Exercises for Chapter 1

1. Construct the massive supermultiplet of $\mathcal{N} = 3$ SUSY for the lowest weight state (Clifford “vacuum” $|\Omega_0\rangle$) having spin 0. (Use the notation where the raising operator on this state produces a state ($\Box$, 2) where the $\Box$ indicates a 3 of the $SU(3)$ $R$-symmetry and 2 denotes a spin half doublet. You can use the following $SU(3)$ group theory results (keep in mind that $\Box = \Box \leftrightarrow 3$)

\[
\Box \times \Box = \Box + \Box \leftrightarrow 3 \times 3 = 3_A + 6_S , \tag{1}
\]

\[
\Box \times \Box = 1 + \Box \leftrightarrow 3 \times 3 = 1 + 8 . \tag{2}
\]

Check that there are an equal number of bosonic and fermionic states in the supermultiplet. Is this state equivalent to a massive supermultiplet of $\mathcal{N} = 4$?

2. Consider $\mathcal{N} = 4$ SUSY with a $4 \times 4$ central charge matrix $Z$. In a skew diagonal basis we can write

\[
Z = \begin{pmatrix}
Z_1 \epsilon^{ab} & 0 \\
0 & Z_2 \epsilon^{ab}
\end{pmatrix} \tag{3}
\]

where $a = 1, 2$ and $b = 1, 2$. In this basis the SUSY algebra can be written as

\[
\{ Q_{aL}^\alpha, Q_{\beta N}^{\dagger} \} = 2 \sigma_\alpha^\mu P_\mu \delta_\beta^a \delta_L^N , \tag{4}
\]

\[
\{ Q_{aL}^\alpha, Q_{\beta N}^{\dagger} \} = 2 \sqrt{2} \epsilon_{\alpha \beta} \epsilon^{ab} \delta_L^N Z_N , \tag{5}
\]

\[
\{ Q_{aL}^{\dagger}, Q_{\beta N}^{\dagger} \} = 2 \sqrt{2} \epsilon_{\alpha \beta} \epsilon^{ab} \delta_L^N Z_N , \tag{6}
\]

where $L = 1, 2$; $N = 1, 2$; and the repeated index $N$ is not summed over. Defining

\[
A_L^a = \frac{1}{2} \left[ Q_{aL}^{1\dagger} + \epsilon_{\alpha \beta} \left( Q_{\beta L}^{2\dagger} \right) \right] , \tag{7}
\]

\[
B_L^a = \frac{1}{2} \left[ Q_{aL}^{1\dagger} - \epsilon_{\alpha \beta} \left( Q_{\beta L}^{2\dagger} \right) \right] , \tag{8}
\]

reduces the algebra in the rest frame to

\[
\{ A_L^a, A_{\beta N}^{\dagger} \} = \delta_{\alpha \beta} \delta_L^N (M + \sqrt{2} Z_L) , \tag{9}
\]

\[
\{ B_L^a, B_{\beta N}^{\dagger} \} = \delta_{\alpha \beta} \delta_L^N (M - \sqrt{2} Z_L) . \tag{10}
\]

Consider a massive state with $M = \sqrt{2} Z_1 = \sqrt{2} Z_2$ and construct the short multiplet starting with the spin 0 Clifford “vacuum” $|\Omega_0\rangle$. How many raising operators are there? Label the elements of the multiplet with dimension of the $R$-symmetry representation, $d_R$, and the spin degneracy $2j + 1$. 

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