos (external straight lines) and $N - 1$ mesons (dash-dot lines). instanton has 4 gaugino legs (internal wavy lines) and $N + 2$ quark and antiquark legs (internal straight lines)

$$F = N + 2 \longrightarrow F = N + 1$$

effective superpotential agrees with the result for $F = N + 1$:

$$W_{\text{eff}} = \frac{1}{\Lambda^{2N-1}} \left[B^i M'_i{}^j \bar{B}_j - \det M' \right]$$

if and only if

$$c\bar{c} = \frac{\mu}{\Lambda^{2N-1}} \ , \ \frac{\tilde{\Lambda}_{N,N+2}^{4-N}}{\mu^{N+2} m} = \frac{1}{\Lambda^{2N-1}}$$

second relation follows from

$$\tilde{\Lambda}^{3\tilde{N}-F} \Lambda^{3N-F} = (-1)^{F-N} \mu^F$$

Intrinsic Scales

$$\tilde{\Lambda}^{3\tilde{N}-F} \Lambda^{3N-F} = (-1)^{F-N} \mu^F \quad (*)$$

consider generic values of $\langle M \rangle$ in dual, dual quarks are massive pure $SU(\tilde{N} = F - N)$ gauge theory.

$$\tilde{\Lambda}_L^{3\tilde{N}} = \tilde{\Lambda}^{3\tilde{N}-F} \det \left(\frac{M}{\mu} \right)$$

gaugino condensation:

$$\begin{aligned} W_L &= \tilde{N} \tilde{\Lambda}_L^3 = (F - N) \left(\frac{\tilde{\Lambda}^{3\tilde{N}-F} \det M}{\mu^F} \right)^{1/(F-N)} \\ &= (N - F) \left(\frac{\Lambda^{3N-F}}{\det M} \right)^{1/(N-F)} \end{aligned}$$

where we have used eqn (*) Adding mass term $m_j^i M_i^j$ gives:

$$M_i^j = (m^{-1})_i^j (\det m \Lambda^{3N-F})^{1/N}$$

which is the correct result

Dual of Dual

assume that $\tilde{\tilde{\Lambda}} = \Lambda$, (*) implies

$$\Lambda^{3N-F} \tilde{\tilde{\Lambda}}^{3\tilde{N}-F} = (-1)^{F-\tilde{N}} \tilde{\mu}^F$$

since $F - \tilde{N} = N$, we must have for consistency

$$\tilde{\mu} = -\mu$$

composite meson of the dual quarks:

$$N_j^i \equiv \bar{\phi}^i \phi_j$$

dual–dual squarks as d , dual–dual superpotential is

$$W_{dd} = \frac{N_j^j}{\tilde{\mu}} \bar{d}^i d_j + \frac{M_j^i}{\mu} N_i^j$$

Dual of Dual

equations of motion give

$$\begin{aligned}\frac{\partial W}{\partial M_j^i} &= \frac{1}{\mu} N_i^j = 0 \\ \frac{\partial W}{\partial N_i^j} &= \frac{1}{\tilde{\mu}} \bar{d}^i d_j + \frac{1}{\mu} M_j^i = 0\end{aligned}$$

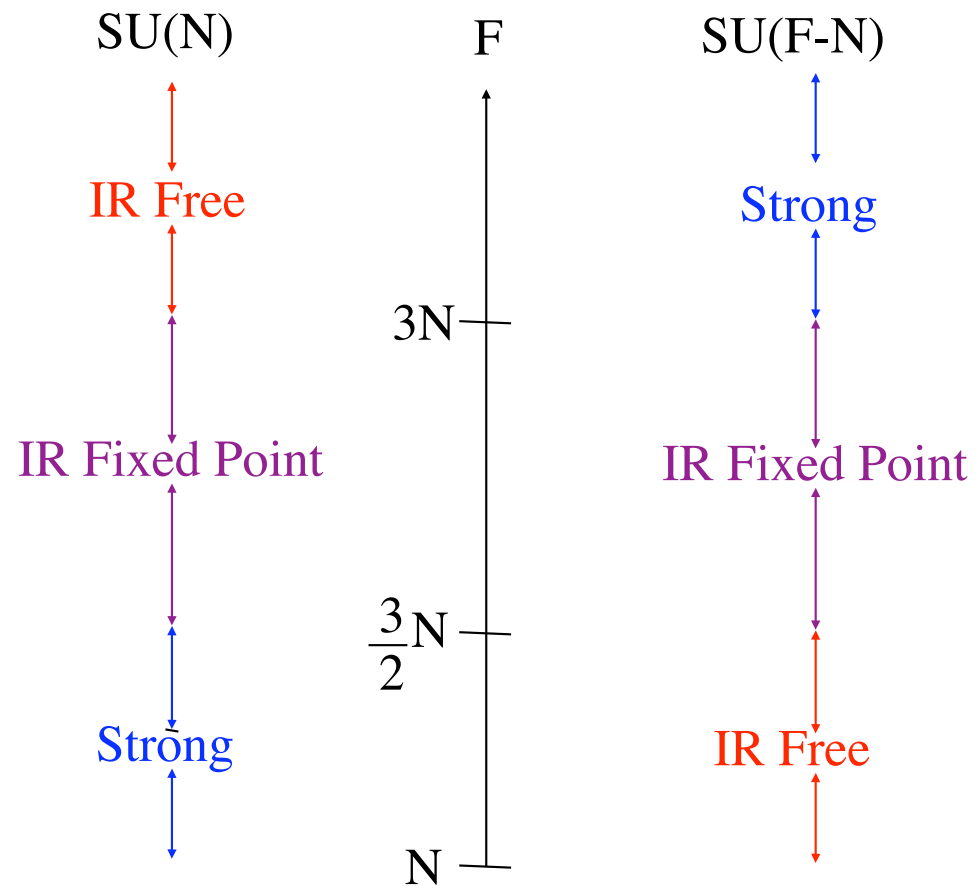
So, since $\tilde{\mu} = -\mu$, we can identify the original squarks with the dual–dual squarks:

$$\Phi_j = d_j .$$

Plugging into the dual–dual superpotential (it vanishes

dual of the dual of SUSY QCD is just SUSY QCD

Duality for SUSY SU(N)

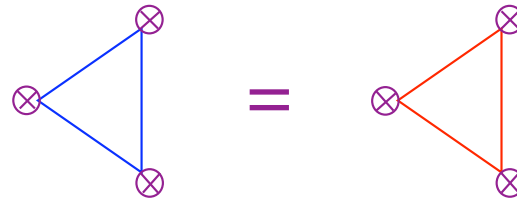


$F=N+1 \rightarrow$ confinement without χ SB

$F=N \rightarrow$ confinement with χ SB

Duality Consistency Checks

- Anomaly Matching**



$$Q, \bar{Q}: SU(N) \quad q, \bar{q}, M: SU(F-N)$$

- Identical Space of Vacua**

$$Q\bar{Q} \quad \longleftrightarrow \quad M$$

$$Q^N, \bar{Q}^N \quad \longleftrightarrow \quad q^{F-N}, \bar{q}^{F-N}$$

- Deformations**

$$SU(N), F \quad \longleftrightarrow \quad SU(F-N), F$$

$$W=m Q_F \bar{Q}_F \quad W=Mq\bar{q} + mM_{FF}$$



$$\langle q \rangle \neq 0, \langle \bar{q} \rangle \neq 0$$

$$SU(N), F-1 \quad \longleftrightarrow \quad SU(F-1-N), F-1$$