

**Cosmological Implications of
Weakly Interacting Massive Particles**

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1. Introduction

In recent years, particle physicists have become increasingly interested in the use of cosmological calculations as tests for their hypotheses about elementary particles. Not only did particles in the early Universe have energies that far exceed the limits of present day accelerators, but also, densities at these times were so large that even weakly interacting particles like neutrinos possessed mean free paths shorter than 3×10^8 m. Under these conditions, particles that interact with normal matter only weakly or gravitationally can produce significant effects, whose repercussions could still be measurable at the present time.

Constraints from Cosmology can usually be obtained by asking the question: could a Universe containing a certain type of particle evolve into the Universe that we presently observe? In this paper we will find mass--lifetime constraints on particles whose strongest interaction is the weak interaction, and mass--coupling constant constraints on particles that interact with normal matter only gravitationally, by requiring that their present energy density not exceed the critical energy density, and that density perturbations in the early Universe be allowed to grow into galaxies.

2. Stable Weakly Interacting Massive Particles

At the present time, observable galaxies appear to be engaged in a rapid expansion; so it would seem that in the past the matter of the Universe was more densely packed than it is now. For the galactic matter to have escaped the gravitational potential well of this denser era, it must have been much more energetic. If we continue to trace this behaviour back further and further in time, we come to more densely packed epochs, with even more energetic particles; and eventually we come to a singularity¹ where all particles are ultra-relativistic. This is the essence of Big Bang Cosmology.

The point that will prove to be central to our discussion of stable² weakly interacting massive particles (WIMPs³) is that these particles were once moving ultra-relativistically, and were so densely packed that reactions occurred very quickly, and hence, at very early times all particles comprised a relativistic gas in thermal equilibrium. This runs counter to the usual cases in thermodynamics where thermal equilibriums are established after some period of time; but in the case of Cosmology, the Universe rapidly goes into a thermal equilibrium which is eventually destroyed. This being the case, we can treat different particles in the early Universe in a manner somewhat analogous to, say, the different modes of vibration and rotation of atoms, or to different radiation modes inside a reflecting cavity.

To begin a discussion of particles in the early Universe, it is helpful to recall some results from Cosmology which are derived using the Robertson-Walker metric⁴. Most importantly, distances between fundamental points (i.e. points following the expansion of space-time) grow in time

proportionally to a scale factor $R(t)$. Since, by the de Broglie relation, momenta are inversely proportional to wavelengths, they decrease with time. More concretely, if a particle has a momentum p_1 at time t_1 , at any subsequent time t , the particles momentum is

$$p(t) = p_1 R(t_1)/R(t) . \quad (2.1)$$

For ultra-relativistic particles, this decrease in energy is often referred to as "red-shifting away." From this relation it is obvious that the energy of a relativistic gas, and hence its temperature, will decrease as the Universe expands. More precisely, since kT , where k is Boltzmann's constant and T is the temperature, is a measure of the average energy of a particle in thermal equilibrium, the temperature of an ultra-relativistic gas is given by

$$T(t) = \text{constant}/R(t) . \quad (2.2)$$

We are now in a position to ask what happens to neutral massive spin $1/2$ particles as the universe expands, and the relativistic gas it contains cools. When the temperature⁵ is much larger than the mass, m_X of a particle X , the two competing processes of creation and annihilation are held in balance: X particles annihilate with anti- X particles, but other particles in the gas have sufficient energy to create more X 's when they annihilate or decay⁶. For any reaction that destroys X 's, there is a reversed reaction that creates X 's, and these two types of reactions occur, on average, equally often. During this period the number of X particles per comoving volume³⁵ is constant, however, as the temperature falls below m_X , particles which have sufficient energy to produce X 's become increasingly rare, in accordance with the Boltzmann factor $\exp(-m_X/kT)$. Thus, although X 's continue to annihilate, their rate of production decreases rapidly with decreasing temperature, and so the number of X particles per comoving volume declines. The annihilation of X particles

does not continue unabated though, since the annihilation rate is proportional to the X number density, n , times the anti-X number density, which is assumed⁷ to be equal to n , so as n decreases, the annihilation rate decreases like n^2 . For this reason, the decrease in n due to annihilation eventually becomes insignificant in comparison to the decrease in n due to the general expansion of the Universe. Qualitatively, it becomes harder and harder for the X's to find anti-X's to annihilate with, and so they eventually behave as if no annihilation is allowed. Mathematically the rate of change of n was expressed by Lee and Weinberg [3] as

$$d n/dt = - 3 (\dot{R}(t)/R(t)) n(t) - \langle \sigma v \rangle n^2(t) + \langle \sigma v \rangle n_{eq}^2(t), \quad (2.3)$$

where $\langle \sigma v \rangle$ is the thermal average of the X anti-X annihilation cross section times the relative velocity, and n_{eq} is the number density of X particles in thermal equilibrium; that is

$$n_{eq}(T) = 2 / (2\pi)^3 \int_0^\infty 4\pi p^2 dp (\exp((p^2 + m_x^2)^{1/2} / kT) + 1)^{-1} \quad (2.4)$$

where the factor 2 comes from assuming X has two spin states (and $h = c = 1$, as throughout).

It can be shown that in a Universe with a flat⁸ Robertson-Walker metric the Hubble parameter is

$$H = \dot{R}/R = (8 \pi \rho G/3)^{1/2}, \quad (2.5)$$

where G is the gravitational constant, and ρ is the energy density of the relativistic gas⁹:

$$\rho = N_f a T^4 = N_f (\pi^2/15) (kT)^4 \quad (2.6)$$

N_f here is the effective number of degrees of freedom,

$$N_f = 1/2 (n_b + 7/8 n_f) \quad (2.7)$$

where n_b and n_f are the total number of internal degrees of freedom¹⁰ for all bosons and fermions present in equilibrium in the gas.

When the temperature is below m_X , the velocities of the X particles are non-relativistic; for Dirac particles this means that the annihilation cross section in the center of mass frame is proportional to $1/v$, therefore $\langle\sigma v\rangle$ is velocity, and hence temperature independent¹¹. If X interacts only weakly, and $m_X \ll M_Z$, the Z boson mass, we can write $\langle\sigma v\rangle$ as

$$\langle\sigma v\rangle = (G_F^2/2\pi) m_X^2 N_A \quad , \quad (2.8)$$

where N_A is a dimensionless factor which takes into account the various channels the annihilation can proceed into¹². Unfortunately eq. (2.8) is not valid for large m_X ; this can be rectified by noting the correspondence evident in the GSW electroweak theory¹³:

$$G_F/\sqrt{2} \rightarrow g^2/(8((4m_X^2 - M_Z^2)^2 + M_Z^2\Gamma_Z^2) \cos^2 \theta_W) \quad , \quad (2.9)$$

$$\text{and} \quad e = g \sin \theta_W \quad . \quad (2.10)$$

So,

$$\langle\sigma v\rangle = e^4 m_X^2 N_A / (64\pi((4m_X^2 - M_Z^2)^2 + M_Z^2\Gamma_Z^2) \cos^4 \theta_W \sin^4 \theta_W) \quad , \quad (2.11)$$

where θ_W is the Weinberg angle, and Γ_Z is the resonance width of the Z boson. We are making the approximation here that all the X particles have the same energy, m_X ; if we took into account the distribution of energies, the peak in the cross section at $m_X = M_Z$ would be lowered and spread out.

We are now ready to attempt a calculation of the number density of X particles that survive to the present time. Eq. (2.3) can be simplified by making the substitution

$$n = f T^3 \quad , \quad n_{eq} = f_{eq} T^3 \quad . \quad (2.12)$$

Using eq. (2.2), this removes the explicit cosmic expansion dependence from the equation, giving

$$df/dt = \langle\sigma v\rangle (45/8\pi^3 N_f k^4 G)^{1/2} (f^2 - f_{eq}^2) \quad . \quad (2.13)$$

Rewriting the temperature as

$$x = kT/m_X \quad (2.14)$$

yields
$$df/dx = b(f^2 - f_{eq}^2) \quad (2.15)$$

where
$$b = \langle \sigma v \rangle (m_X/k^3) (45/8\pi^3 N_f G)^{1/2} . \quad (2.16)$$

The boundary condition for eq. (2.15) is that as $x \rightarrow \infty$, $f(x)$ approaches $f_{eq}(x)$, which, from eq. (2.4), is given by

$$f_{eq}(x) = k^3/(2\pi^2) \int_0^\infty du u^2 (\exp(u^2 + x^{-2})^{1/2} + 1)^{-1} . \quad (2.17)$$

It is expected the number of particles per comoving³⁵ volume, which is proportional to f , remains approximately equal to the number of particles per comoving volume in equilibrium, f_{eq} , until the chemical equilibrium is destroyed at the freezing temperature. The number density of X particles in equilibrium is determined by the temperature, and decreases rapidly as T falls below m_X , but the total number density of X particles can only be reduced by expansion and annihilation(which decreases rapidly with n). The freeze-out, occurs when the rate of change of n due to the cosmic expansion, $-3H n$, becomes much larger than the rate of change of n due to annihilation, $\langle \sigma v \rangle n^2$. This condition is roughly equivalent to requiring that the mean free time of the X particles becomes greater than the characteristic expansion time. In terms of our new variables, Lee and Weinberg [3] defined the freezing temperature, T_f , by

$$df_{eq}/dx = b f_{eq}^2 , \text{ at } x_f = kT_f/m_X . \quad (2.18)$$

Below the freezing temperature, f becomes much larger than f_{eq} , (see fig. 1) so eq. (2.15) can be approximated by

$$df/dx = b f^2 , \quad x < x_f \quad . \quad (2.19)$$