Quantized Klein-Gordon Field in a Cavity of Variable Length (*).

M. RAZAVY and J. TERNING

Theoretical Physics Institute, Department of Physics, University of Alberta
Alberta, Edmonton, T6G 2J1, *Canada*

(ricevuto il 21 Settembre 1984)

PACS. 03.70. – The theory of quantized fields.

Summary. – An effective Hamiltonian is found for the Klein-Gordon field enclosed in a one-dimensional cavity with a moving wall. From this Hamiltonian the number of particles created from the vacuum by the motion of the boundary is determined.

The normal-mode decomposition of a quantized field inside a cavity with a moving wall provides an interesting example of an effective Hamiltonian formulation of an open system. The case of a massless field has received considerable attention in recent years ([1-6]) partly in connection with the problem of creation of particles by black boles ([2,3]) and partly because of its potential application in the laser physics ([1]). For a massless field the classical equation of motion can be solved in a number of ways including the method of conformal co-ordinate transformation ([1]) and the effective Hamiltonian approach ([6]). One of the advantages of the latter is that it can be used for massless or massive fields and for relativistic as well as nonrelativistic particles. Here we consider the effective Hamiltonian formulation for a Klein-Gordon field which is confined to a one-dimensional cavity of variable length $L(t)$, where the boundaries are perfectly reflecting. This system can be described by the Hamiltonian

$$H = \frac{1}{2}\int_0^{L(t)} [\pi^2(x, t) + \psi_x^2(x, t) + m^2\psi^2(x, t)]\,\mathrm{d}x\,, \tag{1}$$

where ψ_x denotes the partial derivative of ψ with respect to x. The field amplitude ψ

(*) Supported in part by the Natural Sciences and Engineering Research Council of Canada.
(1) G. T. MOORE: *J. Math. Phys.*, **11**, 2679 (1970).
(2) B. S. DE WITT: *Phys. Rep. C*, **19**, 295 (1975).
(3) S. A. FULLING and P. C. W. DAVIES: *Proc. R. Soc. London, Ser. A*, **348**, 393 (1976).
(4) L. H. FORD and A. VILENKIN: *Phys. Rev. D*, **25**, 2569 (1982).
(5) M. CASTAGNINO and R. FERRARO: *Ann. Phys. (N. Y.)*, **154**, 1 (1984).
(6) M. RAZAVY and J. TERNING: submitted for publication.

and its momentum density π satisfy the boundary conditions

$$\psi(x = 0, t) = \psi(x = L(t), t) = 0\,, \tag{2}$$

$$\pi(x = 0, t) = \pi(x = L(t), t) = 0\,, \tag{3}$$

and the equal-time commutation relations

$$[\psi(x, t), \pi(x', t)] = i\delta(x - x')\,. \tag{4}$$

Both $\psi(x, t)$ and $\pi(x, t)$ satisfy the Klein-Gordon equation, and because of the conditions (2), (3) and (4), there is a reciprocity symmetry in this system, *viz.*, the interchange $\psi \to \pi$, $\pi \to -\psi$ leaves the equations of motion and the boundary conditions invariant. Let us define the effective Hamiltonian for this system with the help of the unitarity time-dependent transformation (7)

$$H_{\text{eff}} = \exp[iW]\left[\exp[iV]\left(H - i\frac{\partial}{\partial t}\right)\exp[-iV]\right]\exp[-iW]\,, \tag{5}$$

where

$$V(t) = \log\lambda\int_0^{L(t)} x'\,\frac{\partial\pi}{\partial x'}\,\psi(x')\,\mathrm{d}x' \tag{6}$$

and

$$W(t) = \frac{1}{2}\log\lambda\int_0^{L(t)}\pi\left(\frac{x'}{\lambda}\right)\psi\left(\frac{x'}{\lambda}\right)\frac{\mathrm{d}x'}{\lambda} \tag{7}$$

and where λ is the time-dependent scale factor

$$\lambda(t) = L(t)/L(0)\,. \tag{8}$$

By introducing the variable $\xi = x/\lambda$, we can write

$$\exp[iW]\exp[iV]\psi(x)\exp[-iV]\exp[-iW] = \lambda^{\frac{1}{2}}\psi(\xi) \tag{9}$$

and

$$\exp[iW]\exp[iV]\pi(t)\exp[-iV]\exp[-iW] = \frac{1}{\lambda^{\frac{1}{2}}}\pi(\xi)\,. \tag{10}$$

Hence the commutation relation (4) remains unchanged after transformation, but the boundary conditions (2) and (3) in terms of ξ become simple

$$\psi(\xi = 0, t) = \psi(\xi = L(0), t) = 0\,. \tag{11}$$

---

(7) M. RAZAVY: *Lett. Nuovo Cimento*, **37**, 449 (1983).

By carrying out the transformation indicated in (5) we find H_{eff} to be

$$H_{\text{eff}} = \int_0^{L(0)} \left\{ \frac{1}{2}\left[\pi^2 + \frac{1}{\lambda^2}\psi_\xi^2 + m^2\psi^2\right] - \frac{\dot{\lambda}}{\lambda}\left[\xi\pi_\xi\psi + \frac{1}{2}\pi\psi\right]\right\} \mathrm{d}\xi\,, \tag{12}$$

where ψ and π are now dependent on ξ and t and $\dot{\lambda}$ denotes $\mathrm{d}\lambda/\mathrm{d}t$. From (12) we can derive the equations of motion for $\pi(\xi, t)$ and $\psi(\xi, t)$ and these are identical in form, and for the latter the equation of motion is

$$\frac{1}{\lambda^2}(1 - \dot{\lambda}^2\xi^2)\psi_{\xi\xi} - \psi_{tt} + \frac{2\dot{\lambda}}{\lambda}\xi\psi_{t\xi} + \frac{\dot{\lambda}}{\lambda}\psi_t + \frac{1}{\lambda^2}(\ddot{\lambda}\lambda - 3\dot{\lambda}^2)\xi\psi_\xi + \left(\frac{\ddot{\lambda}}{2\lambda} - \frac{3}{4}\frac{\dot{\lambda}^2}{\lambda^2} - m^2\right)\psi = 0\,. \tag{13}$$

Thus in this transformation the reciprocal symmetry of the field is preserved. To write H_{eff} in terms of the creation and annihilation operators we expand $\psi(\xi, t)$ and $\pi(\xi, t)$ in terms of $\sin\big((k\pi/L(0))\xi\big)$, where k is an integer. We can simplify the result by choosing $L(0) = \pi$ and writing

$$\psi(\xi, t) = \left(\frac{\lambda}{\pi}\right)^{\frac{1}{2}} \sum_{k=1}^{\infty} \frac{1}{\omega_k^{\frac{1}{2}}(t)}(a_k^\dagger + a_k)\sin(k\xi) \tag{14}$$

and

$$\pi(\xi, t) = \frac{i}{(\pi\lambda)^{\frac{1}{2}}} \sum_{k=1}^{\infty} \omega_k^{\frac{1}{2}}(t)(a_k^\dagger - a_k)\sin(k\xi)\,, \tag{15}$$

where

$$\omega_k(t) = \big(k^2 + m^2\lambda^2(t)\big)^{\frac{1}{2}}\,. \tag{16}$$

By substituting (14) and (15) in (12) and carrying out the integration over ξ, we obtain

$$H_{\text{eff}} = \frac{1}{\lambda(t)}\sum_k \omega_k\left(a_k^\dagger a_k + \frac{1}{2}\right) + \frac{i\dot{\lambda}}{\lambda}\sum_{k=1}\sum_{j\neq k}(-1)^{k+j}\cdot \frac{jk}{j^2 - k^2}\left(\frac{\omega_k}{\omega_j}\right)^{\frac{1}{2}}(a_k^\dagger a_j^\dagger - a_k^\dagger a_j + a_k a_j^\dagger - a_k a_j)\,. \tag{17}$$

The equations of motion for a_k and $a_k^\dagger$ can be found from (17), for example for $\mathrm{d}a_k/\mathrm{d}t$ we have the following relation:

$$i\frac{\mathrm{d}a_k}{\mathrm{d}t} = \frac{\omega_k(t)}{\lambda(t)}a_k(t) + iR_k(t)\,, \tag{18}$$

where $R_k(t)$ is given by

$$R_k(t) = \frac{\dot{\lambda}}{\lambda}\sum_{j\neq k}(-1)^{k+j}\frac{kj}{j^2 - k^2}\left[\left(\frac{\omega_j}{\omega_k}\right)^{\frac{1}{2}}(a_j - a_j^\dagger) + \left(\frac{\omega_k}{\omega_j}\right)^{\frac{1}{2}}(a_j^\dagger + a_j)\right)\Big] \tag{19}$$

with an equation similar to (18) for $(\mathrm{d}/\mathrm{d}t)a_k^\dagger$. The number operator for particles in the state k is defined by

$$N_k = a_k^\dagger a_k \tag{20}$$

and from this definition and the equations for $\mathrm{d}a_k/\mathrm{d}t$ and $\mathrm{d}a_k^\dagger/\mathrm{d}t$, we obtain the rate of change of N_k;

$$\frac{\mathrm{d}N_k}{\mathrm{d}t} = a_k^\dagger R_k + R_k^\dagger a_k\,. \tag{21}$$

Now suppose that at $t = 0$, there are no particles in the system, *i.e.*

$$a_k^\dagger(0)a_k(0)|0\rangle = 0 \tag{22}$$

then the number of particles in the state k that are created between $t = 0$ and $t = \infty$ is given by

$$\langle N_k\rangle = \int_0^\infty \langle 0|a_k^\dagger R_k + R_k^\dagger a_k|0\rangle\,\mathrm{d}t\,. \tag{23}$$

An approximate value for $\langle N_k\rangle$ can be found by solving (18) by perturbation, *i.e.* by assuming that the expectation value of $\omega_k(t)a_k(t)/\lambda(t)$ is larger than the expectation value of $iR_k(t)$. In this way we can relate $a_k^{(0)}(t)$, the zeroth-order and $a_k^{(1)}(t)$, the first-order term to the in-field operators $a_k(0)$ and $a_k^\dagger(0)$ in the following way:

$$a_k^{(0)}(t) = \exp[-i\Phi_k(t)]a_k(0) \tag{24}$$

and

$$a_k^{(0)}(t) = \exp[-i\Phi_k(t)]\left[a_k(0) + \int_0^t \exp[i\Phi_k(t')]R_k^{(0)}(t')\,\mathrm{d}t'\right], \tag{25}$$

where $R_k^{(0)}(t)$ is $R_k(t)$ defined by (19), but with a_j and $a_j^\dagger$ being replaced by $a_j^{(0)}(t)$ and $a_j^{\dagger(0)}(t)$, respectively, and where

$$\Phi_k(t) = \int_0^t [\omega_k(t)/\lambda(t)]\,\mathrm{d}t\,. \tag{26}$$

Again from the differential equation satisfied by $a_k^{(1)}$, *i.e.*

$$i\frac{\mathrm{d}}{\mathrm{d}t}a_k^{(1)} = \frac{\omega_k(t)}{\lambda(t)}a_k^{(1)} + iR_k^{(0)}(t) \tag{27}$$

and (21) it follows that

$$\langle N_k\rangle \approx \int_0^\infty \langle 0|a_k^{\dagger(1)}R_k^{(0)} + R_k^{\dagger(0)}a_k^{(1)}|0\rangle\,\mathrm{d}t = 2\,\mathrm{Real}\int_0^\infty \mathrm{d}t\int_0^t \langle 0|R_k^{\dagger(0)}(t')R_k^{(0)}(t)|0\rangle \exp\left[i\big(\Phi_k(t') - \Phi_k(t)\big)\right]\mathrm{d}t'\,. \tag{28}$$

substituting for $R_k^{(0)}(t)$ and $R_k^{\dagger(0)}(t')$ and calculating the vacuum expectation value in (28), we find

$$\langle N_k\rangle = 2\sum_{j\neq k}\left(\frac{kj}{j^2-k^2}\right)^2\int_0^\infty \mathrm{d}t\,\frac{\dot\lambda(t')}{\lambda(t')}\int_0^t \frac{\dot\lambda(t')}{\lambda(t')}\cos\left[\Phi_k(t')+\Phi_j(t')-\Phi_k(t)-\Phi_j(t)\right]\mathrm{d}t'\left[\frac{(\omega_k(t)-\omega_j(t))}{(\omega_k(t)\,\omega_j(t))^{\frac12}}\right]\left[\frac{\omega_k(t')-\omega_j(t')}{(\omega_k(t')\,\omega_j(t'))^{\frac12}}\right]. \tag{29}$$

For a given $\lambda(t)$ we can determine $\Phi_k(t')$ from (26) and by integrating over t' and t and summing over j we find $\langle N_k\rangle$. Now let us determine the condition (s) under which the total number of created particles is finite, *i.e.* $\sum\langle N_k\rangle$ has a well-defined value. For this purpose we write

$$\left\langle\sum_{k=1} N_k\right\rangle = 2\sum_{k=1}\sum_{j\neq k}\left(\frac{kj}{j^2-k^2}\right)^2 K_{kj}\,, \tag{30}$$

where K_{kj} is defined by the two integrals in (29). Since $\lambda(t)$ is a finite positive number, it varies between λ_1 and λ_2, *i.e.* $\lambda_1\leqslant\lambda(t)\leqslant\lambda_2$. Then for any integer j such that $j\gg m\lambda_2$, we have

$$\Phi_j(t) = \int_0^t\left[\left(j^2/\lambda^2(t)\right)+m^2\right]^{\frac12}\mathrm{d}t \approx j\int_0^t\frac{\mathrm{d}t}{\lambda(t)}\,. \tag{31}$$

This relation shows that the asymptotic form of K_{kj} for $k\ll\lambda_2 m$ and $j\gg\lambda_2 m$ has a simple form

$$K_{kj}\xrightarrow[k,j\gg\lambda_2 m]{}\int_0^\infty \mathrm{d}t\,\frac{\dot\lambda(t)}{\lambda(t)}\int_0^t\frac{\dot\lambda(t')}{\lambda(t')}\cos\left[(k+j)(z'-z)\right]\mathrm{d}t'\,, \tag{32}$$

where

$$z' = \int_0^{t'}\frac{\mathrm{d}t_1}{\lambda(t_1)}\qquad\text{and}\qquad z = \int_0^t\frac{\mathrm{d}t_1}{\lambda(t_1)}\,. \tag{33}$$

By changing the variables t and t' in (32) to z and z' defined by (33), we find after some simplification that the asymptotic form of K_{kj} is

$$K_{kj}\to\int_0^\infty \mathrm{d}\zeta\,\cos\left[(k+j)\,\zeta\right]f(\zeta)\,, \tag{34}$$

where

$$f(\zeta) = \int_0^\infty \mathrm{d}z\,\frac{\dot\lambda(z)\dot\lambda(z+\zeta)}{\lambda(z)\lambda(z+\zeta)}\,. \tag{35}$$

From (34) if follows that if $f(\zeta)$ and its first derivative are continuous, and if

$$\left(\frac{\mathrm{d}f(\zeta)}{\mathrm{d}\zeta}\right)_{\zeta=0} = 0\,, \tag{36}$$

then K_{kj} decreases at least as $(k+j)^{-4}$ and when this is the case, then the number of created particles is finite. If we substitute (35) in (36) we find that the condition (36) is equivalent to

$$\mathring{\lambda}(t=0) = 0 \tag{37}$$

and this together with the continuity of $\mathring{\lambda}(z)$ are the conditions for $\left\langle \sum_k N_k \right\rangle$ to be finite. Having obtained $\langle N_k \rangle$, we can determine the energy associated with these particles. The Hamiltonian (17) in the limit $t \to \infty$ reduces to

$$H_{\text{eff}}(t=\infty) = \sum_{k=1}^{\infty} \left(m^2 + k^2/\lambda^2(\infty)\right)^{\frac{1}{2}}(N_k + \tfrac{1}{2}) \tag{38}$$

and, therefore, the change of the energy of the system is

$$E_{\mathrm{f}} - E_{\mathrm{i}} = \sum_{k=1} \left\{\left(m^2 + k^2/\lambda^2(\infty)\right)^{\frac{1}{2}}(\langle 0|N_k|0\rangle + \tfrac{1}{2}) - \tfrac{1}{2}(k^2+m^2)^{\frac{1}{2}}\right\}, \tag{39}$$

where the last term, $\frac{1}{2}\omega_k$, is the zero point energy of the field at $t=0$.